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SOLVABILITY AND STABILITY OF SEMILINEAR WAVE EQUATION WITH GENERAL SOURCE AND NONLINEAR BOUNDARY CONDITIONS

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ABSTRACT. We discuss solvability for the semilinear equation of the vibrating string $x_{tt}(t,y) - \Delta x(t,y) + f(t,y,x(t,y)) = 0$ in bounded domain, infinity time interval and certain type of nonlinearity on the boundary. To this effect we derive new dual variational method. Next we discuss stability of solutions with respect to initial conditions.

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1. INTRODUCTION

Throughout the paper, Ω will be a general open bounded domain in $\mathbf{R}^n$, with boundary Γ, assumed to be sufficiently smooth. The aim of the paper is to study existence and stability questions over an infinite interval $[0,\infty)$. We consider the second order hyperbolic semilinear problem with nonlinear sources terms in the equation and on the boundary:

$$\begin{aligned} &x_{tt}(t,y) - \Delta x(t,y) + l(t,y,x(t,y)) = 0, \text{ in } (0,\infty)\times\Omega, \\ &\partial_\nu x(t,y) + x(t,y) = h(t,y,x(t,y)) \quad \text{on } \Sigma = (0,\infty)\times\Gamma, \\ &x(0,\cdot) = x^0(\cdot) \in H^1(\Omega),\ x_t(0,\cdot) = x^1(\cdot) \in L^2(\Omega). \end{aligned} \tag{1.1}$$

We recall first some facts on history of this problem underlying an influence of structural form of nonlinearity l on global solution to (1.1). The supremum of all T 's for which the solution to (1.1) exists on $[0,T)$ is called the lifespan of the solution. We denote this number by $T_{\max}$. We say the solution is global if $T_{\max} = +\infty$ while it is nonglobal if $T_{\max} < +\infty$: If $0 < T_{\max} < +\infty$ we say that the solution blows up in finite time. The forcing term l, of primary interest is the function $l(y,x) = |x|^{p-1}x - \mu^2 x$ when $p > 1$, the difference of polynomial function and linear (monotonic) function. The model theorem, in literature (see [12], compare also [15], for unbounded domain), for such problem stressing also the role of the size of initial conditions is the following

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Theorem 1.1. *Suppose that there is $\epsilon > 0$ such that for all possible $y \in \mathbf{R}^n$ and real values of x we have $xl(y,x) \leq (2+\epsilon)L(y,x)$ (L is antiderivative of l). Suppose further that for some $T > 0$ (1.1) has a solution. Assume that the initial energy*

$$\mathcal{E}(0) = \frac{1}{2}\int_{\mathbf{R}^n} \left|\nabla x^0(y)\right|^2 dy + \frac{1}{2}\int_{\mathbf{R}^n} \left(x^1(y)\right)^2 dy - \int_{\mathbf{R}^n} L(y, x^0(y))dy$$

is negative. Then $T_{\max} < +\infty$.

There is many results extending and sharpening the theorem especialy in the case of $\mu \neq 0$ and of bounded domain. In the beginin of this centuary there appeared many papers that treat the question of global existence and global nonexistence for solutions of the initial value problem for semilinear wave equations which also have damping terms present:

$$\begin{aligned}(1.2)\qquad & x_{tt}(t,y) - \Delta x(t,y) + g(t,y,x_t(t,y)) + l(t,y,x(t,y)) = 0, \text{ in } (0,T)\times\mathbf{R}^n,\\ & x(t,y) = 0 \quad \text{on } \Sigma = (0,T)\times\Gamma,\\ & x(0,\cdot) = x^0(\cdot) \in H^1(\Omega),\ x_t(0,\cdot) = x^1(\cdot) \in L^2(\mathbf{R}^n).\end{aligned}$$

The interaction between the nonlinear damping and source terms creates diffculties which were overcome, for a bounded domain with Dirichlet boundary conditions, by Georgiev and Todorova, Levine and Serrin, Levine, Pucci and Serrin to mention a few (see e.g. [11], [15], [12], [25] and references therein). Summarizing some of these result (see [14], [23], [24]) we can formulate the following

Theorem 1.2. *Assume $x^0 \in H^1$, $x^1 \in L^2$ have compact support and $l(y,x) = |x|^{p-1}x - \mu^2 x$, $g(t,y,x_t) = a\,|x_t|^{m-1}x_t$.*

(a) *If $1 \leq m < p$, then every local solution is global.*
(b) *If $\mu = 0$, $\mathcal{E}(0) < 0$ and $p > m > np/(n+p+1)$, then the local solution blows up in finite time.*

This result shows that both nonlinear damping and the stabilisation term $\mu^2 x$ are insuffcient to prevent the blow up effect of the source for all negative initial energy $\mathcal{E}(0)$ in the above Cauchy problem. Therefore, this result and the blow-up result of Levine [12] in the case of linear damping indirectly show that no condition other than $1 \leq m < p$ should be needed to guarantee the blow-up of the solution, i.e. the restrictions for the initial energy $\mathcal{E}(0)$ and m are method driven. The nature of propagation of singularities and related regularity is very different for bounded domains. The analysis must take into consideration the role of the boundary and the type of boundary conditions imposed on it. It is thus expected that both the results and the methods should depend on the behavior of solutions near the boundary. The model under consideration is equipped with the Neumann nonlinear boundary conditions. It is known that the Lopatinski condition fails for the Neumann problems, causing the loss of 1/3 derivative in linear dynamics driven by boundary sources (unless the dimension of Ω is equal to one). It is thus expected that the boundary and boundary conditions will play a prominent role in the analysis. While internal sources, up to the critical level, do not pose problems with the treatment of local well-posedness, boundary sources, even mildly nonlinear, are problematic and require much more subtle analysis. This is due to the "loss of derivatives" in the linear dynamics. Boundary damping does restore some of the loss of the regularity incurred due to the failure of the Lopatinski condition. The interaction between boundary damping and source has been further exploited, where local existence of solutions was established for boundary sources of a polynomial structure (with the exclusion of super-supercritical exponents) interacting with sufficiently high nonlinear damping. The main goal of the papers, writen in last two years, was twofold: (i) to study well-posedness of the system given by (1.1) on the finite energy space, i.e. $H^1(\Omega) \times L^2(\Omega)$, and (ii) to derive uniform decay rates of the energy when $t \to \infty$. The well-posedness included existence and uniqueness of both local and global solutions. However as it is shown in [7] if we want to admit more general nonlinearities then we cannot expect well-posedness in Hadamard sense. The main difficulty and the novelty of the problem considered is related to the presence on the boundary a nonlinear term $h(t, y, x)$. This difficulty has to do with the fact that Lopatinski condition does not hold for the Neumann problem. The above translates into the fact that in the absence of the damping, the linear map $h \to (x(t), x_t(t))$, is not bounded from $L^2(\Sigma) \to H^1(\Omega) \times L^2(\Omega)$, unless the dimension of Ω is equal to one. More details on this problem is discussed in [7].

The aim of the paper is to study existence and stability questions over an infinite interval $[0, \infty)$ for more general nonlinearities l, h, (l is a difference of continuous and monotonic functions) than in [7], [2] and so we are not interested in well-posedness in Hadamard sense. However, we want still study stability of the system with respect to initial conditions i.e. continuous dependence (in some new sense) with respect to initial conditions.

The importance of the problem with nonlinearity on the boundary appears in optimal control theory see e.g. [16], [17], [18], [7] and the references therein. The problems like (1.1) - with damping terms, were studied mostly by topological method, semigroup theory or monotone operator theory (see [7] for discussion about that). As it is known to the author there is no papers studying (1.1) (with nonlinearity on the boundary) by variational method. We would like to stress that just using the variational approach the dissipation term on the boundary has no influence on the existence and regularity of solution to (1.1). This is why we do not include dissipations in the equation and on the boundary as it is usually done while approaching other methods (see e.g. [7], [2]). What is essential in the method used here is that, following a trick of Galewski [9], [10] we consider first the equation

$$\partial_\nu x(t,y) + x(t,y) = h(t,y,v(t,y)) \text{ on } \Sigma = (0,\infty)\times\Gamma,$$

with v any given function from $C([0,\infty);H^1(\Gamma))$ and put $H = \frac{1}{2}x^2 - x\bar{h}$, which is **convex function**, where $\bar{h}(x,y) = h(t,y,v(t,y))$. Convexity is exploited in the paper in different forms. Just convexity and special structure of nonlinearities eliminate need for dissipations to get global solutions to (1.1). The general nonlinearity of interior source has the form:

$$l(t,y,x) = \pm F_x(t,y,x) - G_x(t,y,x),\ x\in\mathbf{R},\ (t,y)\in(0,\infty)\times\Omega \tag{1.3}$$

where F, G are C^1 in x and F is additionally convex function in x. It turns out that in this case a size of initial conditions is not essential, however some restrictions on initial conditions are hidden in (**H**Γ). Instead we impose some restriction on behavior of nonlinearities h (see (**H**Γ)) and l (see (**As**)) below.

We shall study (1.1) by variational method, i.e. we shall consider (1.1) as the Euler-Lagrange equations of two functionals:

$$\begin{aligned} J(x) &= \int_0^\infty\int_\Omega\left(\frac{1}{2}|\nabla x(t,y)|^2 - \frac{1}{2}|x_t(t,y)|^2 + L(t,y,x(t,y))\right)dy\,dt \\ &\quad - \lim_{T\to\infty}\left(x(T,\cdot),x^1(\cdot)\right)_{L^2(\Omega)}, \end{aligned} \tag{1.4}$$

$$\begin{aligned} J_\Gamma(x) = \int_0^\infty\int_\Gamma &(H(t,y,x(t,y)) + H^*(t,y,-\partial_\nu x(t,y)) \\ &- \langle x(t,y), -\partial_\nu x(t,y)\rangle)dy\,dt, \end{aligned} \tag{1.5}$$

where L, is antiderivative of l with respect to the third variable and H^* is the Fenchel conjugate to H with respect to the third variable, defined on some subspace of functions of the space $C([0,\infty);H^1(\Gamma))$ discussed below. We assume that, $H(t,y,\cdot)$ is finite in $\mathbf{R}$, for each $(t,y)\in(0,\infty)\times\Omega$.

Our purpose is to investigate (1.1) by studying critical points of functionals (1.4) and (1.5). To this effect we apply a new duality approach. As it is easily to see, functionals (1.4) and (1.5) are unbounded in $C([0,\infty);H^1(\Omega))$ and this is a reason for which we are looking for critical points of J on some subsets of $C([0,\infty);H^1(\Omega))$. Our aim is to find a nonlinear subspace X of $C([0,\infty);H^1(\Omega))$ and study (1.4) and (1.5) just only on X. The main difficulty in our approach is just to find the set X. We prove also stability of solutions of (1.1) when $x^0(\cdot)$ and $x^1(\cdot)$ are changing in suitable way. To our best knowledge, this is the first result on stability of solutions for wave flows generated by boundary-interior sources terms. In conclusion, the novel contribution of the present work consists of:

(i) Stability of solutions corresponding to interior and boundary sources is established. This property, when combined with global existence, provides quite new result.

