Dynamic Systems and Applications 21 (2012) 377-392

DIFFERENTIAL EQUATIONS WITH IMPULSES AT VARIABLE TIMES IN FRACTIONAL POWER SPACES

Y. PENG[1,2], W. WEI[2], AND X. XIANG[2]

[1]School of Mathematics and Statistics, Wuhan University, Wuhan, Hubei, 430072, China

[2]Department of Mathematics, Guizhou University, Guiyang, Guizhou, 550025, China

ABSTRACT. In this paper, a class of semilinear evolution equations with impulses at variable times and time-varying generating operators in fractional power spaces is considered. Introducing the reasonable α-mild solution, the existence and uniqueness of α-mild solution are given. At the same time, modifying the classical definitions of continuous dependence and Gâteaux differentiability, some results on periodicity, continuous dependence, Gâteaux differentiable of α-mild solution relative to initial value and parameter also are presented.

AMS (MOS) Subject Classification. 34A37, 34G20, 34K30

1. INTRODUCTION

It is well known the theory of impulsive evolution equations has become more important in recent years in some mathematical models of real processes and phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics (see [1]). There are two typical types of impulsive differential systems in which are the systems with impulses at fixed times and the systems with impulses at variable times on the impulsive evolution equations. For the systems with impulses at fixed times, there has been a significant development in impulsive evolution equations. For the basic theory on impulsive evolution equations on finite dimensional spaces, the reader can refer to Benchohra's book [2]. Particularly, Ahmed and Migorski considered optimal control problems of systems governed by impulsive evolution equations (see [3]–[5]). We also gave a series of results on impulsive evolution equations and optimal control problems such as the semilinear (strongly-nonlinear)

† This work is supported by the National Natural Science Foundation of China under grant no. 10961009, 11161009, by the Fok Ying Tung Education Foundation under grant no. 121104. and Introducing Talents Foundation for the Doctor of Guizhou University under grant no. 2010031.

∗ Corresponding author: pengyf0803@163.com(Y.Peng)

Tel.: +86-851-3624299; Fax:+86-851-3620186

Received November 15, 2011 1056-2176 $15.00

impulsive evolution equations, impulsive integro-differential equations, impulsive periodic systems and optimal controls (see [6]–[10]). For the systems with impulses at variable times, Lakshmikantham and Benchohra studied a class of evolution equations

$$\begin{cases} \dot{x}(t) = f(t, x(t)), & t \neq \tau_k(x(t)), \\ \Delta x(t) = I_k(x(t)), & t = \tau_k(x(t)) \end{cases} \tag{1.1}$$

in finite dimensional, and some results on the existence of classical solution and the pulse phenomena are obtained(see [2], [11]). Recently, the existence and uniqueness of the mild solution of evolution equation with impulses at variable times

$$\begin{cases} \dot{x}(t) = Ax(t) + f(t, x(t)), & \{x(t)\} \bigcap Y(t) = \emptyset, \\ x(0) = x_0, & \\ \Delta x(t) = I_k(x(t)), & \{x(t)\} \bigcap Y(t) = \{y_k(t)\} \end{cases} \tag{1.2}$$

was first discussed by us in infinite dimensional spaces (see [9]).

To the best of our knowledge, the evolution equation with impulses at variable times and time-varying generating operators is not researched. In this paper, we consider the following semilinear differential equations with impulses at variable times and time-varying generating operators

$$\begin{cases} \dot{x}(t) + A(t)x(t) = f(t, x(t)), & x(t) \neq y_1, \\ x(0) = x_0, & \\ x\left(t^+\right) = y_2, & x(t) = y_1, \end{cases} \tag{1.3}$$

where $\{A(t)|t \geq 0\}$ is a family of closed linear operators in Banach space X, y_1, $y_2 \in X$. Expressly, when $A(t) \equiv A$, Peng and Xiang [10] established the existence and uniqueness of the mild solution.

Constructing suitable function sets, we introduce the reasonable α-mild solutions for the problem (1.3). The first main result is presented for the existence of α-mild solution on the problem (1.3). Particularly, when right hand function f is independent of time t or is periodic on time t very interesting oscillatory property of α-mild solution is presented. The second main result is proved on the continuous dependence and Gâteaux differentiable of α-mild solution relative to initial value. In a word, the main purpose of this study is to establish qualitative theory on a class of evolution equation with impulse at variable times in fractional power space.

2. EXISTENCE AND PERIODICITY OF SOLUTION

Let X, Y denote a pair of Banach spaces. If X is continuously embedded in Y, we write $X \hookrightarrow Y$, if X is compactly embedded in Y, we write $X \hookrightarrow\hookrightarrow Y$. $\pounds(X)$ is the class of (not necessary bounded) linear operators in X. $\pounds_b(X)$ is a Banch space of bounded linear operators in X. For $A \in \pounds(X)$, let $\rho(A)$ denote the resolvent set and $R(\lambda, A)$ the resolvent corresponding to $\lambda \in \rho(A)$.

Assumption [A]:

(1) Let $\{A(t)|t \geq 0\}$ be a family of closed linear operators in X, the domain $D(A(t)) = D$ of $A(t)$, $t \geq 0$ is dense in X and independent of t.

