Dynamic Systems and Applications 21 (2012) 393-406

APPLICATION OF SEMIMARTINGALE MEASURE TO THE INVESTIGATION OF STOCHASTIC INCLUSION

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ABSTRACT. A random measure associated to a semimartingale is introduced. It is used to investigate properties of the solution set of a stochastic inclusion.

AMS (MOS) Subject Classification. 93C30, 93E03

1. INTRODUCTION

Stochastic differential inclusions were the subject of several studies (see e.g.: N. U. Ahmed [2], G. Da Prato, H. Frankowska [4], M. Kisielewicz [9, 10, 11], M. Kisielewicz, M. Michta, M. Motyl [13, 14], J. Motyl [15]). Some of them used in their research a measure introduced by C. Doléans-Dade in [5]. It was applied in investigation of of set-valued stochastic integrals and stochastic inclusions driven by a square-integrable martingale or a Brownian motion, (see e.g. [4, 11, 14]).

One of properties of the Doléans-Dade measure is an isometry property for stochastic integrals. It allows to calculate the distance of integrals by the distance of integrands. Using this property, we can also estimate the distance, in the case of set-valued stochastic integrals. This estimate, in turn, applies to study of properties of the set of solutions of stochastic inclusions.

In the paper we introduce a random measure associated to a semimartingale. It is related to the Doléans-Dade measure but it is wider. The main advantage of this measure is "semimartingale measure property" (called SMP-property), which allows to estimate the distance between set-valued stochastic integrals. The SMP-property is proved in Lemma 3.1. It allows to study properties of the set of solutions of stochastic inclusions driven by a semimartingale.

In Section 2 we introduce basic definitions and notations used in the paper. Section 3 contains the definition of the semimartingale measure and its SMP-property. In Section 4 we consider some properties of the solution set of a semimartingale stochastic inclusion.

Received November 15, 2011

2. PRELIMINARIES

Let $T \geq 0$ and $[0,T]$ be an arbitrary closed interval. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$ be a completed filtered probability space satisfying the usual hypothesis i.e.: (i) $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$, (ii) $\mathcal{F}_t = \bigcap_{u\geq t} \mathcal{F}_u$, all t, $0 \leq t \leq T$. We consider a stochastic process x on $(\Omega, \mathcal{F}, P)$ as a collection $x = (x_t)_{t\in[0,T]}$ of n-dimensional random variables $x_t : \Omega \to I\!R^n$, $t \in [0,T]$. It is adapted if x_t belongs to $\mathcal{F}_t$ for each $t \in [0,T]$. It is cádlág [cáglád] if it has right continuous with left limits sample paths [left continuous with right limits sample paths].

Let $\mathcal{P}(\{\mathcal{F}_t\}_{t\in[0,T]})$ denote the smallest σ-algebra on $[0,T] \times \Omega$ generated by cáglád adapted processes. It is generated by a class of all subsets of $[0,T] \times \Omega$ of the form $\{0\} \times F_0$ and $(s,t] \times F$, where $F_0 \in \mathcal{F}_0$ and $F \in \mathcal{F}_s$ for $0 \leq s < t \leq T$. If a stochastic process x is $\mathcal{P}(\{\mathcal{F}_t\}_{t\in[0,T]})$-measurable, it is called predictable.

For a Banach space X by $cl(X)$ and $conv(X)$ we denote spaces of all nonempty closed and convex, respectively, subsets of X. By $\text{dist}(a, A)$ we denote the distance of $a \in X$ to the set $A \in cl(X)$. For $A, B \in cl(X)$ let $\overline{h}(A,B) = \sup_{a\in A} \text{dist}(a, B)$ and $H(A,B) = \max\{\overline{h}(A,B),\ \overline{h}(B,A)\}$.

A set-valued function $G : [0,T] \times \Omega \to cl(I\!R^n)$ is $(\beta \otimes \mathcal{F})$-measurable if for every open set $O \subset I\!R^n$, $\{(t,\omega) : G(t,\omega) \cap O \neq \emptyset\} \in \beta \otimes \mathcal{F}$.

By a set-valued stochastic process $G = (G_t)_{t\in[0,T]}$ with values in $cl(I\!R^n)$ we consider a family of $\mathcal{F}$-measurable set-valued mappings $G_t : \Omega \to cl(I\!R^n)$, each $t \in [0,T]$. It is adapted to the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ if G_t is $\mathcal{F}_t$-measurable for each $t \in [0,T]$.

A set-valued process $G = (G_t)_{t\in[0,T]}$ is predictable if it is $\mathcal{P}(\{\mathcal{F}_t\}_{t\in[0,T]})$-measurable and the family of all such processes is also denoted by $\mathcal{P}$. One has $\mathcal{P} \subset \beta \otimes \mathcal{F}$, where β denotes the Borel σ-algebra on $[0,T]$.