(ii) We are also able to treat different interior and boundary sources. Essential part of the type of nonlinearity is not supercriticality but their structure and boundedness on some bounded sets.

(iii) The methods used in the present paper are very different than the ones used before in the literature [5]. We rely on variational methods combined with suitable convex properties and iteration trick of Galewski, rather than monotonicity methods [5] or the compactness used in [26], [27]. This alone allows to extend the range of Sobolev's exponents for which the analysis is applicable (no need for compactness). It is also believed that the methods could be used successfully in order to treat unbounded domains [15].

First we will study the equation

$$\begin{aligned} &\partial_\nu x(t,y) + x(t,y) = h(t,y,v(t,y)) \text{ on } (0,\infty)\times\Gamma, \\ &x(0,\cdot) = x^0(\cdot) \in H^1(\Gamma), \end{aligned} \tag{1.6}$$

with the help of the functional $J_\Gamma(x)$ (see (1.5)) - section 3. Next we will investigate the equation

$$\begin{aligned} &\partial_\nu x(t,y) + x(t,y) = h(t,y,x(t,y)) \text{ on } (0,\infty)\times\Gamma, \\ &x(0,\cdot) = x^0(\cdot) \in H^1(\Gamma), \end{aligned} \tag{1.7}$$

using an ideas of Galewski on iteration of solutions to (1.6), then we solve

$$\begin{aligned} &x_{tt}(t,y) - \Delta x(t,y) + l(t,y,x(t,y)) = 0 \text{ in } (0,\infty)\times\Omega, \\ &x(t,y) = \bar{x}_\Gamma(t,y) \text{ on } (0,\infty)\times\Gamma,\ \bar{x}_\Gamma(0,\cdot) = x^0(\cdot) \text{ on } \Gamma, \\ &x(0,\cdot) = x^0(\cdot) \in H^1(\Omega),\ x_t(0,\cdot) = x^1(\cdot) \in L^2(\Omega), \end{aligned} \tag{1.8}$$

where $\bar{x}_\Gamma$ is a solution of (1.7), with the help of the functional $J(x)$ (see (1.4)) – section 4, first for the case $l(t,y,x) = F_x(t,y,x) - G_x(t,y,w(t,y))$ with fixed $w \in C([0,\infty); H^1(\Omega))$ and next again with the help of Galewski type iteration, the general case (1.3).

2. MAIN RESULTS

We will focus on the case when $n \geq 3$. First we formulate results concerning problem (1.6), its relation to the functional $J_\Gamma(x)$ and problem (1.8). We explain what do we mean by $x^0(\cdot) \in H^1(\Gamma)$ in (1.6), $x^0(\cdot) \in H^1(\Omega)$ in (1.8) and the normal derivative $\partial_\nu x(t,y)$ in (1.6) and in (1.5). Let $\tilde{H}^1(\Gamma) = \{x \in H^1(\Omega) :\ x|_\Gamma \in H^1(\Gamma)\}$, we shall consider the space $C([0,\infty); \tilde{H}^1(\Gamma))$. Therefore by the normal derivative

$\partial_\nu x(t,y)$ in (1.6) and in (1.5) we mean normal derivative of a function $x(\cdot,\cdot) \in C([0,\infty); H^1(\Omega)) \cap C([0,\infty); \tilde{H}^1(\Gamma))$. Thus let $x^0 \in \tilde{H}^1(\Gamma)$. We use a set

$$U^t = \{x : x(t,\cdot) = x^0(\cdot) + \int_0^t \exp(-s) w(\cdot) ds,\ w \in \tilde{H}^1(\Gamma)\},\ t \in [0,\infty)$$

and the functional

$$J^t_\Gamma(x) = \int_\Gamma (H(t,y,x(t,y)) + H^*(t,y,-\partial_\nu x(t,y)) - \langle x(t,y), -\partial_\nu x(t,y)\rangle) dy,\ t \in [0,\infty), \tag{2.1}$$

considered on U^t, $t \in [0,\infty)$. That means we are looking for solutions to (1.6) in U^t, $t \in [0,\infty)$. Therefore, if such a solution belonging to U^t, $t \in [0,\infty)$, exists it is a strong solution to (1.6).

Assumptions concerning problem (1.6):

($\mathbf{H}\Gamma$) *$h(\cdot,\cdot,v)$ is measurable in $[0,\infty) \times \Gamma$ for each $v \in \mathbf{R}$, $h(t,y,\cdot)$ is continuous in $\mathbf{R}$, for $(t,y) \in [0,\infty) \times \Gamma$ and satisfies growth condition*

$$\|h(t,\cdot,x(t,\cdot))\|_{L^2(\Gamma)} \le c(t) \|x(t,\cdot)\|^\alpha_{L^2(\Gamma)},\ t \in [0,\infty),\ x \in U^t,\ t \in [0,\infty), \tag{2.2}$$

where $\alpha \ge 1$ any fixed, $0 < c(t) \le 1$, $t \in [0,\infty)$, $h(t,y,w) = 0$ only if $w = 0$ and $H(t,y,x) = \frac{1}{2}x^2 - xh(t,y,v(t,y))$, $v(\cdot,\cdot)$ any given from U^t. Moreover we assume that

$$0 < \|x^0\|_{L^2(\Gamma)} < 1. \tag{2.3}$$

Remark 2.1. The main restriction on initial data is made on the boundary Γ: $0 < \|x^0\|_{L^2(\Gamma)} < 1$ and we would like to stress that $x^0 \in \tilde{H}^1(\Gamma)$, thus we impose a little bit more regularity on initial condition $x^0 \in H^1(\Omega)$.

Proposition 2.2. $x_\Gamma(t,\cdot) \in U^t$, $t \in [0,\infty)$ is a solution to (1.6) if and only if $x_\Gamma(t,\cdot)$ affords a minimum to the functional J^t_Γ defined on U^t, $t \in [0,\infty)$ and $J^t_\Gamma(x_\Gamma) = 0$.

Proposition 2.3. Under assumption ($\mathbf{H}\Gamma$) there is an $x_\Gamma(t,\cdot) \in U^t$ which affords a minimum to the functional J^t_Γ defined on U^t, $t \in [0,\infty)$ and $J^t_\Gamma(x_\Gamma) = 0$.

Corollary 2.4. The problem (1.6) has a solution $x_\Gamma(t,\cdot) \in U^t$, $t \in [0,\infty)$.

The next step is to use iteration ideas of Galewski to solve

$$\begin{aligned} &\partial_\nu x(t,y) + x(t,y) = h(t,y,x(t,y)) \text{ on } (0,\infty) \times \Gamma, \\ &x(0,\cdot) = x^0(\cdot) \in \tilde{H}^1(\Gamma). \end{aligned} \tag{2.4}$$

To this effect we take for $v(t,y) = x^0(y) \in U^t$, $t \in [0,\infty)$ and solve by the above Corollary 2.4 the problem

$$\begin{aligned} &\partial_\nu x(t,y) + x(t,y) = h(t,y,x^0(y)) \text{ on } (0,\infty) \times \Gamma, \\ &x(0,\cdot) = x^0(\cdot) \in \tilde{H}^1(\Gamma) \end{aligned} \tag{2.5}$$

getting $x_1(t,\cdot) \in U^t$, $t \in [0,\infty)$. Next putting in the right hand side of (2.5) $x_1(t,\cdot) \in U^t$, $t \in [0,\infty)$ we get $x_2(t,\cdot) \in U^t$, $t \in [0,\infty)$ and so on. In this way we receive a sequence of function $\{x_n(t,\cdot)\} \subset U^t$, $t \in [0,\infty)$ and we come to the following theorem

Theorem 2.5. *The sequence $\{x_n(t,\cdot)\} \subset U^t$, $t \in [0,\infty)$ is convergent to some $\bar{x}_\Gamma(t,\cdot) \in U^t$, $t \in [0,\infty)$ which is a solution to (2.4) and $J^t_\Gamma(\bar{x}_\Gamma) = 0$, $t \in [0,\infty)$.*

We introduce the definition of a weak solution to (1.8):

Definition 2.6. (weak solution). By a weak solution to (1.8), defined on each subinterval $[0,T]$ of $[0,\infty)$, we mean a function $x \in U_T$, where

$$U_T = \Big\{x : x \in C([0,T];H^1(\Omega)),\ \frac{\partial x}{\partial t} \in C([0,T];L^2(\Omega)),\ x(0,y) = x^0(y),$$
$$x_t(0,y) = x^1(y),\ y \in \Omega,\ x(t,y) = \bar{x}_\Gamma(t,y),\ (t,y) \in [0,T]\times\Gamma\Big\},$$

such that for all $\varphi \in U_\varphi$

$$\int_0^T\int_\Omega(-x_t\varphi_t + \nabla x\nabla\varphi)dy\,dt + \int_0^T\int_\Omega x_t\varphi dy\,dt + \int_0^T\int_\Omega l\varphi dy\,dt$$
$$= -\int_\Omega x_t(T,y)\varphi^0(y)dy + \int_\Omega x_t(0,y)\varphi(0,y)dy,$$

where

$$U_\varphi = \Big\{\varphi : \varphi \in C([0,T];H^1(\Omega)),\ \frac{\partial\varphi}{\partial t} \in C([0,T];L^2(\Omega)),$$
$$\varphi(T,y) = \varphi^0(y),\ y \in \Omega,\quad \varphi(t,y) = 0,\ (t,y) \in [0,T]\times\Gamma\Big\}.$$

Remark 2.7. The weak solution considered in this paper is stronger that e.g. in [4] where $x \in C_w([0,T];H^1(\Omega))$, $\frac{\partial x}{\partial t} \in C([0,T];L_2(\Omega))$ ($C_w([0,T];Y)$ denotes the space of weakly continuous functions with values in a Banach space Y).

The main contribution of this paper is a relaxation of assumptions on nolinearities l (but still having the structure of difference of two fuctions: continuous and monotonic) and h. Moreover, in spite of lack of uniqueness results we have global existence theorem to (1.1) and continuous dependence of solutions with respect to initial data in the following sense (because of lack of uniqueness):

Definition 2.8. For given sequences $\{x^0_n\} \subset H^1(\Omega) \cap \tilde{H}^1(\Gamma)$, $\{x^1_n\} \subset L^2(\Omega)$, converging to $\bar{x}^0$ in $H^1(\Omega) \cap \tilde{H}^1(\Gamma)$, $\bar{x}^1$in $L^2(\Omega)$, respectively, there is a subsequence of $\{x_n\}$ - solutions to (1.1) corresponding to $\{x^0_n\}$, $\{x^1_n\}$, which we denote again by $\{x_n\}$ weakly convergent in $H^1((0,\infty)\times\Omega)$ and strongly in $L^2((0,\infty)\times\Omega)$ to an element $\overline{x} \in C([0,\infty);H^1(\Omega))$ $\cap L^1(0,\infty;H^1(\Omega))$, being a solution to (1.1) corresponding to $\bar{x}^0$, $\bar{x}^1$.