(2) For $t \geq 0$, the resolvent $R(\lambda, A(t))$ of $A(t)$ exists for all λ with $\text{Re}\lambda \leq 0$ and there exists a constant $M > 0$ such that

$$\|R(\lambda,\, A(t))\|_{\pounds(X)} \leq \frac{M}{|\lambda| + 1}, \text{ for } t \geq 0.$$

(3) There exist constants $L > 0$ and $0 < \alpha \leq 1$ such that

$$\left\|(A(t) - A(s))A(\tau)^{-1}\right\|_{\pounds(X)} \leq L|t - s|^{\alpha}, \text{ for } s, t, \tau \geq 0.$$

(4) The operators $A(t)A^{-1}(s)$ are uniformly bounded for $0 \leq s, t < \infty$ and there exists a closed operator $A(\infty)$ with domain D such that

$$\lim_{t\to\infty} \left\|(A(t) - A(\infty))A^{-1}(0)\right\|_{\pounds(X)} = 0.$$

Let $X_1 = \{D, \|\cdot\|_1\}$, where $\|x\|_1 = \|Ax\|$. X_1 is a Banach space and $X_1 \hookrightarrow X$. More generally, in a usual way we introduce the fractional power operator $A^{\alpha}(t)$ $(\alpha \in (0,1))$, having dense domain $D\left(A^{\alpha}(t)\right)$, which we also assume to be independent of t and denote $D(A) = D(A(t))$, $D\left(A^{\alpha}\right) = D\left(A^{\alpha}(t)\right)$. Let $\|x\|_{\alpha} = \|A^{\alpha}x\|$ for $x \in D\left(A^{\alpha}(t)\right)$ and denote the Banach space $\{D\left(A^{\alpha}\right), \|\cdot\|_{\alpha}\}$ as X_{α}. Then it is clear that $X_{\beta} \hookrightarrow X_{\alpha}$ for $0 \leq \alpha \leq \beta \leq 1$.

For the initial value problem

$$\begin{cases} \dot{x}(t) + A(t)x(t) = 0, & t \in (0, T], \\ x(0) = x_0, \end{cases} \tag{2.1}$$

it is well-known that (2.1) has a unique classical solution x provided that the assumption [A] is satisfied. Further, there exists a unique evolution operator $U(t,s) \in \pounds_b(X)$, $0 \leq s \leq t$, such that every solution of the equation (2.1) can be represented in the form

$$x(t) = U(t, 0)x_0.$$

For y_1, $y_2 \in X_{\alpha}$ be fixed, $y_1 \neq y_2$, define $PC_{y_1,y_2}\left([0,T), X_{\alpha}\right) = \{x|x$ is a mapping from $[0,T)$ to X_{α} such that x is continuous at t for $x(t) \neq y_1$, left continuous at t and exists right limit $x(t+) = y_2$ for $x(t) = y_1\}$. For $x \in PC_{y_1,y_2}\left([0,T), X_{\alpha}\right)$, if $x(t) = y_1$, then the time t is called an irregular point of x, t is said to be a regular point of x otherwise.

We introduce a reasonable α-mild solution in $PC_{y_1,y_2}\left([0,T), X_{\alpha}\right)$ and present the existence result.

Definition 2.1. A function $x \in PC_{y_1,y_2}([0,T), X_\alpha)$ is said to be a α-mild solution of the evolution equation (1.3), if x satisfies the following integral equation

$$x(t) = U(t,0)x_0 + \int_0^t U(t,s)f(s,x(s))ds + \sum_{x(t_i)=y_1, 0\le t_i<t} U(t,t_i)(y_2 - y_1),$$

where t_i is the irregular point of x.

Assumption [F]:

(1) $f : [0,+\infty) \times X_\alpha \longrightarrow X$ is measurable in t on $[0,+\infty)$ and locally Lipschitz continuous in x on X_α, i.e, for every $\rho > 0$, there exists a constant $L(\rho)$ such that

$$\|f(t,x) - f(t,y)\| \le L(\rho)\|x - y\|_\alpha \text{ for } x, y \in B_\rho = \{z \in X_\alpha | \|z\|_\alpha \le \rho\}.$$

(2) There exists a constant $k > 0$, such that

$$\|f(t,x)\| \le k(1 + \|x\|_\alpha) \text{ for every } x \in X_\alpha \text{ and } t \ge 0.$$

For convenience, we cite the following important existence theorem (see Theorem 3.A of [12]).

Proposition 2.2. *Assume that $x_0 \in X_\beta$ $(0 < \alpha < \beta < 1)$. Under assumptions [A], [F], the Cauchy problem*

$$\begin{cases} \dot{x}(t) + A(t)x(t) = f(t,x(t)), t > 0, \\ x(0) = x_0, \end{cases}$$

has a unique α-mild solution $x \in C([0,+\infty), X_\alpha)$ which satisfies the following integral equation

$$x(t) = U(t,0)x_0 + \int_0^t U(t,s)f(s,x(s))ds.$$

Now we turn to discuss the existence of α-mild solution for (1.3).

Theorem 2.3. *Suppose that x_0, $y_2 \in X_\beta$, $y_1 \in X_\alpha(\alpha < \beta)$ and $y_1 \neq y_2$. Under the assumptions [A], [F], the evolution equation (1.3) has a unique α-mild solution $x \in PC_{y_1,y_2}([0,+\infty), X_\alpha)$. Further, the α-mild solution must satisfy one of the following three case:*

Case (I): $x \in C([0,\infty), X_\alpha)$,

Case (II): $x \in PC_{y_1,y_2}([0,+\infty), X_\alpha)$ *has a unique irregular point,*

Case (III): $x \in PC_{y_1,y_2}([0,+\infty), X_\alpha)$ *and there exist $t_1 = \inf\{t \in [0,+\infty) | x(t) = y_1\}$ and $t_2 = \inf\{t \in (t_1,+\infty) | x(t) = y_1\}$ such that $x(t_i) = y_1$ $(i = 1, 2)$. Further there exists a $\tilde{t} \ge 0$ such that x is a periodic function on $[\tilde{t},+\infty)$, provided that there exists a $T_0 > 0$ such that $A(t + T_0) = A(t)$ for $t \ge 0$, $t_2 - t_1 = hT_0$ for some positive rational number h and one of the following two conditions is satisfied:*
(a) *f is independent of time t, i.e. $f(t,x) = g(x)$ for all $t \ge 0$,*

(b) *$f(t+T_1,x) = f(t,x)$ for all $x \in X$ and $t_2 - t_1 = h_1 T_1$ for some positive rational number h_1.*

Proof. It can be seen from the Proposition 2.2 that if there exists a positive number $\delta > 0$, such that for any $s \geq 0$, the following evolution equation

$$\begin{cases} \dot{x}(t) + A(t)x(t) = f(t, x(t)), \ t > s, \\ x(s) = y_2, \end{cases} \tag{2.2}$$

has a unique α-mild solution $x \in C([s, s+\delta], X_\alpha)$ and $y(t) \neq y_1$, for $t \in [s, s+\delta]$, then the evolution equation (1.3) has a unique α-mild solution $x \in PC_{y_1,y_2}([0,+\infty), X_\alpha)$.