We denote by $|\cdot|$ an Euclidean norm on $I\!R^n$. Other norms are denoted with respect to a space on which they are defined, e.g.: $\|\cdot\|_{L^2(\Omega)}$ for the norm in $L^2(\Omega)$.

By $L^2(\Omega)$, $[L^\infty(\Omega)]$ we denote the space $L^2(\Omega, \mathcal{F}, P; I\!R^n)$, $[L^\infty(\Omega, \mathcal{F}, P; I\!R^n)]$.

3. SEMIMARTINGALE MEASURE

Let $Z = (Z_t)_{t\in[0,T]}$ denote a one-dimensional semimartingale, $Z_0 = 0$, with decomposition $Z = N + A$, where N is a local martingale, A is a cádlág process with path of finite variation on compacts (FV-process). $[Z,Z]$ is a quadratic variation process of Z and $|dA_t(\omega)|$ denotes a random measure induced by the paths of the process A (see e.g.: [17]).

Let $\mathcal{H}^p$ denote a space of semimartingales with a norm

$$\|Z\|_{\mathcal{H}^p} = \inf_{Z=N+A} j_p(N, A),$$

where $p = 2$ or $p = \infty$,

$$j_p(N, A) = \|[N, N]_T^{1/2} + \int_0^T |dA_\tau|\|_{L^p(\Omega)}.$$

and infimum is taken over all possible decompositions $Z = N + A$.
$\mathcal{H}^p$ is a Banach space (see e.g. [17]).

By $\mathcal{H}^2_n$ we denote a space of n-dimensional semimartingales $Z = (Z^1, \ldots, Z^n)$, $Z^i \in \mathcal{H}^2$, $i = 1, \ldots, n$, with a norm

$$\|Z\|_{\mathcal{H}^2_n} = (\textstyle\sum_{i=1}^n \|Z^i\|^2_{\mathcal{H}^2})^{1/2}.$$

Let $p = 2$. For a semimartingale $Z \in \mathcal{H}^2$ we define a measure μ_Z as follows.

Let M be a right continuous and square-integrable martingale. We remind a construction of a Doléans-Dade measure μ_M on $\mathcal{P}$. On a rectangle $(s, t] \times B$ in $[0, T] \times \Omega$ with B being $\mathcal{F}_s$-measurable, $s \leq t$, we define a set function λ_M

$$\lambda_M((s, t] \times B) = E(1\!\!1_B(M_t - M_s)^2),$$

where $1\!\!1_B$ denotes the characteristic function of B, and extend it to a unique σ-finite measure μ_M on $\mathcal{P}$ (see e.g.: [3] Section 2.4).

By [17] Cor.II.6.4 this measure can be also defined for a local martingale $N \in \mathcal{H}^2$. We denote it by μ_N.

Now, for an FV-process A we define a measure ν_A on $\mathcal{P}$ as follows.
Let $\alpha(\omega, dt)$ denotes a kernel of a random measure defined on $[0, T]$ by

$$\alpha(\omega, dt) := c_A(\omega)|dA_t(\omega)|,$$

where $c_A(\omega) = \int_0^T |dA_t(\omega)|$ denotes the total variation of a random measure $|dA_t(\omega)|$ induced by the paths of the process A.

Let D be a predictable subset of $[0, T] \times \Omega$. A measure ν_A is defined by

$$\nu_A(D) = \int_\Omega \int_0^T 1\!\!1_D(\omega, t)\alpha(\omega, dt)P(d\omega),$$

For an $\mathcal{H}^2$-semimartingale Z we define a measure $\mu_Z = \mu_N + \nu_A$.

Let $Z \in \mathcal{H}^2$ and $f : [0, T] \times \Omega \to I\!\!R^n$. We define a space

$$L^2_{\mu_Z} = \{f \in \mathcal{P} : \int_{\Omega \times [0,T]} |f|^2 d\mu_Z < \infty\}.$$

$L^2_{\mu_Z}$ endowed with a norm

$$\|f\|_{L^2_{\mu_Z}} = (\textstyle\int_{\Omega \times [0,T]} |f|^2 d\mu_Z)^{\frac{1}{2}}$$

is a Banach space.

Now we present a property (SMP-property) of the above introduced measure. SMP-property allows to get similar results for set-valued stochastic integrals driven by a semimartingale as for set-valued integrals driven by a square-integrable martingale, (see e.g.: [16]).