To formulate assumptions concerning equation (1.8) we need to recall some theorem from [13] - linear case of (1.8) which we use as a starting point to study (1.8).

Theorem 2.9. *Let* $Q \in L^1(0,\infty;L^2(\Omega))$, $x_\Gamma \in U^t$, $t \in [0,\infty)$, $x^0 \in H^1(\Omega) \cap \tilde{H}^1(\Gamma)$, $x^1 \in L^2(\Omega)$ *with the compatibility condition:* $x_\Gamma(0,y) = x^0(y)$, $y \in \Gamma$. *Then there exists* $\overline{x}$ *being a unique weak solution to*

$$\begin{aligned} & x_{tt}(t,y) - \Delta x(t,y) = Q(t,y), \\ & x(0,y) = x^0(y),\ x_t(0,y) = x^1(y),\ y \in \Omega, \qquad (2.6) \\ & x(t,y) = \bar{x}_\Gamma(t,y),\ (t,y) \in \Sigma \end{aligned}$$

and such that

$$\overline{x} \in C([0,\infty);H^1(\Omega)) \cap L^2(0,\infty;H^1(\Omega)) = U_x,$$

$$\overline{x}_t \in C([0,\infty);L^2(\Omega)) \cap L^2(0,\infty;L^2(\Omega)) = U_{x_t},$$

$$\partial_\nu \bar{x} \in L^2(\Sigma),$$

$$\|\overline{x}\|_{U_x} \le C(\|Q\|_{L^1(0,\infty;L^2(\Omega))} + \|\bar{x}_\Gamma\|_{H^1(\Sigma)} + \left\|x^0\right\|_{H^1(\Omega)} + \left\|x^1\right\|_{L^2(\Omega)}),$$

$$\|\overline{x}_t\|_{U_{x_t}} \le D(\|Q\|_{L^1(0,\infty;L^2(\Omega))} + \|\bar{x}_\Gamma\|_{H^1(\Sigma)} + \left\|x^0\right\|_{H^1(\Omega)} + \left\|x^1\right\|_{L^2(\Omega)})$$

with some $C > 0$, $D > 0$ *independent on* Q *and* $\|\cdot\|_{U_x} = \|\cdot\|_{C([0,\infty);H^1(\Omega))} + \|\cdot\|_{L^2(0,\infty;H^1(\Omega))}$, $\|\cdot\|_{U_{x_t}} = \|\cdot\|_{C([0,\infty);L^2(\Omega))} + \|\cdot\|_{L^2(0,\infty;L^2(\Omega))}$.

Shortly the solution $\overline{x}$ from the above theorem may be estimated by

$$\|\overline{x}\|_{U_x} \le C\,\|Q\|_{L^1(0,\infty;L^2(\Omega))} + E^w, \qquad (2.7)$$

$$\|\overline{x}_t\|_{U_{x_t}} \le D\,\|Q\|_{L^1(0,\infty;L^2(\Omega))} + A^w, \qquad (2.8)$$

where $E^w = C(\|\bar{x}_\Gamma\|_{H^1(\Sigma)} + \|x^0\|_{H^1(\Omega)} + \|x^1\|_{L^2(\Omega)})$ and $A^w = D(\|\bar{x}_\Gamma\|_{H^1(\Sigma)} + \|x^0\|_{H^1(\Omega)} + \|x^1\|_{L^2(\Omega)})$ and in the paper we will just use the last estimations. Everywhere below the constants C and D will always denote those occurring in (2.7), (2.8).

Let us put for $\bar{x}_\Gamma \in U^t$, $t \in [0,\infty)$

$$\begin{aligned} U \;=\; \big\{ & x : x \in C([0,\infty);H^1(\Omega)) \cap L^2(0,\infty;H^1(\Omega)), \\ & \tfrac{\partial x}{\partial t} \in C([0,\infty);L^2(\Omega)) \cap L^2(0,\infty;L^2(\Omega)),\ x(0,y) = x^0(y), \\ & x_t(0,y) = x^1(y),\ y \in \Omega,\ x(t,y) = \bar{x}_\Gamma(t,y),\ (t,y) \in \Sigma \big\}. \end{aligned}$$

We shall consider U with a topology induced by the norm $\|x\|_U = \|x\|_{U_x} + \|x_t\|_{U_{x_t}}$.

Assumptions concerning equation (1.1).

(**As**) *Let* F *and* G *of the variable* (t,y,x) *be given.* F *and* G *are measurable in* (t,y) *and continuously differentiable with respect to third variable in* $\mathbf{R}$, F *is convex in* x *and satisfies*

$$F(t,y,x) \ge a(t,y)x + b(t,y),$$

for some $a, b \in L^1(0,\infty;L^2(\Omega))$*, for all* $x \in \mathbf{R}$ *and* $(t,y) \in [0,\infty)\times\Omega$*.Assume that our original nonlinearity (see (1.8)) has the form*

$$l = F_x - G_x \tag{2.9}$$

or

$$l = -F_x - G_x$$

and that there exist a ball $\mathbf{F}$ *with center at zero in* U*,* $\hat{x} \in \mathbf{F}$*, constants* E_F*,* E_G*,* E *such that* $\|F_x(x)\|_{L^1(0,\infty;L^2(\Omega))} \le E_F$*,* $\|G_x(x)\|_{L^1(0,\infty;L^2(\Omega))} \le E_G$*,* $\|F(\hat{x})\|_{L^1((0,\infty)\times\Omega)} < \infty$*,* $\|G(x)\|_{L^1((0,\infty)\times\Omega)} \le E$ *(we use the notation* $K_x(z) = K_x(t,y,z(t,y)))$*, for* $x \in \mathbf{F}$*. Put*

$$X^F = \{v \in U : \ \|v\|_U \le C(E_F + E_G) + E^w\} \tag{2.10}$$

Moreover we assume that

$$\|F_x(x) - G_x(x)\|_{L^1(0,\infty;L^2(\Omega))} \le E_F + E_G \tag{2.11}$$

or

$$\|-F_x(x) - G_x(x)\|_{L^1(0,\infty;L^2(\Omega))} \le \ E_F + E_G$$

for $x \in X^F$*.*

In $X^F \times X^F$ define the map $X^F \times X^F \ni (x,w) \to \mathcal{H}(x,w) = v$ where v is a solution to the following problem

$$\begin{gathered} v_{tt}(t,y) - \Delta v(t,y) = -F_x(t,y,x(t,y)) + G_x(t,y,w(t,y)) \ \text{in } (0,\infty)\times\Omega, \\ v(0,y) = x^0(y), \ v_t(0,y) = x^1(y), \ y \in \Omega, \\ v(t,y) = \bar{x}_\Gamma(t,y), \ (t,y) \in \Sigma. \end{gathered}$$

Let $\bar{X}^F = \mathcal{H}(X^F \times X^F)$. Now we are able to formulate the main theorem of the paper.

Theorem 2.10 (Main theorem)**.** *Under* ***(As)*** *there exists* $\overline{x} \in \bar{X}^F$ *such that*

$$J(\overline{x}) = \inf_{x \in \bar{X}^F} J(x)$$

and $\overline{x}$ *is a weak solution to* (1.8)*.*

Remark 2.11. The known results concerning existence of global solution to (1.1) (without uniqueness) assume that nonlinearity l is locally Lipschitz and sublinear at infinity or of polynomial type with additional boundedness for initial data (see e.g. [7]). We do not assume that l is locally Lipschitz – it needs not to be – instead, we assume the special structure of l namely that it is difference of two functions: continuous and monotonic. This structure certainly contains a polynomial type of nonlinearity. Thus, in that, it extends some results of [7]. Moreover we assume boundedness of $\|F_x(\cdot)\|_{L^1(0,\infty;L^2(\Omega))}$, $\|G_x(\cdot)\|_{L^1(0,\infty;L^2(\Omega))}$ in some ball $\mathbf{F} \subset U$, which restrict behavior of F_x, G_x with respect to t. Some type of boudedness for initial data is hidden in inequalities (2.3), (2.11) and that we are looking for soluton in X^F. We would like to stress that l is only continuous in x. We have not problems with subcritical, supercritical or super-supercritical sources as just the special type of l and the assumption (2.3) (in spite that it is enough general) eliminate such cases. Moreover we are looking for solutions in **bounded** set X^F, so all solutions are bounded in the norm of U. This is a case which is often assumed to get global solution (see [7], [4] and references therein).

We have no uniqueness of solutions to (1.1), but a kind of continuous dependence on initial data is still possible.

Theorem 2.12. *Assume (**H**Γ) and (**As**). Let* $\{x_n^0\}$, $\{x_n^1\}$ *be given sequences in* $H^1(\Omega) \cap \tilde{H}^1(\Gamma)$, $L^2(\Omega)$ *respectively, (with* $0 < \|x_n^0\|_{L^2(\Gamma)} < 1$*), converging to* $\bar{x}^0$, $\bar{x}^1$ *in* $H^1(\Omega) \cap \tilde{H}^1(\Gamma)$, $L_2(\Omega)$ *and such that* $\|x_n^0\|_{H^1(\Omega)} \leq C(E_F + E_G) + E^w$, $\|x_n^1\|_{L^2(\Omega)} \leq D(E_F + E_G) + A^w$, $n = 1, 2 \ldots$. *Then there is a subsequence of* $\{x_n\}$ *- solutions to (1.1) corresponding to* $\{x_n^0\}$, $\{x_n^1\}$*, which we denote again by* $\{x_n\}$ *weakly convergent in* $H^1((0,\infty)\times\Omega)$ *and strongly in* $L^2((0,\infty)\times\Omega)$ *to an element* $\bar{x} \in U$ *being a solution to* (1.1) *corresponding to* $\bar{x}^0$, $\bar{x}^1$.

Let F^* be the Fenchel conjugate of F. Define a dual to J functional

$$J_D : L^2(0,\infty; L^2(\Omega)) \times L^2(0,\infty; L^2(\Omega)) \times L^2(0,\infty; L^2(\Omega)) \to \mathbf{R},$$

as:

$$\begin{aligned} J_D(p,q,z) = &-\int_0^\infty \int_\Omega F^*(t,y,\Big(\lim_{T\to\infty} p_t(T-t,y) + div\ q(t,y) + z(t,y)\Big))dy\,dt \\ &- \frac{1}{2}\int_0^\infty \int_\Omega |q(t,y)|^2\,dy\,dt + \frac{1}{2}\int_0^\infty \int_\Omega \Big|\lim_{T\to\infty} p(T-t,y)\Big|^2\,dy\,dt \\ &+ \Big(x^0(\cdot), \lim_{T\to\infty} p(T,\cdot)\Big)_{L_2(\Omega)} - \int_\Sigma \bar{x}_\Gamma(t,y)\,(q(t,y),\nu(y))\,dy\,dt, \end{aligned}$$

where $\nu = (\nu_1,\ldots,\nu_n)$ is the unit outward normal to Γ. Now we can formulate theorem which gives us additional informations on solutions to (1.1) important in classical mechanics. This theorem is absolutely new for problem (1.1).