Since $y_1 \neq y_2$, we have $\|y_1 - y_2\|_\alpha > 0$. Using the strong continuity of evolution system $\{U(t,s)|t \geq s \geq 0\}$, there exists a constant $\delta_1 > 0$ such that

$$\|U(t,s)y_2 - y_2\|_\alpha \leq \frac{\|y_1 - y_2\|_\alpha}{4} \text{ for } s \leq t \leq s + \delta_1.$$

By Proposition 2.2, the Cauchy problem (2.2) has a unique α-mild solution $y \in C([s, s+1], X_\alpha)$ for every $s \geq 0$ and y satisfies the following integral equation

$$y(t) = U(t,s)y_2 + \int_s^t U(t,\tau) f(\tau, y(\tau)) d\tau, \ t \in [s, s+1].$$

Further, we have

$$\begin{aligned} \|y(t)\|_\alpha &\leq \|U(t,s)y_2\|_\alpha + \int_s^t \|U(t,\tau)f(\tau,y(\tau))\|_\alpha d\tau \\ &\leq C(\beta,\alpha)\|y_2\|_\beta + kC(\alpha,\gamma)\frac{1}{1-\gamma} \\ &\quad + kC(\alpha,\gamma)\int_s^t (t-\tau)^{-\gamma}\|y(\tau)\|_\alpha d\tau. \end{aligned}$$

By Gronwall inequality with singularity (see Lemma 2.1 of [12]), there exists $M > 0$ such that

$$\|y(t)\|_\alpha \leq M\left(C(\beta,\alpha)\|y_2\|_\beta + kC(\alpha,\gamma)\frac{1}{1-\gamma}\right).$$

Furthermore, there is $\delta_2 > 0$ such that

$$\int_s^t \|U(t,\tau)f(\tau,y(\tau))\|_\alpha d\tau \leq \frac{\|y_1 - y_2\|_\alpha}{4} \text{ for } s \leq t \leq s + \delta_2.$$

Set $\delta = \min\{\delta_1, \delta_2, 1\}$, for $0 \leq t - s < \delta$, we have

$$\begin{aligned} \|y(t) - y_1\|_\alpha &\geq \|y_2 - y_1\|_\alpha \\ &\quad - \left[\|U(t,s)y_2 - y_2\|_\alpha + \int_s^t \|U(t,\tau)f(\tau,y(\tau))\|_\alpha d\tau\right] \\ &\geq \frac{\|y_2 - y_1\|_\alpha}{2}, \end{aligned}$$

i.e, $y(t) \neq y_1$, for $t \in [s, s+\delta]$. Thus, the impulsive evolution equation (1.3) has a unique α-mild solution $x \in PC_{y_1,y_2}([0,+\infty), X_\alpha)$ given by

$$x(t) = U(t,0)x_0 + \int_0^t U(t,\tau)f(\tau, x(\tau))d\tau + \sum_{x(t_i)=y_1, 0\le t_i<t} U(t,t_i)(y_2 - y_1).$$

We have only three possibilities:

(I) x has not irregular point,

(II) x has a unique irregular point,

(III) x has two irregular points at least.

In case (I) is satisfied, then the α-mild solution x satisfies following integral equation

$$x(t) = U(t,0)x_0 + \int_0^t U(t,\tau)f(\tau, x(\tau))d\tau, \tag{2.3}$$

and $x(t) \neq y_1$ for all $t \geq 0$. Hence $x \in C([0,\infty), X_\alpha)$.

In case (II) is satisfied, then $x \in PC_{y_1,y_2}([0,+\infty), X_\alpha)$ satisfies the following integral equation

$$x(t) = U(t,0)x_0 + \int_0^t U(t,\tau)f(\tau, x(\tau))d\tau + U(t,t_1)(y_2 - y_1),$$

and for all $t \in [0,+\infty)$ and $t \neq t_1$, we have $x(t) \neq y_1$.

For the case (III), without loss of generality, we assume $h = h_1 = 1$ and denote by $T = t_2 - t_1$. Now, we show that $x(t+T) = x(t)$ for all $t \geq t_1$.

The condition (a) implies that for $t \in (t_1, t_2]$, the α-mild solution satisfies

$$x(t) = U(t,t_1)y_2 + \int_{t_1}^t U(t,s)g(x(s))ds.$$

For $t + T \in (t_2, t_2 + (t_2 - t_1)]$, we have

$$\begin{aligned} x(t+T) &= U(t+T, t_1+T)y_2 + \int_{T+t_1}^{T+t} U(t+T,s)g(x(s))ds \\ &= U(t,t_1)y_2 + \int_{t_1}^t U(t,s)g(x(s+T))ds. \end{aligned}$$

It is easy to see that by assumption [F](2) there exists $\rho > 0$ such that $\|x(t)\|_\alpha$, $\|x(T+t)\|_\alpha \leq \rho$ for every $t \in [t_1, t_2]$. Furthermore, we obtain

$$\|x(t+T) - x(t)\|_\alpha \leq L(\rho)C(\alpha,\gamma)\int_{t_1}^t (t-s)^{-\gamma}\|x(s+T) - x(s)\|_\alpha ds.$$