Lemma 3.1. *Let $Z \in \mathcal{H}^2$ and $f \in L^2_{\mu_Z}$. For $s, t \in [0,T]$, $s < t$, we have*

$$\| \int_s^t f_\tau dZ_\tau \|^2_{L^2(\Omega)} \leq 2 \int_{\Omega \times (s,t]} |f|^2 d\mu_Z.$$

Proof. Let $Z \in \mathcal{H}^2$, $f \in L^2_{\mu_Z}$ and $s, t \in [0,T]$, $s < t$. Using the inequality $(a+b)^2 \leq 2(a^2+b^2)$ we get

$$\begin{aligned}\| \int_s^t f_\tau dZ_\tau \|^2_{L^2(\Omega)} &= E| \int_s^t f_\tau dZ_\tau|^2 = E|\int_s^t f_\tau dN_\tau + \int_s^t f_\tau dA_\tau|^2 \\ &\leq 2(E|\int_s^t f_\tau dN_\tau|^2 + E|\int_s^t f_\tau dA_\tau|^2) \\ &= 2(E\int_s^t |f_\tau|^2 d[N,N]_\tau + E|\int_s^t f_\tau dA_\tau|^2) \\ &\leq 2(E\int_s^t |f_\tau|^2 d[N,N]_\tau + E|\int_s^t |f_\tau||dA_\tau||^2).\end{aligned}$$

By the Kunita-Watanabe inequality and the definition of μ_Z-measure we get

$$\begin{aligned}\| \int_s^t f_\tau dZ_\tau \|^2_{L^2(\Omega)} &\leq 2(E\int_s^t |f_\tau|^2 d[N,N]_\tau + E(\int_s^t |dA_\tau| \int_s^t |f_\tau|^2 |dA_\tau|)) \\ &\leq 2E\int_s^t |f_\tau|^2 d[N,N]_\tau + 2E(c_A(\omega) \cdot \int_s^t |f_\tau|^2 |dA_\tau|) \\ &= 2(E\int_s^t |f_\tau|^2 d[N,N]_\tau + E(\int_s^t |f_\tau|^2 \alpha(\omega, dt)) \\ &= 2(\int_{\Omega \times (s,t]} |f_\tau|^2 d\mu_N + \int_{\Omega \times (s,t]} |f_\tau|^2 d\nu_A) = 2(\int_{\Omega \times (s,t]} |f|^2 d\mu_Z).\end{aligned}$$

This completes the proof. □

Considering a stochastic integral as a stochastic process its SMP-property will take the following form.

Corollary 3.2. *For $Z \in \mathcal{H}^2$ and $f \in L^2_{\mu_Z}$ we have*

$$\| \int f_\tau dZ_\tau \|^2_{\mathcal{H}^2_n} \leq 2 \|f\|^2_{L^2_{\mu_Z}}.$$

4. STOCHASTIC DIFFERENTIAL INCLUSION

At the beginning of this section we present properties of the set $\mathcal{S}_{\mu_Z}(G)$ and definitions of set-valued stochastic integrals driven by a semimartingale Z, which are used in the second part of this section. Thanks to Lemma 3.1 and Corollary 3.2 proofs of these properties are similar to the proofs presented in [16] for the case of set-valued integrals driven by a square-integrable martingale and therefore they are omitted. The second part of this section contains properties of the solution set of the stochastic inclusion driven by a semimartingale.

Definition 4.1 ([16])**.** For an $\mathcal{H}^2$-semimartingale Z and a predictable set-valued process G, we define a set $\mathcal{S}_{\mu_Z}(G)$ by

$$\mathcal{S}_{\mu_Z}(G) := \{f \in L^2_{\mu_Z} \ : \ f(t,\omega) \in G(t,\omega) \ \mu_Z\text{-a.e.}\}.$$

A predictable set-valued process G is integrable with respect to a semimartingale measure μ_Z, if the set $\mathcal{S}_{\mu_Z}(G)$ is nonempty.

G is μ-integrably bounded if there exists a process $m \in L^2_{\mu_Z}$ such that $H(G,\{0\}) \leq m \ \mu_Z$-a.e..

Lemma 4.2. *For an $\mathcal{H}^2$-semimartingale Z, $Z_0 = 0$ and a predictable μ-integrably bounded set-valued process G we get*

- *the set $\mathcal{S}_{\mu_Z}(G)$ is a nonempty closed and bounded subset of $L^2_{\mu_Z}$,*
- *if G takes on convex values, $\mathcal{S}_{\mu_Z}(G)$ is convex and weakly compact in $L^2_{\mu_Z}$.*

Definition 4.3 ([16])**.** Let Z be an $\mathcal{H}^2$-semimartingale, $Z_0 = 0$. Let G be a predictable μ-integrably bounded set-valued process.

A set-valued stochastic integral $\int G_\tau dZ_\tau$ of G with respect to Z is defined by $\int G_\tau dZ_\tau = \{\int g_\tau dZ_\tau \ : g \in \mathcal{S}_{\mu_Z}(G)\}$.