Theorem 2.13 (Variational principle and duality result)**.** *Assume (**As**). Let* $\overline{x} \in \bar{X}^F$ *be such that* $J(\overline{x}) = \inf_{x\in\bar{X}^F} J(x)$. *Then there exists* $(\bar{p},\bar{q}) \in L^2(0,\infty; L^2(\Omega)) \times L^2(0,\infty; L^2(\Omega))$ *such that for a.e.* $(t,y) \in (0,\infty)\times\Omega$,

$$\lim_{T\to\infty} \bar{p}(T-t,y) = \bar{x}_t(t,y), \tag{2.12}$$

$$\bar{q}(t,y) = \nabla\bar{x}(t,y), \tag{2.13}$$

$$\lim_{T\to\infty} \bar{p}_t(T-t,y) + div\ \overline{q}(t,y) - l(t,y,\bar{x}(t,y)) = 0 \tag{2.14}$$

and

$$J(\bar{x}) = J_D(\bar{p}, \bar{q}, \bar{z}),$$

where

$$\bar{z} = G_x(t, y, \bar{x}(t, y)). \tag{2.15}$$

The proofs of theorems are given in Sections 3, 4. They consist of several steps. First we prove Propositions 1, 2 and Corollary 3, i.e. we solve problem (1.6). Next we prove Theorem 2.10 (Main theorem). First for nonlinearity l consisting only of one function F_x, next by iteration method of Galewski for general case.

3. PROOF OF EXISTENCE FOR PROBLEM (1.6)

As mentioned before, the main difficulty of the problem under study is the fact that the Neumann problem does not satisfy Lopatinski condition and therefore, the map from the boundary data in $L^2(\Sigma)$ into finite energy space is not bounded (unless dimension of Ω is equal to one). In order to cope with this problem in [7], [4] a regularizing term - strongly monotone dissipation is introduced, whose effect is to 'force' the Lopatinski condition. We follow, in quite, different way. We convert the problem (1.6) into variational one and this allow us to omit the meaning of the dissipation term. However, the price we pay for that is the type of nonlinearity for the boundary source term h (see ($\mathbf{H}\Gamma$)).

Being inspired by Brezis-Ekeland [6] (see also [21]) we formulate the variational principle for problem (1.6).

Proposition 3.1. $\bar{x}(t, \cdot) \in U^t$, $t \in [0, \infty)$ is a solution to (1.6) if and only if $\bar{x}$ affords a minimum to the functional J^t_Γ defined on U^t, $t \in [0, \infty)$ and $J^t_\Gamma(\bar{x}) = 0$, $t \in [0, \infty)$.

Proof. It is a simple consequence of the equivalence of the following two relations in $(0, \infty) \times \Gamma$:

$$\textit{(i)} \ -\partial_\nu \bar{x}(t, y) \in \partial H(t, y, \bar{x}(t, y)), \tag{3.1}$$

$$\textit{(ii)} \ \ H(t, y, \bar{x}(t, y)) + H^*(t, y, -\partial_\nu \bar{x}(t, y)) = \langle \bar{x}(t, y), -\partial_\nu \bar{x}(t, y)\rangle \tag{3.2}$$

and the inequality

$$H(t, y, x(t, y)) + H^*(t, y, -\partial_\nu x(t, y)) \geq \langle x(t, y), -\partial_\nu x(t, y)\rangle$$

which holds for all $x(t, \cdot) \in U^t$, $t \in [0, \infty)$. □

Let us notice that the sets U^t are nonempty. Instead of consider the functional (1.5) we will study the functional $J^t_\Gamma(x)$ on U^t, $t \in [0, \infty)$ and suitable Proposition 2.2, which proof we omit. We prove under hypothesis ($\mathbf{H}\Gamma$) that there exists a minimum to the functional J^t_Γ defined on U^t, $t \in [0, \infty)$, i.e. the proof of Proposition 2.3.

Proof. (of Proposition 2.3). Let us notice that $J_\Gamma^t(x(t,\cdot))$ is bounded below in U^t, $t\in[0,\infty)$, in fact, $J_\Gamma^t(x(t,\cdot))\geq 0$ and weakly lower semicontinuous in U^t, $t\in[0,T]$. Moreover, let us observe, by **(H**Γ**)**, $J_\Gamma^t(x(t,\cdot))\to\infty$ when $\|x(t,\cdot)\|_{H^1(\Gamma)}\to\infty$. Really, it is enough to notice that for $t\in[0,\infty)$

$$\int_\Gamma H(t,y,x(t,y))dy=\int_\Gamma(\frac{1}{2}x(t,y)^2-x(t,y)\bar{h}(t,y))dy, \tag{3.3}$$

$$\begin{aligned}\int_\Gamma H^*(t,y,-\partial_\nu x(t,y))dy &= \int_\Gamma \frac{1}{2}\left(\partial_\nu x(t,y)+\bar{h}(t,y)\right)^2 dy \\ &\geq \int_\Gamma(\frac{1}{2}\left||\partial_\nu x(t,y)|^2-\left(\bar{h}(t,y)\right)^2\right|)dy \\ &\geq \frac{1}{2}\left(\|\nabla x(t,\cdot)\nu\|_{L^2}^2-\left\|\bar{h}(t,\cdot)\right\|_{L^2}^2\right).\end{aligned} \tag{3.4}$$

From (3.3) and (**H**Γ) we infer that $\|x_n(t,\cdot)\|_{L^2(\Gamma)}$ is bounded for minimizing sequence $\{x_n(t,\cdot)\}$ of J_Γ^t and next from (**??**) that $\|\nabla x_n(t,\cdot)\|_{L^2(\Gamma)}$ is bounded. Thus, there is a subsequence of $\{x_n(t,\cdot)\}$ (which we again denote by $\{x_n(t,\cdot)\}$) such that it is weakly in $H^1(\Gamma)$ convergent to some x_Γ and pointwise convergent to it. Therefore, for each $t\in[0,\infty)$,

$$\liminf_{n\to\infty} J_\Gamma^t(x_n(t,\cdot))\geq J_\Gamma^t(x_\Gamma(t,\cdot)).$$

Now let us define the dual functional to J_Γ^t, $t\in[0,\infty)$, by

$$\begin{aligned}J_{\Gamma D}^t(x)=\int_\Gamma&(H(t,y,-x(t,y))+H^*(t,y,\partial_\nu x(t,y)-g(t,y,-x_t(t,y))) \\ &-\langle x(t,y),\partial_\nu x(t,y)\rangle)dy,\ t\in[0,\infty).\end{aligned} \tag{3.5}$$

It is clear that $J_\Gamma^t(x(t,\cdot))=J_{\Gamma D}^t(-x(t,\cdot))$ for all $x(t,\cdot)\in U^t$ and so

$$\inf_{x\in U^t} J_\Gamma^t(x(t,\cdot))=\inf_{x\in U^t} J_{\Gamma D}^t(x(t,\cdot)),\ t\in[0,\infty).$$

By the duality theory for convex functionals (see [8] and [21]) we have that $\inf_{x\in U^t} J_\Gamma^t(x(t,\cdot))=-\inf_{x\in U^t} J_{\Gamma D}^t(x(t,\cdot))$, $t\in[0,\infty)$. Hence we infer that $J_\Gamma^t(x_\Gamma(t,\cdot))=0$. □

Proof. (of Theorem 2.5) Let us take in (1.6) for $v(t,y)=x^0(y)$. Then by Corollary 2.4 there exists $x_1(t,\cdot)\in U^t$, $t\in[0,\infty)$ being a solution to (1.6). Putting again in (1.6) for $v(t,y)=x_1(t,y)$ we obtain $x_2(t,\cdot)\in U^t$, $t\in[0,T]$ being a solution to (1.6) with this v. Continuing this substitutions we obtain a sequence $x_n(t,\cdot)\in U^t$, $t\in[0,\infty)$, $n=0,1,2,\ldots$. We assert that this sequence has the following properties:

$$h(t,y,x_{n-1}(t,y))=x_n(t,y),\ n=1,2,\ldots. \tag{3.6}$$

Really, by the form of H,

$$H^*(t,y,-\partial_\nu x(t,y))=\frac{1}{2}\left(\partial_\nu x(t,y)+\bar{h}(t,y)\right)^2$$

and (3.1), (3.2) we have for the sequence $x_n(t,\cdot)\in U^t$, $t\in[0,\infty)$, $n=0,1,2,\dots$,

$$\partial_\nu x_n(t,y)-h(t,y,x_{n-1}(t,y))=-x_n(t,y),$$

$$\partial_\nu x_n(t,y)+h(t,y,x_{n-1}(t,y))=x_n(t,y).$$

Taking into account both equalities we get (3.6). Hence and by the assumption (2.2) on h and on x^0 we infer that $\|x_n(t,\cdot)\|_{L^2(\Gamma)}$ is bounded. Following in the same way as in the proof of Proposition 2.3 we obtain that $\|\nabla x_n(t,\cdot)\|_{L_2(\Gamma)}$ is bounded. Thus there is a subsequence of $\{x_n(t,\cdot)\}$, which we again denote by $\{x_n(t,\cdot)\}$, that is weakly in $H^1(\Gamma)$ and pointwise in Γ convergent to some $\bar{x}_\Gamma(t,\cdot)$. Therefore, by $(\mathbf{H}\Gamma)$, there exists the limit $\lim_{n\to\infty}(-h(t,y,x_{n-1}(t,y))+x_n(t,y))=-h(t,y,\bar{x}_\Gamma(t,y))+\bar{x}_\Gamma(t,y)$ and so $\bar{x}_\Gamma(t,\cdot)$ satisfies the equation (2.4) and relation (3.2). □

4. PROOF OF EXISTENCE FOR PROBLEM (1.8)

4.1. **Auxilliary problem with $l(t,y,x)=F_x(t,y,x)-\bar{G}_x(t,y)$.** First consider another equation (with $\bar{x}_\Gamma\in H^1(\Sigma)$ being a solution to (2.4))

$$\begin{aligned}&x_{tt}(t,y)-\Delta x(t,y)+F_x(t,y,x(t,y))-\bar{G}_x(t,y)=0,\ \text{in }(0,\infty)\times\Omega,\\&x(t,y)=\bar{x}_\Gamma(t,y),\ (t,y)\in\Sigma,\\&x(0,\cdot)=x^0(\cdot)\in H^1(\Omega)\cap\tilde{H}^1(\Gamma),\ x_t(0,\cdot)=x^1(\cdot)\in L^2(\Omega),\end{aligned}\tag{4.1}$$

where $\bar{G}_x(t,y)=G_x(t,y,w(t,y))$ with $w(\cdot,\cdot)$ being any fixed function from X^F and corresponding to (4.1) functional

$$\begin{aligned}J^F(x)\ =\ &\int_0^\infty\int_\Omega\left(\frac{1}{2}|\nabla x(t,y)|^2-\frac{1}{2}|x_t(t,y)|^2\right)dy\,dt\\&+\int_0^\infty\int_\Omega F(t,y,x(t,y))-x(t,y)\,\bar{G}_x(t,y))dy\,dt\\&-\lim_{T\to\infty}\left(x(T,\cdot),x^1(\cdot)\right)_{L^2(\Omega)}\end{aligned}\tag{4.2}$$

defined on U. (4.1) is the Euler-Lagrange equation for the action functional J^F.