Using Gronwall inequality with singularity, one can verify that

$$x(t+T) = x(t) \text{ for } t \in [t_1, t_2].$$

Consequently, we have

$$x(t) = x(T+t) \text{ for } t \geq t_1.$$

Hence $x \in PC_{y_1,y_2}([0,+\infty), X_\alpha)$ is a periodic function on the interval $[t_1,+\infty)$, and x satisfies the following integral equation

$$x(t) = U(t,0)x_0 + \int_0^t U(t,\tau)g(x(\tau))d\tau + \sum_{\substack{0\le t_1+kT<t,\\ k=0,1,2,\cdots}} U(t,(t_1+kT))(y_2-y_1).$$

The condition (b) implies that for $t \in (t_1,t_2]$, then $t+T \in (t_2, t_2+T]$ and

$$\begin{aligned} x(t+T) &= U(t+T,t_2)\,y_2 + \int_{t_2}^{t+T} U(t+T,\tau)f(\tau,x(\tau))d\tau \\ &= U(t,t_1)\,y_2 + \int_{t_1}^{t} U(t,\tau)f(\tau+T,x(T+\tau))d\tau \\ &= U(t,t_1)\,y_2 + \int_{t_1}^{t} U(t,\tau)f(\tau,x(T+\tau))d\tau. \end{aligned}$$

Furthermore,

$$x(t+T)-x(t) = \int_{t_1}^{t} U(t,\tau)\left[f(\tau,x(\tau+T)) - f(\tau,x(\tau))\right]ds.$$

Similarly as before, x is a periodic function on the interval $[t_1,+\infty)$, and satisfies the following integral equation

$$x(t) = U(t,0)x_0 + \int_0^t U(t,\tau)f(\tau,x(\tau))d\tau + \sum_{\substack{0\le t_1+kT<t,\\ k=0,1,2,\cdots}} U(t-(t_1+kT))(y_2-y_1).$$

The proof is completed. □

3. CONDITION DEPENDENCE OF SOLUTION

Throughout this section, we shall consider the continuity of α-mild solution x of (1.3) with respect to the initial value x_0 on the interval $J=[0,T]$. For this purpose, we need to consider the following auxiliary Cauchy problem

$$\begin{cases} \dot{x}(t) + A(t)x(t) = f(t,x(t)), & t>0, \\ x(0) = x_0, \end{cases} \tag{3.1}$$

whose α-mild solution we denote by $x(t;0,x_0)$ given by

$$x(t;0,x_0) = U(t,0)\,x_0 + \int_0^t U(t,\tau)f(\tau,x(\tau;0,x_0))\,d\tau. \tag{3.2}$$

Since we can never expect to have the continuity of $x(\cdot;0,x_0)$ with respect to $(0,x_0)$ at $t=t^*$ where $x(t^*;0,x_0)=y_1$, we introduce approximate α-mild solution and have to modify the classical definition of continuous dependence.

Definition 3.1. The function $x_\varepsilon(\cdot; s, \eta)$ is said to be an approximate α-mild solution of the following impulsive equation

$$\begin{cases} \dot{x}(t) + A(t)x(t) = f(t, x(t)), & x(t) \neq y_1, \\ x(s) = \eta, & \\ x(t^+) = y_2, & x(t) = y_1, \end{cases} \tag{3.3}$$

if $x_\varepsilon(\cdot; s, \eta)$ satisfies the following impulsive integral equation

$$\begin{aligned} x_\varepsilon(t; s, \eta) &= U(t,s)\eta + \int_s^t U(t,\tau) f(\tau, x_\varepsilon(\tau; s, \eta)))d\tau \\ &\quad + \sum_{\substack{s \le t_i < t, \\ x(t_i; s, \eta) \in B\left(y_1, \frac{\varepsilon}{2}\right)}} U(t - t_i)(y_2 - y_1) \end{aligned}$$

for some sufficient small $\varepsilon \geq 0$, where $B\left(y_1, \frac{\varepsilon}{2}\right) = \left\{\eta \in X_\alpha \middle| \|\eta - y_1\|_\alpha \leq \frac{\varepsilon}{2}\right\}$. It is clear that when $\varepsilon = 0$, the approximate α-mild solution is α-mild solution and α-mild solution is an approximate α-mild solution. In addition, for any fixed $\varepsilon > 0$, the impulsive differential equation (1.3) has a unique approximate α-mild solution.

Definition 3.2. The α-mild solution $x(\cdot; 0, x_0)$ of (1.3) is said to have continuous dependence relative to $(0, x_0)$ iff (I):

$$\lim_{\theta \to 0, s \to 0, \eta \to x_0} x_\theta(t; s, \eta) = x(t; 0, x_0) \text{ if } x(t; 0, x_0) \neq y_1, \tag{3.4}$$

and (II): given any $\varepsilon > 0$ there is a closed $J_\varepsilon \subseteq J$ and a $\delta > 0$ such that $m(J \setminus J_\varepsilon) < \varepsilon$ and

$$\|x_\theta(t; s, \eta) - x(t; 0, x_0)\|_\alpha < \varepsilon \text{ for } t \in J_\varepsilon \tag{3.5}$$

provided $\theta + s + \|\eta - x_0\|_\beta < \delta$, where m denote the Lebesgue measure.

We can now prove the following result.

Theorem 3.3. *Assume that the hypotheses of Theorem 2.3 hold. Then the α-mild solution $x(\cdot) = x(\cdot; 0, x_0)$ of (1.3) have continuous dependence relative to $(0, x_0)$ in the sense of Definition 3.2.*

Proof. By Theorem 2.3, the α-mild solution $x(t) = x(t; 0, x_0)$ of (1.3) meets the warning line $y = y_1$ at most a finite number of times on the interval $[0, T]$. Here, we have two possibilities:

(1) x has not irregular point on the interval $[0, T]$,

(2) x has one irregular point on the interval $[0, T]$ at last.