For fixed $0 \leq s < t \leq T$ we also define $\int_s^t G_\tau dZ_\tau = \{\int_s^t g_\tau dZ_\tau \ : g \in \mathcal{S}_{\mu_Z}(G)\}$.

Let $F : [0,T] \times I\!R^n \to cl\, conv(I\!R^n)$ be a multifunction $(\beta \otimes \mathcal{F})$-measurable.

By $L^2([0,T] \times \Omega; I\!R^n)$ we denote a space of $I\!R^n$-valued stochastic processes with a norm $\|x\|_{L^2([0,T]\times\Omega;I\!R^n)} = (E\int_0^T |x_t|^2 dt)^{1/2}$.

Let $S^2([0,T])$ denote a space of $I\!R^n$-valued adapted cádlág processes with a norm $\|x\|_{S^2} = \|\sup_{t\in[0,T]} |x_t|\|_{L^2(\Omega)}$.

For any $x \in S^2([0,T])$ and a multifunction F, by $F \circ x_-$, where $x_{t-} = \lim_{s\uparrow t} x_s$, we denote a set-valued process $(F(t, x_{t-}(\omega)))_{t\in[0,T]}$.

Let $x \in S^2([0,T])$, $Z \in \mathcal{H}^\infty$. Let $F(t,\cdot)$ be continuous for any $t \in [0,T]$. If a process $F \circ x_-$ is μ-integrably bounded, then the set $\mathcal{S}_{\mu_Z}(F \circ x_-)$ is nonempty in $L^2_{\mu_Z}$. It follows from Lemma 4.2.

Definition 4.4. Let Z be an $\mathcal{H}^\infty$-semimartingale, $Z_0 = 0$, $F : [0,T] \times I\!R^n \to cl\, conv(I\!R^n)$ and $s,t \in [0,T]$, $s < t$. We consider the stochastic inclusion

$$x_t - x_s \in cl_{L^2(\Omega)}(\textstyle\int_s^t F(\tau, x_{\tau-}) dZ_\tau) \tag{SI}$$

with $x_0 = \xi \in L^2(\Omega, \mathcal{F}_0, P; I\!R^n)$.

A process $x \in S^2([0,T])$ is a solution of the stochastic inclusion (SI), if $x_0 = \xi$ and for any $s,t \in [0,T]$, $s < t$ a random variable $x_t - x_s$ belongs to the set

$$cl_{L^2(\Omega)}(\int_s^t F(\tau, x_{\tau-})dZ_\tau).$$

A set of all solutions of the stochastic inclusion (SI) is denoted by

$$\mathcal{T}(\xi, Z, F) = \{x \in S^2([0,T]) : x \text{ is a solution of (SI)}\}.$$

We say that $F : [0,T] \times I\!R^n \to cl\, conv(I\!R^n)$ is a Lipschitz multifunction if there exists a constant D such that for all $t \in [0,T]$ and $x, y \in I\!R^n$

$$H(F(t,x), F(t,y)) \leq D|x - y|.$$

We say that $F : [0,T] \times I\!R^n \to cl\, conv(I\!R^n)$ is a Carathéodory-type multifunction if for any $x \in I\!R^n$ $F(\cdot, x)$ is β-measurable and for any $t \in [0,T]$ $F(t, \cdot)$ is continuous.

Assumption 4.5. Let $F : [0,T] \times I\!R^n \to cl\, conv(I\!R^n)$ be a multifunction satisfying

- $F : [0,T] \times I\!R^n \to cl\, conv(I\!R^n)$ is $(\beta \otimes \mathcal{F})$-measurable,
- $F : [0,T] \times I\!R^n \to cl\, conv(I\!R^n)$ is a Lipschitz multifunction,
- for any $x \in S^2([0,T])$ a set-valued process $F \circ x_-$ is μ-integrably bounded.

Now we prove non-emptiness and closedness of the set of solutions $\mathcal{T}(\xi, Z, F)$.

Theorem 4.6. *Let Z be an $\mathcal{H}^\infty$-semimartingale, $Z_0 = 0$. Let $F : [0,T] \times I\!R^n \to cl\, conv(I\!R^n)$ satisfies the Assumption* 4.5*. Then for any $\xi \in L^2(\Omega, \mathcal{F}_0, P; I\!R^n)$ the set $\mathcal{T}(\xi, Z, F)$ is nonempty.*

Proof. In the proof we will use the Covitz-Nadler Theorem, (see e.g.: [12] Th.II.4.4). Let $N + A$ be a decomposition of the semimartingale Z, i.e. $Z = N + A$. Let us divide the interval $[0,T]$ by $0 = t_0 < t_1 < \ldots < t_{k-1} < t_k = T$. Let $c_Z^i = (\int_{t_{i-1}}^{t_i} d[N,N]_\tau)^{1/2} + \int_{t_{i-1}}^{t_i} |dA_\tau|$, for $i = 1, \ldots, k$. We choose the points t_i such that

$$Dc_2 \|c_Z^i\|_{L^\infty(\omega)} < 1,$$

where a constant c_2 comes from [17] Th.V.2.2.