The dual functional, now reads

$$\begin{aligned}&J_D^F(p,q,z)\\&=-\textstyle\int_0^\infty\int_\Omega F^*(t,y,(\lim_{T\to\infty}p_t(T-t,y)+div\ q(t,y)+z(t,y)))\,dy\,dt\\&-\tfrac{1}{2}\textstyle\int_0^\infty\|q(t,\cdot)\|^2_{L^2(\Omega)}dt+\tfrac{1}{2}\int_0^\infty\|\lim_{T\to\infty}p(T-t,\cdot)\|^2_{L^2(\Omega)}dt\\&+\lim_{T\to\infty}(x^0(\cdot),p(T,\cdot))_{L^2(\Omega)}-\textstyle\int_\Sigma\bar{x}_\Gamma(t,y)\,(q(t,y),\nu(y))\,dy\,dt,\end{aligned}\tag{4.3}$$

for a.e. $t\in[0,\infty)$, $z(t,y)=\bar{G}_x(t,y)$, $\lim_{T\to\infty}p_t(T-t,\cdot)+div\ q(t,\cdot)$ is an element of $L^2(\Omega)$ and $F^*(t,y,\cdot)$ is the Fenchel conjugate to $x\to F(t,y,x)-\ x\bar{G}_x(t,y)$, i.e. $F^*(t,y,h)=\sup_{d\in\mathbf{R}}\{hd-\big(F(t,y,d)-d\bar{G}_x(t,y)\big)\}$, for $(t,y)\in[0,\infty)\times\Omega$, $h\in\mathbf{R}$, hence

$$\int_0^\infty\int_\Omega F^*(t,y,\lim_{T\to\infty}p_t(T-t,y)+div\ q(t,y)+z(t,y)))dy\,dt$$

$$= \sup_{v\in L^2((0,\infty)\times\Omega)} \{\int_0^\infty \int_\Omega (\lim_{T\to\infty} p_t\,(T-t,y) + divq\,(t,y))v(t,y)dy\,dt$$
$$- \int_0^\infty \int_\Omega (F\,(t,y,v\,(t,y)) - v\,(t,y)\,z(t,y))\,dy\,dt\}$$

and $J_D^F : U_D \to \mathbf{R}$, where

$$U_D = \Big\{(p,q) : p \in C([0,\infty); L^2(\Omega)),\ \ \lim_{T\to\infty} p_t\,(T-\cdot,\cdot)$$
$$+div\ q\,(\cdot,\cdot) \in L^1(0,\infty; L^2(\Omega))\ ,$$
$$p(0,\cdot) = x^1(\cdot),\ q \in L^2(0,\infty; \big(L^2(\Omega)\big)^n)\Big\}\,.$$

We prove the following

Lemma 4.1. *There exist constants C_1^w, C_2^w, C_3^w, C_4^w independent on $x \in X^F$ such that*

$$\begin{aligned}\|v\|_{L^2(0,\infty;H^1(\Omega))} &\le C_2^w,\ \|v_t\|_{L^2(0,\infty;L^2(\Omega))} \le C_1^w,\\ \|v_{tt}\|_{L^2(0,\infty;H^{-1}(\Omega))} &\le C_3^w,\ \|\Delta v\|_{L^2(0,\infty;H^{-1}(\Omega))} \le C_4^w,\end{aligned}$$
$$\|v\|_{U_x} \le C(E_F + E_G) + E^w,$$
$$\|v_t\|_{U_{x_t}} \le D(E_F + E_G) + A^w$$

where v is the weak solution of the problem

$$v_{tt}(t,y) - \Delta v(t,y) = -F_x(t,y,x(t,y)) + \bar{G}_x(t,y)\ in\ (0,\infty)\times\Omega, \tag{4.4}$$
$$v(0,y) = x^0(y),\ v_t(0,y) = x^1(y),\ y \in \Omega,$$
$$v(t,y) = \bar{x}_\Gamma(t,y),\ (t,y) \in \Sigma,$$

with $x \in X^F$.

Proof. Fix arbitrary $x \in X^F$. Since $x \in U$ and by the assumptions on F and G, see (**As**), it follows that $F_x(\cdot,\cdot,x(\cdot,\cdot))$, $\bar{G}_x(\cdot,\cdot) \in L^1(0,\infty; L_2(\Omega))$. Hence by Theorem 2.9 and (2.7) there exists a unique solution $v \in U$ of our problem for the equation (4.4) satisfying $\|v\|_{U_x} \le CE + E^w$. Taking into account the definition of the set X^F we get the following estimation, independent on $x \in X^F$

$$\|v\|_{L^2(0,\infty;H^1(\Omega))} \le C(E_F + E_G) + E^w,$$

by (2.8)

$$\|v_t\|_{L^2(0,\infty;L^2(\Omega))} \le D(E_F + E_G) + A^w$$

and since for some E_1: $\|\Delta v\|_{L^2(0,\infty;H^{-1}(\Omega))} \le E_1 \|v\|_{L^2(0,\infty;H^1(\Omega))}$ thus

$$\|v_{tt}\|_{L^2(0,\infty;H^{-1}(\Omega))} \le E_1 C(E_F + E_G) + E_1 E^w + (E_F + E_G).$$

Hence, putting

$$C_1^w = D(E_F + E_G) + A^w,$$

$$C_2^w = C(E_F + E_G) + E^w,$$

$$C_3^w = E_1 C(E_F + E_G) + E_1 E^w + (E_F + E_G)$$

and

$$C_4^w = E_1 C(E_F + E_G) + E_1 E^w$$

we infer the first assertion of the lemma. Further by (2.7) we get

$$\|v\|_{U_x} \le C(E_F + E_G) + E^w,$$

$$\|v_t\|_{U_{x_t}} \le D(E_F + E_G) + A^w$$

and we get the last assertion of the lemma. □

Proposition 4.2. For every $x \in X^F$ the weak solution $\widetilde{x}$ of the problem

$$\widetilde{x}_{tt}(t,y) - \Delta\widetilde{x}(t,y) = -F_x(t,y,x(t,y)) + \bar{G}_x(t,y), \tag{4.5}$$

$$\widetilde{x}(0,y) = x^0(y),\ \widetilde{x}_t(0,y) = x^1(y),\ y \in \Omega,$$

$$\widetilde{x}(t,y) = \bar{x}_\Gamma(t,y),\ (t,y) \in \Sigma$$

belongs to X^F.

Proof. Fix arbitrary $x \in X^F$, $F_x(\cdot,\cdot,x(\cdot,\cdot)) \in L^1(0,\infty;L^2(\Omega))$. Hence by Theorem 2.9 there exists a unique weak solution $\widetilde{x} \in U$ of problem (4.5). Moreover $\widetilde{x}_{tt} - \Delta\widetilde{x} \in L^1(0,\infty;L^2(\Omega))$. By the definition of the set X^F, it follows that

$$\left\|F_x(\cdot,\cdot,x(\cdot,\cdot)) - \bar{G}_x(\cdot,\cdot)\right\|_{L^1(0,\infty;L^2(\Omega))} \le E_F + E_G.$$

Further by (2.7) we get

$$\|\widetilde{x}\|_{U_x} \le C(E_F + E_G) + E^w.$$

Thus for an arbitrary $x \in X^F$ there exists an $\widetilde{x} \in X^F$. □

Remark 4.3. Proposition 4.2 apparently may suggest that it is more convenient to apply a suitable fixed point theorem in order to get the existence of weak solutions to problem (1.1). Indeed, the above mentioned proposition states that the map $\mathcal{H}_w$ assigning to $x \in X^F$ a weak solution $\widetilde{x} \in X^F$ of (4.5) has the property $\mathcal{H}_w(X^F) \subset X^F$. However this is only starting point in fixed point theory. In order to proceed with the so called topological method we must prove that the map $\mathcal{H}_w$ and set $\mathcal{H}_w(X^F)$ possess suitable properties. However the assumptions **(As)** do not imply directly (if at all) neither that $\mathcal{H}_w$ is contraction nor that $\mathcal{H}_w(X^F)$ is convex or relatively compact. We have chosen variational approach which ensures not only the existence of solutions but also certain variational properties of solutions which are absolutely new in that case.

Let us put $\bar{X}^F_w = \mathcal{H}_w(X^F)$. Of course, $\bar{X}^F = \cup_{w \in X^F} \bar{X}^F_w$.

Lemma 4.4. *Each $x \in \bar{X}^F$ has the weak derivatives x_{tt} and Δx belonging to $L^2(0,\infty; H^{-1}(\Omega))$. Moreover, sets*

$$X_{tt} = \left\{x_{tt} : x \in \bar{X}^F\right\}, \ X_\Delta = \left\{\Delta x : x \in \bar{X}^F\right\}$$

are bounded in $L^2(0,\infty;H^{-1}(\Omega))$. Hence each sequence $\left\{x_{tt}^j\right\}$ from X_{tt} has subsequence converging weakly in $L^2(0,\infty;H^{-1}(\Omega))$ to a certain element from set X_{tt} and sequence $\left\{x_t^j\right\}$ converges strongly in $L^2(0,\infty;L^2(\Omega))$.

Proof. It is clear, by Lemma 4.1, that $\left\{x_{tt}^j\right\}$ from X_{tt} has a subsequence converging weakly in $L^2(0,\infty;H^{-1}(\Omega))$. By the same lemma corresponding sequence $\{x^j\}$ is also weakly (up to some subsequence) converging in $L^2(0,\infty;H^1(\Omega))$ to some element $\bar{x}$. By the definition of X^F and uniqueness of weak solutions of (4.5) we infer that $\left\{x_{tt}^j\right\}$ has subsequence weakly convergent to $\bar{x}_{tt}$. The last, the fact that $\left\{x_t^j\right\}$ is bounded in $L^2(0,\infty;L^2(\Omega))$ and known theorem imply that for this subsequent $\left\{x_t^j\right\}$ is convergent strongly in $L^2(0,\infty;L_2(\Omega))$ to $\bar{x}_t$. □

We observe that functionals J^F is well defined on X^F. Moreover by **(As)** and convexity of $F(t,y,\cdot)$ we have boundedness of

$$x \to \int_0^\infty \int_\Omega F(t,y,x(t,y))dy\,dt \text{ and } x \to \int_0^\infty \int_\Omega G(t,y,x(t,y))dy\,dt \tag{4.6}$$

in X^F.

Lemma 4.5. *The functional J^F attains its minimum in $\bar{X}_w^F$ i.e.*

$$\inf_{x\in\bar{X}_w^F} J^F(x) = J^F(\overline{x}),$$

where $\overline{x} \in \bar{X}_w^F$.