In case (1), we can find $\delta > 0$ such that, for all $s + \|\eta - x_0\|_\beta < \delta$, the following equation

$$\begin{cases} \dot{x}(t) + A(t)x(t) = f(t, x(t)), & t > s \geq 0, \\ x(s) = \eta, & \end{cases} \tag{3.6}$$

has a unique α-mild solution $x(\cdot; s, \eta) \in C([s, T], X_\alpha)$ given by

$$x(t; s, \eta) = U(t, s)\eta + \int_s^t U(t, \tau) f(\tau, x(\tau; s, \eta)) d\tau.$$

Furthermore, there exists $\rho > 0$ such that

$$\sup_{0 \le t \le T} \|x(t; 0, x_0)\|_\alpha \le \rho, \qquad \sup_{0 \le s \le t \le T} \|x(t; s, \eta)\|_\alpha \le \rho,$$

and we have

$$\begin{aligned} &\|x(t; s, \eta) - x(t; 0, x_0)\|_\alpha \\ &\le \left[k(1+\rho) C(\alpha, \gamma) \frac{s^{1-\gamma}}{1-\gamma} + C(\beta, \alpha) \|\eta - x_0\|_\beta + C(\beta, \alpha, \gamma) \|x_0\|_\alpha s^\gamma \right] \\ &\quad + C(\alpha, \gamma) L(\rho) \int_s^t (t-\tau)^{-\gamma} \|x(\tau; s, \eta) - x(\tau; 0, x_0)\|_\alpha d\tau. \end{aligned}$$

By Gronwall inequality with singularity, there exists $M > 0$ such that

$$\|x(t; s, \eta) - x(t; 0, x_0)\|_\alpha \le M \left[s^{1-\gamma} + s^\gamma + \|\eta - x_0\|_\beta \right] \longrightarrow 0$$

as $s \to 0$, $\eta \to x_0$.

For the case (2), if $x_0 = y_1$, we only study the α-mild solution $x(\cdot; 0^+, y_2)$. Consequently, we may assume that $x(\cdot; 0, x_0)$ meets the warning line $y = y_1$, p times in $[0, T]$ and let t_j, $j = 1, 2, \ldots, p$, be the moments where $x(\cdot, 0, x_0)$ hits y_1, that is, $0 < t_1 < t_2 < \cdots < t_p < T$. By Theorem 2.3, we can prove that there exists $\delta > 0$ such that the following differential equation with impulses

$$\begin{cases} \dot{x}(t) + A(t)x(t) = f(t, x(t)), & x(t) \ne y_1, \\ x(s) = \eta, & x(t) = y_1, \end{cases} \tag{3.7}$$

has a unique approximate α-mild solution $x_\theta(\cdot; s, \eta)$ corresponding to (s, η) which satisfies $\theta + s + \|\eta - x_0\|_\beta < \delta$. The approximate α-mild solution $x_\theta(\cdot; s, \eta)$ has p irregular points in $[0, T]$ and let $t_j(\theta, s, \eta) \equiv \bar{t}_j$, $j = 1, 2, \ldots, p$, be the moments where $x(\cdot; s, \eta)$ hits y_1, that is, $0 < \bar{t}_1 < \bar{t}_2 < \cdots < \bar{t}_p < T$, and

$$\sup_{0 \le s \le t \le T} \|x_\theta(t; s, \eta)\|_\alpha \le \rho$$

for some constant $\rho > 0$. Given any $\varepsilon > 0$, by the case (1), we have

$$\lim_{\substack{\theta \to 0 \\ s \to 0 \\ \eta \to x_0}} \bar{t}_1 = t_1$$

and

$$\lim_{\substack{s \to 0 \\ \eta \to x_0}} x_\theta(t; s, \eta) = x(t; 0, x_0) \text{ for all } t \in \left[0, t_1 - \frac{\varepsilon}{4p}\right]$$

for some sufficient small θ. Further, we have

$$\|x_\theta(t; s, \eta) - x(t; 0, x_0)\|_\alpha$$

$$\leq C(\beta,\alpha,\gamma)\left\|y_2\right\|_\beta \left|t_1^s-t_1\right|^\gamma + C(\alpha,\gamma)(2+\rho)\frac{\left|t_1^s-t_1\right|^{1-\gamma}}{1-\gamma}$$
$$+ C(\alpha,\gamma)L(\rho+1)\int_{\max\left\{t_1,t_1^s\right\}}^{t} \left\|x(\tau;s,\eta)-x\left(\tau;0,x_0\right)\right\|_\alpha d\tau$$

for all $t\in\left[t_1+\frac{\varepsilon}{4p},t_2-\frac{\varepsilon}{4p}\right]$. By Gronwall inequality with singularity, we have

$$\left\|x_\theta(t;s,\eta)-x\left(t;0,x_0\right)\right\|_\alpha \longrightarrow 0 \text{ for all } t\in\left[t_1+\frac{\varepsilon}{4p},t_2-\frac{\varepsilon}{4p}\right]$$

as $s\rightarrow 0$, $\eta\rightarrow x_0$. Repeating the preceding argument, for some sufficient small θ, one can obtain

$$\lim_{\substack{\theta\to 0\\ s\to 0\\ \eta\to x_0}} \bar{t}_{i+1}=t_{i+1},\ \lim_{\substack{\theta\to 0\\ s\to 0\\ \eta\to x_0}} x_\theta(t;s,\eta)=x(t;0,x_0) \text{ for all } t\in\left[t_i+\frac{\varepsilon}{4p},t_{i+1}-\frac{\varepsilon}{4p}\right],$$

and

$$\lim_{\substack{\theta\to 0\\ s\to 0\\ \eta\to x_0}} x_\theta(t;s,\eta)=x(t;0,x_0) \text{ for all } t\in\left[t_p+\frac{\varepsilon}{4p},T\right],$$

where $i=1,2,\ldots,p-1$.