First, we construct a solution of the stochastic inclusion (SI) on $[0, t_1]$. For any $\xi \in L^2(\Omega, \mathcal{F}_0, P; I\!R^n)$ and $x \in S^2([0, t_1])$ we define a map Γ by

$$\Gamma(x) = \{y \in S^2([0,t_1]) : \; y_t = \xi + \int_0^t f_\tau dZ_\tau,$$
$$\text{where } f \in \mathcal{S}_{\mu_Z}(F \circ x_-), \text{ for } (t,\omega) \in [0,t_1] \times \Omega\}$$

Let x be an arbitrary element of $S^2([0,t_1])$. Thanks to μ-integrably boundedness of $F \circ x_-$ and Lemma 4.2 it follows that $\Gamma(x)$ is nonempty.

$\Gamma(x)$ is not necessarily a closed set in $S^2([0,t_1])$ (in a sense of $\|\cdot\|_{S^2}$-norm).

Let us consider a set $cl_{S^2}(\Gamma(x))$. It is a closure of the set $\Gamma(x)$ in $S^2([0,t_1])$. This set is a nonempty, bounded and closed subset of $S^2([0,t_1])$.

We show that a map $x \to cl_{S^2}(\Gamma(x))$ is a set-valued contraction in $S^2([0,t_1])$.

Let u and v be arbitrary elements of $S^2([0,t_1])$. We show that there exists a constant $K \in [0,1)$ such that

$$H_{S^2}(cl_{S^2}(\Gamma(u)), cl_{S^2}(\Gamma(v))) \le K\|u-v\|_{S^2}.$$

Let y be an arbitrary element of $cl_{S^2}(\Gamma(u))$. For any $\epsilon > 0$ there exists a stochastic process $\tilde{y} \in \Gamma(u)$ such that $\|y-\tilde{y}\|_{S^2} < \epsilon$. It can be represented as $\tilde{y}_t = \xi + \int_0^t f_\tau dZ_\tau$ for some $f \in \mathcal{S}_{\mu_Z}(F \circ u_-)$ on $[0,t_1] \times \Omega$. It follows from the definition of the set $\Gamma(u)$.

From the Filippov Theorem (see e.g.: [12] Th.II.3.12) there exists $\bar{f} \in \mathcal{S}_{\mu_Z}(F \circ v_-)$ such that

$$|f(t,\omega) - \bar{f}(t,\omega)| \le \operatorname{dist}(f(t,\omega), F(t, v(t-,\omega))) + \epsilon, \tag{4.1}$$

for any $t \in [0,t_1]$ and a.a. $\omega \in \Omega$. Let $\bar{y}_t = \xi + \int_0^t \bar{f}_\tau dZ_\tau$ for $t \in [0,t_1]$. From the definition of the set $\Gamma(v)$ we get $\bar{y} \in \Gamma(v)$.

Let us estimate the distance between y and $\bar{y}$ in $S^2([0,t_1])$. We get

$$J = \|y-\bar{y}\|_{S^2} \le \|y-\tilde{y}\|_{S^2} + \|\tilde{y}-\bar{y}\|_{S^2} \le \epsilon + \|\int (f_\tau - \bar{f}_\tau) dZ_\tau\|_{S^2}.$$

For an arbitrary $f \in F \circ u_-$ we have

$$\operatorname{dist}(f(t,\omega), F(t, v(t-,\omega))) \le H(F(t,u_{t-}), F(t,v_{t-})),$$

for any $t \in [0,t_1]$ and a.a. $\omega \in \Omega$, so by (4.1) we get

$$J \le \epsilon + c_2 \|c_Z^1\|_{L^\infty(\omega)} \| \sup_{t\in[0,t_1]} (H(F(t,u_{t-}), F(t,v_{t-})) + \epsilon)\|_{L^2(\Omega)}.$$