Proof. By definition of the set $\bar{X}_w^F$ and (4.6) we see that the functional J^F is bounded in X^F. We denote by $\{x^j\}$ a minimizing sequence for J^F in $\bar{X}_w^F$. This sequence has a subsequence which we denote again by $\{x^j\}$ converging weakly in $L^2(0,\infty;H^1(\Omega))$ and strongly in $L^2(0,\infty;L^2(\Omega))$, hence also strongly in $L^2((0,\infty)\times\Omega;\mathbf{R})$ to a certain element $\overline{x} \in U$. Moreover $\{x^j\}$ is also convergent almost everywhere. Thus by the construction of the set $\bar{X}_w^F$ and uniqueness of weak solutions of (4.5) we observe that $\overline{x} \in \bar{X}_w^F$. Hence

$$\liminf_{j\to\infty} J^F\left(x^j\right) \geq J^F(\bar{x}).$$

Thus

$$\inf_{x\in\bar{X}_F} J^F\left(x^j\right) = J^F(\bar{x}).$$

□

To consider properly the dual action functional let us put

$$W_t^{1F} = \{p \in C([0,\infty); L^2(\Omega)) : \\ p_t \in L^2(0,\infty; H^{-1}(\Omega)),\ p(0,\cdot) = x^1(\cdot)\}$$

and

$$W_y^{1F} = \Big\{q \in L^2(0,\infty; (L^2(\Omega))^n) : \text{there exists } p \in W_t^{1F} \\ \text{such that } \lim_{T\to\infty} p_t(T-\cdot,\cdot) + div\ q(\cdot,\cdot) \in L^2(0,\infty; L^2(\Omega))\Big\}.$$

and define a set on which a dual functional will be considered.

Definition of X^{Fd} : We say that an element $(p,q) \in W_t^{1F} \times W_y^{1F}$ belongs to X^{Fd} provided that there exists $x \in \bar{X}_w^F$ such that for a.e. $t \in (0,\infty)$

$$-\lim_{T\to\infty} p_t(T-t,\cdot) - div\ q(t,\cdot) = -F_x(t,\cdot,x(t,\cdot)) + \bar{G}_x(t,\cdot) \tag{4.7}$$

with

$$q(t,\cdot) = \nabla x(t,\cdot)$$

or else

$$\lim_{T\to\infty} p(T-t,\cdot) = x_t(t,\cdot) \text{ with } q(t,\cdot) = \nabla x(t,\cdot). \tag{4.8}$$

Remark 4.6. The definition of X^{Fd} says that for each $x \in \bar{X}_w^F$ there exist in X^{Fd} two pairs of (p,q): one defined by (4.7), second defined by (4.8).

We observe that neither $\bar{X}_w^F$ nor X^{Fd} is a linear space. Thus even standard calculations using convexity arguments are rather difficult. What helps us is a special structure of the sets $\bar{X}_w^F$ and X^{Fd} which despite their nonlinearity makes these calculations possible.

Now we may state the first step to main result of the paper which is the following existence theorem.

Theorem 4.7. *There is* $\overline{x} \in \bar{X}_w^F$, *that* $\inf_{x\in\bar{X}_w^F} J^F(x) = J^F(\overline{x})$. *Moreover, there is* $(\overline{p},\overline{q}) \in X^{Fd}$ *such that*

$$J_D^F(\overline{p},\overline{q},z) = \inf_{x\in\bar{X}_w^F} J^F(x) = J^F(\overline{x}) \tag{4.9}$$

and the following system holds, for $t \in [0,\infty)$,

$$\overline{x}_t(t,\cdot) = \lim_{T\to\infty} \overline{p}(T-t,\cdot), \tag{4.10}$$

$$\nabla\overline{x}(t,\cdot) = \overline{q}(t,\cdot), \tag{4.11}$$

$$-\lim_{T\to\infty} \overline{p}_t(T-t,\cdot) - div\ \overline{q}(t,\cdot) = -F_x(t,\cdot,\overline{x}(t,\cdot)) + \bar{G}_x(t,\cdot). \tag{4.12}$$

4.1.1. *Variational Principle.* We state the necessary conditions. We observe that due to the construction of the set $\bar{X}_w^F$ and by the second Remark following Proposition 4.2 it follows that a minimizing sequence in $\bar{X}_w^F$ for the functional J^F may be assumed to be weakly convergent in $L^2(0,\infty;H^1(\Omega))$ and strongly in $L^2(0,\infty;L^2(\Omega))$.

Theorem 4.8. *Let* $\inf_{x\in\bar{X}_w^F} J^F(x) = J^F(\overline{x})$*, where* $\overline{x} \in L^2(0,\infty;L^2(\Omega))$ *is a limit, strong in* $L^2(0,\infty;L^2(\Omega))$ *and weak in* $L^2(0,\infty;H^1(\Omega))$*, of a minimizing sequence* $\{x^j\} \subset \bar{X}_w^F$*. Then there exist* $(\bar{p},\bar{q}) \in X^{Fd}$ *such that for a.e.* $t \in (0,\infty)$*,*

$$\lim_{T\to\infty} \bar{p}(T-t,\cdot) = \bar{x}_t(t,\cdot), \tag{4.13}$$

$$\bar{q}(t,\cdot) = \nabla\bar{x}(t,\cdot), \tag{4.14}$$

$$-\lim_{T\to\infty} \bar{p}_t(T-t,\cdot) - div\ \overline{q}(t,\cdot) + F_x(t,\cdot,\bar{x}(t,\cdot)) - \bar{G}_x(t,\cdot) = 0 \tag{4.15}$$

and

$$J^F(\bar{x}) = J_D^F(\bar{p},\bar{q},z),$$

where $z(t,y) = G_x(t,y,w(t,y))$*.*

Proof. Let $\bar{x} \in \bar{X}_w^F$ be such that $J^F(\bar{x}) = \inf_{x^j\in\bar{X}_w^F} J^F(x^j)$. Define

$$-\lim_{T\to\infty} \bar{p}_t(T-t,\cdot) = div\ \bar{q}(t,\cdot) - F_x(t,\cdot,\bar{x}(t,\cdot)) + \bar{G}_x(t,\cdot),\ t\in(0,\infty) \tag{4.16}$$

with $\bar{q}$ given by

$$\bar{q}(t,y) = \nabla\bar{x}(t,y) \text{ for all } (t,y)\in(0,\infty)\times\Omega. \tag{4.17}$$

By the definitions of J^F, J_D^F, relations (4.17), (4.16) and the Fenchel-Young inequality it follows that

$$\begin{aligned}
J^F(\bar{x}) = &\int_0^\infty \int_\Omega \Big(\frac{1}{2}|\nabla\bar{x}(t,y)|^2 - \frac{1}{2}|\bar{x}_t(t,y)|^2 \\
&+ F(t,y,\bar{x}(t,y)) - \bar{x}(t,y)\bar{G}_x(t,y)\Big) dy\,dt \\
&- \lim_{T\to\infty}\left(\bar{x}(T,\cdot), x^1(\cdot)\right)_{L^2(\Omega)} \\
\le &-\int_0^\infty \int_\Omega \Big\langle \bar{x}_t(t,y), \lim_{T\to\infty}\bar{p}(T-t,y)\Big\rangle dy, dt \\
&+ \int_0^\infty\int_\Omega \langle\nabla\bar{x}(t,y),\bar{q}(t,y)\rangle\, dy\,dt - \frac{1}{2}\int_0^\infty \|\bar{q}(t,\cdot)\|^2_{L^2(\Omega)}\,dt \\
&+ \frac{1}{2}\int_0^\infty \Big\|\lim_{T\to\infty}\bar{p}(T-t,\cdot)\Big\|^2_{L^2(\Omega)}\,dt \\
&+ \int_0^\infty\int_\Omega \left(F(t,y,\bar{x}(t,y)) - \bar{x}(t,y)\bar{G}_x(t,y)\right) dy\,dt \\
= &-\frac{1}{2}\int_0^\infty \|\bar{q}(t,\cdot)\|^2_{L^2(\Omega)}\,dt + \lim_{T\to\infty}\left(x^0(\cdot),\bar{p}(T,\cdot)\right)_{L^2(\Omega)}
\end{aligned}$$

$$-\int_0^\infty \int_\Omega F^*\left(t, y, (\lim_{T\to\infty} \bar{p}_t\,(T-t,y) + div\ \bar{q}\,(t,y) + z(t,y))\right) dt$$
$$+\frac{1}{2}\int_0^\infty \left\|\lim_{T\to\infty} \bar{p}\,(T-t,\cdot)\right\|^2_{L^2(\Omega)} dt - \int_\Sigma \bar{x}_\Gamma\,(t,y)\,(\bar{q}\,(t,y)\,,\nu(y))\,dy\,dt.$$

Therefore we get that

$$J^F(\bar{x}) \le J^F_D\,(\bar{p},\bar{q},z)\,. \tag{4.18}$$

Let $\{p^j,q^j\} \subset X^{Fd}$ denote the sequences corresponding to $\{x^j\}$ accordingly to the definition of the set X^{Fd}. It is clear that the above $(\overline{p},\overline{q})$ is a limit of the sequence $\{p^j,q^j\} \in X^{Fd}$. We observe that there are two possible forms for the sequence $\{p^j,q^j\} \subset X^{Fd}$ corresponding to the sequence$\{x^j\}$ accordingly to the definition of the set X^{Fd} with $q^j = \nabla x^j$. Namely, for $(t,y) \in (0,\infty)\times\Omega$, $j = 1,2,\ldots,$

$$q^j\,(t,y) = \nabla x^j\,(t,y)\,,\ \lim_{T\to\infty} p^j\,(T-t,y) = x^j_t\,(t,y) \tag{4.19}$$

or

$$\begin{aligned} -\lim_{T\to\infty} p^j_t\,(T-t,\cdot) &= div\ q^j\,(t,\cdot) - F_x(t,\cdot,x^j(t,\cdot)) + \bar{G}_x(t,\cdot), \\ q^j\,(t,y) &= \nabla x^j\,(t,y)\,. \end{aligned} \tag{4.20}$$

First we investigate the convergence of both sequences. As for the sequence (4.19) we obviously get, since $\{x^j\}$ converges strongly in U, $\left\{x^j_t\right\}$ weakly in $L^2(0,\infty;L^2(\Omega))$ and $\left\{p^j_t\right\}$ weakly in $L^2(0,\infty;H^{-1}(\Omega))$

$$x^j_t \rightharpoonup \overline{x}_t = \overline{p},\ \nabla x^j \to \nabla\overline{x} = \overline{q}.$$