Let

$$J_\varepsilon=\left[0,t_1-\frac{\varepsilon}{4p}\right]\bigcup\left(\bigcup_{i=1}^{p-1}\left[t_i+\frac{\varepsilon}{4p},t_{i+1}-\frac{\varepsilon}{4p}\right]\right)\bigcup\left[t_p+\frac{\varepsilon}{4p},T\right],$$

we have

$$m\left(J\setminus J_\varepsilon\right)=\frac{\varepsilon}{2}<\varepsilon$$

and

$$\lim_{\substack{s\to 0\\ \eta\to x_0}} x_\theta(t;s,\eta)=x(t;0,x_0) \text{ for all } t\in J_\varepsilon$$

for some sufficient small θ. The proof is complete. □

Remark 3.4. Let $x(t,x_0)$, $x(t,x_0+\varepsilon(\bar{x}-x_0))$ be α-mild solution of (1.1) corresponding to x_0, $x_0+\varepsilon(\bar{x}-x_0)$, respectively. $x(t_i,x_0)=y_1$ implies there exist $\delta_i^1>0$, $\delta_i^2>0$ such that for any $\varepsilon\in[-\delta_i^1,\delta_i^1]$, the equation $x(t,x_0+\varepsilon(\bar{x}-x_0))=y_1$ has a unique solution $h_i(\varepsilon)\in[t_i-\delta_i^2,t_i+\delta_i^2]$, that is, there is a unique function $h_i:[-\delta_i^1,\delta_i^1]\longrightarrow[t_i-\delta_i^2,t_i+\delta_i^2]$ which is continuous, and moreover $h_i(0)=t_i$.

4. GÂTEAUX DIFFERENTIABILITY OF SOLUTION

For the following differential equation with impulses at fixed times

$$\begin{cases} \dot{x}(t)+A(t)x(t)=f(t,x(t)), & t\in(0,T]\backslash\Lambda,\\ x(0)=x_0, & \\ x\left(t_k^+\right)=J_k\left(x\left(t_k\right)\right)+x\left(t_k\right), & t_k\in\Lambda, \end{cases} \tag{4.1}$$

where $\Lambda = \{t_k \in [0,T] | 0 = t_0 < t_1 < t_2 < \cdots < t_n < t_{n+1} + T\}$, one can show that the equation (4.1) has a unique α-mild solution $x(\cdot, x_0) \in PC([0,T], X)$ given by

$$x(t, x_0) = U(t,0)x_0 + \int_0^t U(t,\tau) f(\tau, x(\tau, x_0))\, d\tau + \sum_{0<t_k<t} U(t, t_k)\, J_k(x(t_k, x_0))$$

provided that the assumption [F] is hold and $J_k : X_{\beta_i} \longrightarrow X_{\beta_i}$ ($\beta_i > \alpha$) maps bounded set to bounded set ($i = 1, 2, \ldots, n$) (see Theorem 3.B of [12]). If the functions J_k, f are continuously Frechet differential at x and $f_x(\cdot) = f_x(\cdot, x(\cdot, x_0)) \in L^1([0,T], \pounds(X))$, $J_{kx}(x(\cdot, x_0)) \in \pounds_b(X_{\beta_i})$, α-mild solution $x(\cdot, x_0)$ is Gâteaux differentiable, and the Gâteaux derivative φ of $x(\cdot, x_0)$ at x_0 in the direction $\bar{x} - x_0$ is the α-mild solution of the following impulsive evolution equation

$$\begin{cases} \dot{\varphi}(t) + A(t)\varphi(t) = f_x(t, x(t, x_0))\varphi(t), & t \in (0,T]\ \Lambda, \\ \varphi(0) = \bar{x} - x_0, \\ \varphi(t_k^+) = J_{kx}(x(t, x_0))\varphi(t_k) + \varphi(t_k), & t_k \in \Lambda. \end{cases} \tag{4.2}$$

Let us next consider the impulsive differential equation (1.3). Since in general we cannot even expect to have continuity of solution of (1.3) at points t for which satisfy $x(t) = y_1$, we have to introduce a suitable notion for Gâteaux differentiability of solution with respect to initial values.

Definition 4.1. The α-mild solution $x(\cdot) = x(\cdot, x_0)$ of (1.3) is said to be Gâteaux differentiable relative to x_0 if Gâteaux derivative $x_{x_0}(t, x_0)$ of $x(t, x_0)$ exists at x_0 in the direction $\bar{x} - x_0$ for all $t \in [0,T]$ such that $x(t, x_0) \neq y_1$, otherwise

$$x_{x_0}(t, x_0) = \lim_{s \to t^-} x_{x_0}(s, x_0), \tag{4.3}$$

where

$$x_{x_0}(t, x_0) = \lim_{\varepsilon \to 0} \frac{x(t, x_0 + \varepsilon(\bar{x} - x_0)) - x(t, x_0)}{\varepsilon}. \tag{4.4}$$

Now we are in a position to prove the following result.

Theorem 4.2. *Assume that the hypotheses of Theorem* 3.2 *hold and* y_1, $y_2 \in X_1$. *If the function* f *is continuously Frechet differential at* x *and* $f_x(\cdot) = f_x(\cdot, x) \in L^1([0,T], \pounds(X))$, $\{f(t, y_1), f(t, y_2) | t \in [0,T]\} \subset X$ *is a bounded set, then the* α*-mild solution* $x(\cdot, x_0)$ *of* (1.3) *is Gâteaux differentiable at* x_0 *in the direction* $\bar{x} - x_0 \in X_\beta$ *and furthermore, its Gâteaux derivative* $\frac{\partial}{\partial x_0} x(\cdot, x_0)$ *is the* α*-mild solution of the following impulsive differential equation*

$$\begin{cases} \dot{\varphi}(t) + A(t)\varphi(t) = f_x(t, x(t))\varphi(t), & x(t) \neq y_1, \\ \varphi(t^+) = h_t'(0)\Big[A(t)y_1 - A(t)y_2 + f(t, y_1) & \\ \qquad -f(t^+, y_2)\Big] + \varphi(t), & x(t) = y_1, \\ \varphi(0) = \bar{x} - x_0, & \end{cases} \tag{4.5}$$

provided that when $x(t,x_0)=y_1$, there exists function $h_t\in C^1$ such that $x(h_t(\varepsilon),x_0+\varepsilon(\bar{x}-x_0))=y_1$ for all ε with $|\varepsilon|$ being sufficiently small and $h_t(\varepsilon)\in O(t)$ which is a neighborhood of t.