From the Lipschitz condition for the multifunction F we get

$$\begin{aligned} J &\le \epsilon + c_2 \|c_Z^1\|_{L^\infty(\omega)} \| \sup_{t\in[0,t_1]} (D|u_t - v_t| + \epsilon)\|_{L^2(\Omega)} \\ &\le \epsilon + Dc_2 \|c_Z^1\|_{L^\infty(\omega)} \| \sup_{t\in[0,t_1]} |u_t - v_t| \|_{L^2(\Omega)} + c_2 \|c_Z^1\|_{L^\infty(\omega)} \epsilon \\ &\le Dc_2 \|c_Z^1\|_{L^\infty(\omega)} \|u-v\|_{S^2} + \epsilon_1, \end{aligned}$$

where $\epsilon_1 = (c_2\|c_Z^1\|_{L^\infty(\omega)} + 1)\epsilon$. Thus there exists a constant $K = Dc_2\|c_Z^1\|_{L^\infty(\omega)}$ which does not depend on the choice of the element y from the set $cl_{S^2}(\Gamma(u))$. Therefore,

$$\|y_t - \bar{y}_t\|_{S^2} \le K\|u-v\|_{S^2} + \epsilon_1.$$

Since $\epsilon > 0$ was arbitrarily chosen, the distance from an element $y \in cl_{S^2}(\Gamma(u))$ to the set $cl_{S^2}(\Gamma(v))$ can be estimated by

$$\operatorname{dist}_{S^2}(y, cl_{S^2}(\Gamma(v))) \le K\|u-v\|_{S^2}.$$

Thus

$$H_{S^2}(cl_{S^2}(\Gamma(u)), cl_{S^2}(\Gamma(v))) \le K\|u-v\|_{S^2}.$$

The constant $K = Dc_2\|c_Z^1\|_{L^\infty(\omega)}$ is a nonnegative number less than 1, and therefore, the map $cl_{S^2}(\Gamma(x))$ is a set-valued contraction in $S^2([0,t_1])$.

From the Covitz-Nadler Theorem we conclude there exists a process $y \in S^2([0,t_1])$ such that $y \in cl_{S^2}(\Gamma(y))$. For any $\epsilon > 0$ we can choose $y^\epsilon \in \Gamma(y)$ satisfying

$$\|y - y^\epsilon\|_{S^2} < \epsilon.$$

By the definition of the set $\Gamma(y)$ there exists $f^\epsilon \in \mathcal{S}_{\mu_Z}(F \circ y_-)$ such that $y_t^\epsilon = \xi + \int_0^t f_\tau^\epsilon dZ_\tau$ for any $t \in [0,t_1]$. Therefore,

$$\| \sup_{t\in[0,t_1]} |y_t - (\xi + \int_0^t f_\tau^\epsilon dZ_\tau)| \|_{L^2(\Omega)} < \epsilon.$$

Thus for any $t \in [0,t_1]$

$$\|y_t - (\xi + \int_0^t f_\tau^\epsilon dZ_\tau)\|_{L^2(\Omega)} < \epsilon. \tag{4.2}$$

Now we show that the above process y is a solution of the stochastic inclusion (SI) on $[0,t_1]$. We do this by checking that $y_t - y_s \in cl_{L^2(\Omega)}(\int_s^t F(\tau, y_{\tau-})dZ_\tau)$ for any $s,t \in [0,t_1]$, $s < t$.

By (4.2) we get that for every $s,t \in [0,t_1]$, $s < t$

$$\|y_t - y_s - \int_s^t f_\tau^\epsilon dZ_\tau\|_{L^2(\Omega)} < \epsilon.$$

Since $\epsilon > 0$ was arbitrarily chosen, we obtain

$$y_t - y_s \in cl_{L^2(\Omega)}(\int_s^t F(\tau, y_{\tau-})dZ_\tau), \text{ for any } s,t \in [0,t_1], \quad s < t.$$

Let $i = 2$. The proof is similar to the $i = 1$ case. We should only change the interval $[0,t_1]$ into $(t_1,t_2]$ and take the starting point of the constructed solution equal to y_{t_1}. In a similar way we obtain a process $y \in S^2((t_1,t_2])$ and for an arbitrary $\epsilon > 0$ a process $f^\epsilon \in \mathcal{S}_{\mu_Z}(F \circ y_-)$ such that

$$\|y_t - (y_{t_1} + \int_{t_1}^t f_\tau^\epsilon dZ_\tau)\|_{L^2(\Omega)} < \epsilon,$$

for any $t \in (t_1,t_2]$. The above inequality means that for every $s,t \in (t_1,t_2]$, $s < t$, the stochastic process y is an element of the closure in a sense of an $L^2(\Omega)$-norm of the set

$$\int_s^t F(\tau, y_{\tau-})dZ_\tau,$$

Therefore, y is a solution of the inclusion (SI) on the interval $(t_1,t_2]$.

When we repeat the above construction for $i = 2,3,\ldots,k-1$, taking starting points of the constructed solutions equal to y_{t_i}, we get solutions of the inclusion (SI) on the intervals $(t_i,t_{i+1}]$.