Since x^j converges pointwise to $\overline{x}$ therefore

$$\left\{F_x\left(t,y,x^j\,(t,y)\right)\right\} \text{ converges pointwise to } F_x\,(t,y,\bar{x}\,(t,y))$$

too. Therefore by the Fenchel-Young equality we have

$$\begin{aligned} J^F(\bar{x}) &= \inf_{x\in\bar{X}^F_w} J^F\,(x) = \liminf_{j\to\infty} J^F(x^j) \\ &\ge \liminf_{j\to\infty}\left(\int_0^\infty \int_\Omega \left(\frac{1}{2}|\nabla x^j(t,y)|^2 - \frac{1}{2}\left|x^j_t\,(t,y)\right|^2\right) dy\,dt \right.\\ &\quad \left. - \lim_{T\to\infty}\left(x^j\,(T,\cdot)\,,x^1\,(\cdot)\right)_{L^2(\Omega)}\right) \\ &\quad + \liminf_{j\to\infty}\int_0^\infty \int_\Omega \left(F\left(t,y,x^j\,(t,y)\right) - x^j\,(t,y)\,z(t,y)\right) dy\,dt \\ &\ge -\frac{1}{2}\int_0^\infty \|\bar{q}\,(t,\cdot)\|^2_{L^2(\Omega)}\,dt + \int_0^\infty \int_\Omega \langle\nabla\bar{x}\,(t,y)\,,\bar{q}\,(t,y)\rangle\,dy\,dt \\ &\quad + \frac{1}{2}\liminf_{j\to\infty}\left(\lim_{T\to\infty}\int_0^T \|p_j\,(T-t,\cdot)\|^2_{L^2}\,dt\right) \\ &\quad + \liminf_{j\to\infty}(\int_0^\infty \left\langle x^j\,(t,\cdot)\,,\lim_{T\to\infty} p^j_t\,(T-t,\cdot)\right\rangle_{H^1,H^{-1}} dt \end{aligned}$$

$$
\begin{aligned}
&+\lim_{T\to\infty}\left(x^{0}\left(\cdot\right),p^{j}\left(T,\cdot\right)\right)\\
&+\int_{0}^{\infty}\int_{\Omega}\left(F\left(t,y,\bar{x}\left(t,y\right)\right)-\bar{x}\left(t,y\right)z(t,y)\right)dy\,dt\\
\geq &+\frac{1}{2}\int_{0}^{\infty}\left\|\lim_{T\to\infty}\bar{p}\left(T-t,\cdot\right)\right\|_{L^{2}(\Omega)}^{2}dt-\frac{1}{2}\int_{0}^{\infty}\left\|\bar{q}\left(t,\cdot\right)\right\|_{L^{2}(\Omega)}^{2}dt\\
&+\int_{0}^{\infty}\int_{\Omega}\left(F\left(t,y,\bar{x}\left(t,y\right)\right)-\bar{x}\left(t,y\right)z(t,y)\right)dy\,dt\\
&-\int_{0}^{\infty}\int_{\Omega}\left\langle\bar{x}\left(t,y\right),\left(\lim_{T\to\infty}\bar{p}_{t}\left(T-t,y\right)+div\ \bar{q}\left(t,y\right)\right)+z(t,y)\right\rangle_{H^{1},H^{-1}}dy\,dt\\
&-\int_{\Sigma}\bar{x}_{\Gamma}\left(t,y\right)\left\langle\bar{q}\left(t,y\right),\nu(y)\right\rangle dy\,dt+\lim_{T\to\infty}\left(x^{0}\left(\cdot\right),\bar{p}\left(T,\cdot\right)\right)_{L^{2}(\Omega)}\\
\geq &\ J_{D}^{F}\left(\bar{p},\bar{q},z\right).
\end{aligned}
$$

Inequalities $J^{F}(\bar{x})\geq J_{D}^{F}\left(\bar{p},\bar{q},z\right)$, (4.18) imply equality $J^{F}(\bar{x})=J_{D}^{F}\left(\bar{p},\bar{q},z\right)$. $J^{F}(\bar{x})=J_{D}^{F}\left(\bar{p},\bar{q},z\right)$, implies by standard convexity argument

$$
\begin{aligned}
&\frac{1}{2}\int_{0}^{\infty}\left\|\lim_{T\to\infty}\bar{p}\left(T-t,\cdot\right)\right\|_{L^{2}(\Omega)}^{2}dt+\frac{1}{2}\int_{0}^{\infty}\int_{\Omega}\left|\bar{x}_{t}\left(t,y\right)\right|^{2}dy\,dt\\
&\quad=\int_{0}^{\infty}\int_{\Omega}\left(\bar{x}_{t}\left(t,y\right),\lim_{T\to\infty}\bar{p}\left(T-t,y\right)\right)_{L^{2}(\Omega)}dy\,dt.
\end{aligned}
$$

Hence by Fenchel-Young transform we obtain that $\lim_{T\to\infty}\bar{p}\left(T-t,y\right)=\bar{x}_{t}\left(t,y\right)$ i.e. (4.13). ☐

4.2. **Auxilliary problem with** $l(t,y,x)=-F_{x}(t,y,x)-\bar{G}_{x}(t,y)$**.** A similar theorem is true for the problem

$$
\begin{aligned}
&x_{tt}(t,y)-\Delta x(t,y)-F_{x}(t,y,x(t,y))-\bar{G}_{x}(t,y)=0,\ \text{in }(0,\infty)\times\Omega,\\
&x(t,y)=\bar{x}_{\Gamma}(t,y),\ (t,y)\in\Sigma,\\
&x(0,\cdot)=x^{0}(\cdot),\ x_{t}(0,\cdot)=x^{1}(\cdot)
\end{aligned}
\tag{4.21}
$$

and corresponding to it functional

$$
\begin{aligned}
J^{F-}\left(x\right)=&\int_{0}^{\infty}\int_{\Omega}\left(\frac{1}{2}|\nabla x(t,y)|^{2}-\frac{1}{2}\left|x_{t}\left(t,y\right)\right|^{2}-F(t,y,x(t,y))\right)dy\,dt\\
&-\int_{0}^{\infty}\int_{\Omega}x(t,y)\bar{G}_{x}(t,y)dy\,dt-\lim_{T\to\infty}\left(x\left(T,\cdot\right),x^{1}\left(\cdot\right)\right)_{L^{2}(\Omega)}
\end{aligned}
\tag{4.22}
$$

defined on U with same hypotheses **(As)** and the sets X^{F}, $\bar{X}_{w}^{F}$, $\bar{X}^{F}$. Really, Lemmas 4.1–4.5 are still valid as sign of F does not change their proofs, also the proof of the above theorem does not change. Hence we get for (4.21) the following theorem.

Theorem 4.9. *There is* $\overline{x}\in\bar{X}_{w}^{F}$*, that* $\inf_{x\in\bar{X}_{w}^{F}}J^{F-}\left(x\right)=J^{F-}\left(\overline{x}\right)$*. There is* $(\bar{p},\bar{q})\in L^{2}(0,\infty;L^{2}(\Omega))\times L^{2}(0,\infty;L^{2}(\Omega))$ *such that for a.e.* $(t,y)\in(0,\infty)\times\Omega$*,*

$$
\lim_{T\to\infty}\bar{p}(T-t,y)=\bar{x}_{t}(t,y),
\tag{4.23}
$$

$$\bar{q}(t,y) = \nabla \bar{x}(t,y), \tag{4.24}$$

$$-\lim_{T\to\infty} \bar{p}_t(T-t,\cdot) - div\ \overline{q}\,(t,\cdot) - F_x\,(t,\cdot,\bar{x}\,(t,\cdot)) - \bar{G}_x(t,\cdot) = 0 \tag{4.25}$$

and

$$J^{F-}(\bar{x}) = J_D^{F-}\,(\bar{p},\bar{q},z)\,,$$

where

$$\begin{aligned} J_D^{F-}\,(\bar{p},\bar{q},z) = & \int_0^\infty \int_\Omega F^*\,(t,y,(\bar{p}_t\,(T-t,y) + div\ \bar{q}\,(t,y) + z(t,y)))\,dy\,dt \\ & + \frac{1}{2}\int_0^\infty \left\|\lim_{T\to\infty} \bar{p}\,(T-t,\cdot)\right\|^2_{L^2(\Omega)} dt - \frac{1}{2}\int_0^\infty \|\bar{q}\,(t,\cdot)\|^2_{L^2(\Omega)}\,dt \\ & - \int_\Sigma \bar{x}_\Gamma\,(t,y)\,(\bar{q}\,(t,y)\,,\nu(y))\,dy\,dt + \lim_{T\to\infty}\left(x^0\,(\cdot)\,,\bar{p}\,(T,\cdot)\right)_{L^2(\Omega)}. \end{aligned}$$

4.3. **The case of nonlinearity** $l = F_x - G_x$**.** Now we consider our original problem i.e.

$$\begin{aligned} & x_{tt}(t,y) - \Delta x(t,y) - G_x(t,y.x(t,y)) + F_x(t,y,x(t,y)) = 0, \text{ in } (0,\infty)\times\Omega, \\ & x(t,y) = \bar{x}_\Gamma(t,y), (t,y)\in\Sigma, \\ & x(0,\cdot) = x^0(\cdot),\ x_t(0,\cdot) = x^1(\cdot) \end{aligned} \tag{4.26}$$

and corresponding to it functional

$$\begin{aligned} J^{FG}\,(x) = & \int_0^\infty \int_\Omega \left(\frac{1}{2}|\nabla x(t,y)|^2 - \frac{1}{2}\,|x_t\,(t,y)|^2 - G(t,y,x(t,y))\right) dy\,dt \\ & + \int_0^\infty \int_\Omega F(t,y,x(t,y))dy\,dt - \lim_{T\to\infty}\left(x\,(T,\cdot)\,,x^1\,(\cdot)\right)_{L^2(\Omega)}, \end{aligned} \tag{4.27}$$

defined in U.

Proof. (of main Theorem and Theorem 2.13) Let us take any $x_0(\cdot,\cdot) \in \bar{X}^F$. By Theorem 4.8 the following problem

$$\begin{aligned} & x_{tt}(t,y) - \Delta x(t,y) - G_x(t,y.x_0(t,y)) + F_x(t,y,x(t,y)) = 0, \text{ in } (0,\infty)\times\Omega, \\ & x(t,y) = \bar{x}_\Gamma(t,y),\ (t,y)\in\Sigma, \\ & x(0,\cdot) = x^0(\cdot),\ x_t(0,\cdot) = x^1(\cdot) \end{aligned} \tag{4.28}$$

has a weak solution $x_1(\cdot,\cdot) \in \bar{X}^F_{x_0}$. Next consider

$$\begin{aligned} & x_{tt}(t,y) - \Delta x(t,y) - G_x(t,y.x_1(t,y)) + F_x(t,y,x(t,y)) = 0, \text{ in } (0,\infty)\times\Omega, \\ & x(t,y) = \bar{x}_\Gamma(t,y), (t,y)\in\Sigma, \\ & x(0,\cdot) = x^0(\cdot),\ x_t(0,\cdot) = x^1(\cdot). \end{aligned} \tag{4.29}$$