Proof. Consider the following differential equation

$$\begin{cases} \dot{u}(t)+A(t)u(t)=f(t,u(t)), \quad t>0, \\ u(0)=u_0\in X_\beta. \end{cases} \tag{4.6}$$

By Proposition 2.1, this equation has a unique α-mild solution $u(\cdot,u_0)$ given by

$$u(t,u_0)=U(t,0)u_0+\int_s^t U(t,\tau)f(\tau,u(\tau,u_0))\,d\tau.$$

Since the function f is continuously Frechet differentiable, $u(\cdot,u_0)$ is Gâteaux differentiable at u_0 in the direction $\bar{u}-u_0\in X_\beta$ and

$$v(\cdot,u_0)=u_{u_0}(t,u_0)$$

is the α-mild solution of the following differential equation

$$\begin{cases} \dot{v}(t)+A(t)v(t)=f_u(t,u(t))v(t), \quad t>0, \\ v(0)=\bar{u}-u_0. \end{cases} \tag{4.7}$$

For any $x_0\in X_\beta$, Theorem 2.3 tells us that the α-mild solution $x(t)=x(t,x_0)$ of (1.3) meets the warning line $y=y_1$ at most finite number of times on the interval $[0,T]$ and therefore we suppose that $x(t)$ meets $y=y_1$ at t_j, $j=1,2,\dots,p$, where $0<t_1<t_2<\cdots<t_p<T$. Let $t\in[0,t_1)$ be fixed. Then $x(t,x_0+\varepsilon(\bar{x}-x_0))$ does not hit the warning line $y=y_1$ for all ε which being sufficiently small and therefore $x(t,x_0+\varepsilon(\bar{x}-x_0))=u(t,x_0+\varepsilon(\bar{x}-x_0))$ which implies that

$$x_{x_0}(t,x_0)=u_{x_0}(t,x_0)$$

exists. Let

$$x_\varepsilon(t)=\frac{x(t,x_0+\varepsilon(\bar{x}-x_0))-x(t,x_0)}{\varepsilon}, \tag{4.8}$$

for $t\in[0,t_1]$, we have

$$\lim_{\varepsilon\to 0}x_\varepsilon(t)=\frac{\partial}{\partial x_0}x(t,x_0)\equiv\varphi(t)$$

which satisfies the following integral equation

$$\varphi(t)=U(t,0)(\bar{x}-x_0)+\int_0^t U(t,\tau)f_x(\tau,x(\tau))\,\varphi(\tau)d\tau.$$

Set $h_1=h_{t_1}$, if $h_1(\varepsilon)>t_1$, we have

$$\begin{aligned}&\lim_{\varepsilon\to 0}\frac{x(h_1(\varepsilon),x_0+\varepsilon(\bar{x}-x_0))-x(t_1,x_0)}{\varepsilon}\\ &=\lim_{\varepsilon\to 0}\frac{x(h_1(\varepsilon),x_0+\varepsilon(\bar{x}-x_0))-x(t_1,x_0+\varepsilon(\bar{x}-x_0))}{\varepsilon}\end{aligned}$$

$$
\begin{aligned}
&+\lim_{\varepsilon\to 0}\frac{x\left(t_1,x_0+\varepsilon\left(\bar{x}-x_0\right)\right)-x\left(t_1,x_0\right)}{\varepsilon}\\
&=\dot{h}_1(0)\left[-A\left(t_1\right)x\left(t_1,x_0\right)+f\left(t_1,x\left(t_1,x_0\right)\right)\right]+\varphi\left(t_1\right)\\
&=\dot{h}_1(0)\left[-A\left(t_1\right)y_1+f\left(t_1,y_1\right)\right]+\varphi\left(t_1\right).
\end{aligned}
$$

If $h_1(\varepsilon)<t_1$, we also have

$$
\lim_{\varepsilon\to 0}\frac{x\left(h_1(\varepsilon),x_0+\varepsilon\left(\bar{x}-x_0\right)\right)-x\left(t_1,x_0\right)}{\varepsilon}=\dot{h}_1(0)\left[-A\left(t_1\right)y_1+f\left(t_1,y_1\right)\right]+\varphi\left(t_1\right).
$$

Hence, $x_\varepsilon\left(h_1(\varepsilon)\right)$ is differentiable at $\varepsilon=0$, and

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{x\left(h_1(\varepsilon),x_0+\varepsilon\left(\bar{x}-x_0\right)\right)-x\left(t_1,x_0\right)}{\varepsilon}\\
&=\dot{h}_1(0)\left[-A\left(t_1\right)y_1+f\left(t_1,y_1\right)\right]+\varphi\left(t_1\right).
\end{aligned}
\tag{4.9}
$$

Furthermore, if $h_1(\varepsilon)>t_1$, we have

$$
\begin{aligned}
\lim_{\varepsilon\to 0}x_\varepsilon(h_1^+(\varepsilon))&=\lim_{\varepsilon\to 0}\frac{x\left(h_1^+(\varepsilon),x_0+\varepsilon\left(\bar{x}-x_0\right)\right)-x\left(h_1(\varepsilon),x_0\right)}{\varepsilon}\\
&=\lim_{\varepsilon\to 0}\frac{x\left(h_1(\varepsilon),x_0+\varepsilon\left(\bar{x}-x_0\right)\right)-x\left(t_1,x_0\right)}{\varepsilon}\\
&\quad-\lim_{\varepsilon\to 0}\frac{U\left(h_1(\varepsilon),t_1\right)y_2-y_2}{\varepsilon}\\
&\quad-\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{t_1}^{h_1(\varepsilon)}U\left(h_1(\varepsilon),\tau\right)f\left(\tau,x\left(\tau,x_0\right)\right)d\tau\\
&=-\dot{h}_1(0)\left[A\left(t_1\right)y_1+A\left(t_1\right)y_2-f\left(t_1,y_1\right)+f\left(t_1^+,y_2\right)\right]+\varphi\left(t_1\right).
\end{aligned}
\tag{4.10}
$$