The solution of the inclusion (SI) for $s,t \in [0,T]$, $s<t$ is a composition of the solutions constructed on the intervals $[0,t_1]$ and $(t_i,t_{i+1}]$, $i=1,\dots,k-1$. □

Theorem 4.7. *Let Z be an $\mathcal{H}^\infty$-semimartingale, $Z_0=0$, decomposed into a sum $Z=N+A$, where N is a local martingale and A is a deterministic FV-process. Let $F:[0,T]\times I\!R^n \to cl\ conv(I\!R^n)$ satisfies the Assumption 4.5. Then for any $\xi \in L^2(\Omega,\mathcal{F}_0,P;I\!R^n)$ the set $\mathcal{T}(\xi,Z,F)$ is closed in $S^2([0,T])$.*

Proof. Let $\{x^k\}_{k\geq 1}$ be a sequence of elements of the set of all solutions of the stochastic inclusion (SI), which converges to the limit x in $S^2([0,T])$.

We have to show that the limit x belongs to the set $\mathcal{T}(\xi,Z,F)$, i.e.: for any $s,t\in[0,T]$, $s<t$

$$\operatorname{dist}_{L^2(\Omega)}(x_t-x_s,\int_s^t F(\tau,x_{\tau-})dZ_\tau)=0.$$

Observe that

$$\begin{aligned} I &= \operatorname{dist}_{L^2(\Omega)}(x_t-x_s,\int_s^t F(\tau,x_{\tau-})dZ_\tau)\\ &\leq \|x_t-x_s-(x_t^k-x_s^k)\|_{L^2(\Omega)}\\ &\quad + H_{L^2(\Omega)}(\int_s^t F(\tau,x^k_{\tau-})dZ_\tau,\int_s^t F(\tau,x_{\tau-})dZ_\tau)=I_1+I_2,\end{aligned}$$

and

$$I_1=\|x_t-x_s-(x_t^k-x_s^k)\|_{L^2(\Omega)}\leq 2\|x-x^k\|_{S^2}\to 0,$$

while $k\to\infty$.

In order to analyze I_2, let g be an arbitrary element of $\mathcal{S}_{\mu_Z}(F\circ x^k_-)$. Using Lemma 3.1 and [8] Th. 2.2, we get

$$\begin{aligned} \inf_{f\in\mathcal{S}_{\mu_Z}(F\circ x_-)}\|\int_s^t g_\tau dZ_\tau-\int_s^t f_\tau dZ_\tau\|^2_{L^2(\Omega)} &\leq 2\inf_{f\in\mathcal{S}_{\mu_Z}(F\circ x_-)}\int_{(s,t]\times\Omega}|g_\tau-f_\tau|^2 d\mu_Z\\ = 2\int_{(s,t]\times\Omega}\inf_{y\in F(\tau,x_{\tau-})}|g_\tau-y|^2 d\mu_Z &= 2\int_{(s,t]\times\Omega}\operatorname{dist}^2(g_\tau,F(\tau,x_{\tau-}))d\mu_Z.\end{aligned}$$

Observe that we get similar result for $\inf_{f\in\mathcal{S}_{\mu_Z}(F\circ x^k_-)}\|\int_s^t g_\tau dZ_\tau-\int_s^t f_\tau dZ_\tau\|^2_{L^2(\Omega)}$, when $g\in\mathcal{S}_{\mu_Z}(F\circ x_-)$. Moreover,

$$\begin{aligned} \sup_{g\in\mathcal{S}_{\mu_Z}(F\circ x^k_-)}\inf_{f\in\mathcal{S}_{\mu_Z}(F\circ x_-)}&\|\int_s^t g_\tau dZ_\tau-\int_s^t f_\tau dZ_\tau\|^2_{L^2(\Omega)}\\ &\leq 2\int_{(s,t]\times\Omega}\overline{h}^2(F(\tau,x^k_{\tau-}),F(\tau,x_{\tau-}))d\mu_Z\end{aligned}$$

and similar for $\sup_{f\in\mathcal{S}_{\mu_Z}(F\circ x_-)}\inf_{g\in\mathcal{S}_{\mu_Z}(F\circ x^k_-)}\|\int_s^t g_\tau dZ_\tau - \int_s^t f_\tau dZ_\tau\|^2_{L^2(\Omega)}$. Finally, we get

$$(I_2)^2 \le 2\int_{(s,t]\times\Omega} H^2(F(\tau, x^k_{\tau-}), F(\tau, x_{\tau-}))d\mu_Z. \tag{4.3}$$

Using the Lipschitz condition for the set-valued process F, we get

$$\begin{aligned}(I_2)^2 &\le 2\cdot D^2\int_{(s,t]\times\Omega}|x^k_{\tau-}-x_{\tau-}|^2 d\mu_Z\\ &= 2\cdot D^2\cdot\left(E\int_s^t|x^k_{\tau-}-x_{\tau-}|^2 d[N,N]_\tau + E\left(c_A\cdot\int_s^t|x^k_{\tau-}-x_{\tau-}|^2|dA_\tau|\right)\right).\end{aligned}$$