By the same argument (4.29) has a weak solution $x_2(\cdot,\cdot) \in \bar{X}^F_{x_1}$. In this way we obtain a sequence of weak solutions $\{x_n\} \subset \bar{X}^F$ to problems, in $(0,\infty)\times\Omega$,

$$\begin{aligned} & x_{ntt}(t,y) - \Delta x_n(t,y) - G_x(t,y,x_{n-1}(t,y)) + F_x(t,y,x_n(t,y)) = 0, \\ & x_n(t,y) = \bar{x}_\Gamma(t,y), (t,y)\in\Sigma, \\ & x_n(0,\cdot) = x^0(\cdot),\ x_{nt}(0,\cdot) = x^1(\cdot),\ n = 1,2,\ldots, \end{aligned} \tag{4.30}$$

which satisfy the relations

$$\lim_{T\to\infty} p_n(T-t,\cdot) = x_{nt}(t,\cdot), \tag{4.31}$$

$$q_n(t,\cdot) = \nabla x_n(t,\cdot), \tag{4.32}$$

$$-\lim_{T\to\infty} p_{nt}(T-t,\cdot) - div\ q_n\,(t,\cdot) + F_x\,(t,\cdot,x_n\,(t,\cdot)) - G_x(t,\cdot,x_{n-1}(t,\cdot)) = 0 \tag{4.33}$$

and

$$J^F(x_n, x_{n-1}) = J_D^F\,(p_n, q_n, z_{n-1})\,, \tag{4.34}$$

where

$$z_{n-1} = G_x(t,y,x_{n-1}(t,y)),$$

$$\begin{aligned} J^F\,(x_n, x_{n-1}) = &\int_0^\infty \int_\Omega \left(\frac{1}{2}|\nabla x_n(t,y)|^2 - \frac{1}{2}\,|x_{nt}\,(t,y)|^2 - G(t,y,x_{n-1}(t,y))\right) dy\,dt \\ &+ \int_0^\infty \int_\Omega F(t,y,x_n(t,y))dy\,dt - \lim_{T\to\infty} \left(x_n\,(T,\cdot)\,, x^1\,(\cdot)\right)_{L_2(\Omega)}, \end{aligned}$$

$$\begin{aligned} J_D^F\,(p_n, q_n, z_{n-1}) = &-\int_0^\infty \int_\Omega F^*\left(t,y,\left(\lim_{T\to\infty} p_{nt}\,(T-t,y) + div\ q_n\,(t,y) + z_{n-1}(t,y)\right)\right) dy\,dt \\ &- \frac{1}{2}\int_0^T \int_\Omega |q_n\,(t,y)|^2\,dy\,dt + \frac{1}{2}\int_0^T \int_\Omega \left|\lim_{T\to\infty} p_n\,(T-t,y)\right|^2 dy\,dt \\ &- \int_\Sigma \bar{x}_\Gamma\,(t,y)\,(q_n\,(t,y)\,,\nu(y))\,dy\,dt + \lim_{T\to\infty}\left(x^0\,(\cdot)\,, p_n\,(T,\cdot)\right)_{L_2(\Omega)}. \end{aligned}$$

Since $x_{n-1}(\cdot,\cdot) \in \bar{X}^F_{x_{n-2}}$ therefore x_n satisfies the estimations of Lemma 4.1. Hence the sequence $\{x_n\}$ is weakly convergent in $L^2(0,\infty;H^1(\Omega))$ to some $\overline{x}(\cdot,\cdot) \in \bar{X}^F$ and strongly in $L^2(0,\infty;L^2(\Omega))$ (see also Lemma 4.4). Moreover $\{x_{nt}\}$ is convergent strongly in $L^2(0,\infty;L^2(\Omega))$ and $\{p_{nt}\}$ weakly in $L^2(0,\infty;H^{-1}(\Omega))$

$$x_{nt} \to \overline{x}_t = \overline{p}, \quad \nabla x^j \to \nabla\overline{x} = \overline{q}.$$

Following the same way as in the proof of Theorem 4.8 we obtain that the sequence of relations (4.31)-(4.34) converges to the relations:

$$\lim_{T\to\infty} \bar{p}(T-t,\cdot) = \bar{x}_t(t,\cdot),$$

$$\bar{q}(t,\cdot) = \nabla\bar{x}(t,\cdot),$$

$$-\lim_{T\to\infty} \bar{p}_t(T-t,\cdot) - div\ \overline{q}\,(t,\cdot) + F_x\,(t,\cdot,\bar{x}\,(t,\cdot)) - G_x(t,\cdot,\bar{x}\,(t,\cdot)) = 0$$

and for $\bar{z}\,(t,y) = G_x(t,y,\bar{x}\,(t,y))$

$$J^{FG}(\bar{x}) = J_D^F\,(\bar{p},\bar{q},\bar{z})\,.$$

Thus $\bar{x}$ is a weak solution to (4.26) and so we have proved our main theorem and Theorem 2.13). □

4.4. **The case of nonlinearity $l = -F_x - G_x$.** Now we consider our original problem i.e.

$$\begin{aligned}&x_{tt}(t,y) - \Delta x(t,y) - G_x(t,y.x(t,y)) - F_x(t,y,x(t,y)) = 0, \text{ in } (0,\infty)\times\Omega,\\&x(t,y) = \bar{x}_\Gamma(t,y), (t,y)\in\Sigma,\\&x(0,\cdot) = x^0(\cdot),\ x_t(0,\cdot) = x^1(\cdot),\end{aligned}$$

and corresponding to it functional

$$\begin{aligned}J^{FG}(x) &= \int_0^\infty\int_\Omega\left(\frac{1}{2}|\nabla x(t,y)|^2 - \frac{1}{2}|x_t(t,y)|^2 - G(t,y,x(t,y))\right)dy\,dt\\&\quad-\int_0^\infty\int_\Omega F(t,y,x(t,y))dy\,dt - \lim_{T\to\infty}\left(x(T,\cdot),x^1(\cdot)\right)_{L_2(\Omega)},\end{aligned}$$

defined in U.

The justification of this case is identical as in the former section the case $l = F_x - G_x$ taking into account the result of the section "Simply case II".

4.5. **Proof of stability - Theorem 2.12.** In this section we prove some stability results to our nonlinear case i.e. Theorem 2.12. To this effect let us assume we are given a sequence $\{x_n^0\}$ in $\tilde{H}^1(\Gamma)$ converging to $\bar{x}^0$ in $\tilde{H}^1(\Gamma)$. Let $\{\bar{x}_{\Gamma n}\}$ be a sequence of solutions to (1.6) corresponding to $\{x_n^0\}$. Then by Theorem 2.5 and its proof we know that $J_\Gamma^t(\bar{x}_{\Gamma n}) = 0$, $t\in[0,\infty)$, $n = 1,2,\ldots$ and that $J_\Gamma^t(x)\geq 0$ for all $x\in U^t$, $t\in[0,\infty)$ (moreover J_Γ^t is weakly lower semicontinuous in U^t, $t\in[0,\infty)$). Therefore the sequence $\{\bar{x}_{\Gamma n}\}$ is bounded in $\tilde{H}^1(\Gamma)$ and hence it has a subsequence (which we shall denote again by $\{\bar{x}_{\Gamma n}\}$) converging weakly in $\tilde{H}^1(\Gamma)$ to some $\bar{x}_\Gamma\in\tilde{H}^1(\Gamma)$ and $J_\Gamma^t(\bar{x}_\Gamma) = 0$ on U^t, $t\in[0,\infty)$. The last implies that $\bar{x}_\Gamma$ is a solution to (1.6) corresponding to $\bar{x}^0$.

Next let us assume that $\{x_n^0\}$, $\{x_n^1\}$ given sequences in $H^1(\Omega)\cap\tilde{H}^1(\Gamma)$, $L^2(\Omega)$, respectively and such that $\|x_n^0\|_{C([0,T];H^1(\Omega))}\leq C(E_F+E_G)+E^w$, $\|x_n^1\|_{C([0,T];L_2(\Omega))}\leq C(E_F+E_G)+E^w$, $n = 1,2\ldots$ converging to $\bar{x}^0$, $\bar{x}^1$ in $H^1(\Omega)\cap\tilde{H}^1(\Gamma)$, $L^2(\Omega)$, respectively and $x_n^0(y) = \bar{x}_{\Gamma n}(0,y)$ on Γ, $n = 1,2,\ldots$. Thus we assume hypotheses **(HΓ)**, **(As)** to be considered and satisfied.

In consequence, all the assertions of Theorem 2.13 are true for all n with estimations independent on n. Because, we have nonlinear problem we cannot expect the same type of continuous dependence as in [13] for linear case.

First note that for each x_n^0, x_n^1 there exists a solution $\bar{x}_n\in\bar{X}^F\subset U$ to (1.1) determined by x_n^0 and x_n^1, $n = 1,2,\ldots$. Therefore, choosing suitable subsequence, let $\bar{x}\in\bar{X}^F$ be a weak limit in $H^1((0,\infty)\times\Omega)$ and strong in $L^2((0,\infty)\times\Omega)$ of $\{\bar{x}_n\}$. We know also that for all n, $J^{FG}(\bar{x}_n) = J_D^F(\bar{p}_n,\bar{q}_n,\bar{z}_n)$ where $\{\bar{p}_n,\bar{q}_n,\bar{z}_n\}$ denote the sequences corresponding to $\{\bar{x}_n\}$ and satisfying

$$\bar{x}_{nt}(t,\cdot) = \lim_{T\to\infty}\bar{p}_n(T-t,\cdot),$$

$$\nabla \bar{x}_n(t,\cdot) = \bar{q}_n(t,\cdot),$$
$$\lim_{T\to\infty} \bar{p}_{nt}(T-t,\cdot) + div\ \bar{q}_n(t,\cdot) - l(t,\cdot,\bar{x}_n(t,\cdot)) = 0,$$
$$\bar{z}_n(t,y) = G_x(t,y,\bar{x}_n(t,y)).$$

$\{\bar{x}_n\}$ is weakly convergent in $H^1((0,\infty)\times\Omega)$ and strongly in $L^2(0,\infty;L^2(\Omega))$, $\bar{x}_{nt} \to \overline{x}_t = \overline{p}$, $\nabla\bar{x}_n \to \nabla\overline{x} = \overline{q}$ in $L^2(0,\infty;L^2(\Omega))$, therefore $\bar{x}_n$ converges pointwise to $\overline{x}$. Hence $\{\bar{p}_{nt} + div\ \bar{q}_n\}$ converges pointwise to $\bar{p}_t + div\ \bar{q}$. Thus we infer that

$$J^{FG}(\bar{x}) = \lim_{n\to\infty} J^{FG}(\bar{x}_n) = \lim_{n\to\infty} J_D^F(\bar{p}_n,\bar{q}_n,\bar{z}_n) = J_D^F(\bar{p},\bar{q},\bar{z})$$

with

$$\bar{z}(t,y) = G_x(t,y,\bar{x}(t,y))$$

and in consequence

$$\overline{x}_t(t,\cdot) = \lim_{T\to\infty} \overline{p}(T-t,\cdot),$$
$$\nabla\overline{x}(t,\cdot) = \overline{q}(t,\cdot),$$
$$\lim_{T\to\infty} \overline{p}_t(T-t,\cdot) + div\ \overline{q}(t,\cdot) - l(t,\cdot,\overline{x}(t,\cdot)) = 0$$

and so we get the assertion of the theorem.

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