At the same time, if $h_1(\varepsilon)<t_1$, we have

$$
\begin{aligned}
\lim_{\varepsilon\to 0}x_\varepsilon(t_1^+)&=\lim_{\varepsilon\to 0}\frac{x\left(t_1,x_0+\varepsilon\left(\bar{x}-x_0\right)\right)-x\left(t_1^+,x_0\right)}{\varepsilon}\\
&=\lim_{\varepsilon\to 0}U\left(t_1,h_1(\varepsilon)\right)\frac{x\left(t_1,x_0+\varepsilon\left(\bar{x}-x_0\right)\right)-x\left(t_1,x_0\right)}{\varepsilon}\\
&\quad+\lim_{\varepsilon\to 0}\frac{U\left(t_1,h_1(\varepsilon)\right)y_2-y_2}{\varepsilon}\\
&\quad+\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{h_1(\varepsilon)}^{t_1}U\left(t_1,\tau\right)f\left(\tau,x\left(\tau,x_0+\varepsilon\left(\bar{x}-x_0\right)\right)\right)d\tau\\
&=-\dot{h}_1(0)\left[A\left(t_1\right)y_1+A\left(t_1\right)y_2-f\left(t_1,y_1\right)+f\left(t_1^+,y_2\right)\right]+\varphi\left(t_1\right).
\end{aligned}
\tag{4.11}
$$

For $t\in(t_1,t_2]$, we also have

$$
\lim_{\varepsilon\to 0}x_\varepsilon(t)=\frac{\partial}{\partial x_0}x\left(t,x_0\right)\equiv\varphi(t).
$$

Obviously, φ is the α-mild solution of the following Cauchy problem

$$\left\{\begin{aligned} &\dot{\varphi}(t)+A(t)\varphi(t)=f_x(t,x(t,x_0))\,\varphi(t), && t_1<t\le t_2,\\ &\varphi\left(t_1^+\right)=-\dot{h}_1(0)\Big[A(t_1)\,y_1+A(t_1)\,y_2\\ &\qquad\qquad +f\left(t_1^+,y_2\right)-f(t_1,y_1)\Big]+\varphi(t_1). \end{aligned}\right. \tag{4.12}$$

By repeating the same argument for $t\in(t_m,t_{m+1})$, $m=1,2,\dots,p$, we can see that $x_{x_0}(t,x_0)=\varphi(t)$ exists for all $t\neq t_m$, $m=1,2,\dots,p$, and is α-mild solution of the following equation

$$\left\{\begin{aligned} &\dot{\varphi}(t)+A(t)\varphi(t)=f_x(t,x(t,x_0))\varphi(t), && t_m<t\le t_{m+1},\\ &\varphi\left(t_m^+\right)=-\dot{h}_m(0)\Big[A(t_m)\,y_1+A(t_m)\,y_2\\ &\qquad\qquad +f\left(t_m^+,y_2\right)-f(t_m,y_1)\Big]+\varphi(t_m). \end{aligned}\right. \tag{4.13}$$

Thus, the α-mild solution $x(\cdot,x_0)$ of (1.3) is Gâteaux differentiable with respect to the initial values x_0 and its Gâteaux derivative satisfies the impulsive equation (4.5). The proof is complete. □

Similarly, we can also show the following theorem.

Theorem 4.3. *Assume that the hypotheses of Theorem* 2.3 *hold and* y_1, $y_2\in X_1$, $g\in L^p([0,T],X)$ $(p>1)$. *If the function* f *is continuously Frechet differentiable at* x *and* $f_x(\cdot)=f_x(\cdot,x)\in L^1([0,\infty),\pounds(X))$, $\{f(t,y_1),f(t,y_1)\,|t\in[0,\infty)\}\subset X$ *is a bounded set, then the following differential equation with impulses at variable times*

$$\left\{\begin{aligned} &\dot{x}(t)+A(t)x(t)=f(t,x(t))+g(t), && x(t)\neq y_1,\\ &x(0)=x_0,\\ &x\left(t^+\right)=y_2, && x(t)=y_1, \end{aligned}\right. \tag{4.14}$$

has a unique α*-mild solution* $x(\cdot,g)$ *which is Gâteaux differentiable at* g *in the direction* $\bar{g}-g$ *and furthermore,* $\frac{\partial}{\partial g}x(\cdot,g)$ *is the mild solution of the following impulsive differential equation*

$$\left\{\begin{aligned} &\dot{y}(t)+A(t)y(t)=f_x(t,x(t))y(t)+\bar{g}(t)-g(t), && x(t)\neq y_1,\\ &y\left(t^+\right)=-h_t'(0)\Big[A(t)y_1+A(t)y_2-f(t,y_1)\\ &\qquad\qquad +f\left(t^+,y_2\right)\Big]+y(t), && x(t)=y_1,\\ &y(0)=0, \end{aligned}\right. \tag{4.15}$$

provided that $\alpha<\frac{p-1}{p}$, *and when* $x(t,g)=y_1$, *there exist function* $h_t\in C^1$ *such that* $x(h_t(\varepsilon),g+\varepsilon(\bar{g}-g))=y_1$ *for all* ε *with* $|\varepsilon|$ *being sufficiently small,* $h_t(\varepsilon)\in U(t)$ *which is a neighborhood of* t.

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