Since A is a deterministic FV-process, we get

$$\begin{aligned}(I_2)^2 \le 2\cdot D^2\cdot\max\{1,c_A\}\cdot(E&\int_s^t|x^k_{\tau-}-x_{\tau-}|^2 d[N,N]_\tau\\ &+E\int_s^t|x^k_{\tau-}-x_{\tau-}|^2|dA_\tau|).\end{aligned}$$

Using Emery's inequality, we get

$$\begin{aligned}(I_2)^2 &\le 2\cdot D^2\cdot\max\{1,c_A\}\cdot\||x^k-x|^2\|_{S^1}\cdot(\|[N,N]\|_{\mathcal{H}^\infty}+\||dA|\|_{\mathcal{H}^\infty})\\ &= 2\cdot D^2\cdot\max\{1,c_A\}\cdot\|x^k-x\|^2_{S^2}\cdot(\|[N,N]\|_{\mathcal{H}^\infty}+\||dA|\|_{\mathcal{H}^\infty}),\end{aligned}$$

which tends to 0 while $k\to\infty$, so we have the result. □

Theorem 4.7 can be applied only to a stochastic inclusion driven by the semimartingale with a deterministic FV-part. From a mathematical point of view this is a serious restriction. However, real problems are often described by stochastic inclusions of this type.

Example 4.8 ([16])**.** Suppose we have a model of a free-arbitrage market defined on a filtered probability space. The capital of an investor (a writer of a contingent claim) is defined under a self-financing assumption by a relation

$$\xi_t(\omega,u) = \xi_0(\omega,u) + \int_0^t \theta_\tau(\omega,u)dB_\tau(\omega,u) + \int_0^t \gamma_\tau(\omega,u)dS_\tau(\omega,u),\ \ t\in[0,T],$$

where (θ,γ) is an investor's strategy (hedge) process, while B and S are price processes of a bond (an asset with a predictable price) and stock, respectively (see e.g.: [6] for details), u denotes a control parameter taken from a given set U of attainable controls. If the model is based on daily returns of a stock, statistical tests reject hypotheses about normality distribution made in the model of the Black and Scholes type, (one of the most commonly used Gaussian model in financial mathematics). It follows that real prices are usually better characterized by the so-called heavy tailed distributions, skewness property, effects of clusters and so on. Moreover, an empirical study of the German stock price data shows that paths should be modeled by a discontinuous process instead of a continuous one.

Generalizations of the Gaussian model were proposed in many different manners. It was allowed in [1], that the price process has jumps and the resulting equation has the form (in a one dimensional case)

$$\xi_t(\omega, u) = \mu_t \xi_{t-}(\omega, u)dt + \sigma_t \xi_{t-}(\omega, u)dW_t(\omega, u) + \beta \xi_{t-}(\omega, u)dN_t(\omega, u),$$

where N is a point process counting the number of jumps of size β which the relative price $\xi_t(\omega, u)/\xi_{t-}(\omega, u)$ had before time t and W is a standard Wiener process.

Since $(N_t)_{t\geq 0}$ can be treated as a Poisson process with some intensity λ (see e.g.: [17]), then the above problem can be again rewritten equivalently as

$$\xi_t = \xi_0 + \int_0^t f_\tau dZ_\tau; \;\; t \in [0, T],$$

or

$$\xi_t \in \xi_0 + \int_0^t F_\tau dZ_\tau; \;\; t \in [0, T], \tag{4.4}$$

with $f_\tau(\omega, u) = (\mu_\tau \xi_{\tau-}, \sigma_\tau \xi_{\tau-}, \beta, \beta)$, $Z_\tau = (0, W_\tau, N_\tau - \lambda\tau, 0) + (\tau, 0, 0, \lambda\tau) = M_\tau + A_\tau$ and

$$F(\tau, \omega) = \bigcup_{u \in U} f(\tau, \omega, u).$$

The set-valued integral $\int F_\tau dZ_\tau$ driven by a semimartingale appears, instead of a single-valued $\int f_\tau dZ_\tau$, in a natural way, if we consider the control of financial problems connected with the models as those presented above. We obtain the stochastic inclusion (4.4), which describes the discussed financial problem. We can analyze it with respect to the whole set U of attainable controls.

Now we put an another assumption for the multifunction F. It is more general then Assumption 4.5 but it does not ensure the existence of a solution of the stochastic inclusion (SI).

Assumption 4.9. Let $F : [0, T] \times I\!\!R^n \to cl\, conv(I\!\!R^n)$ be a multifunction satisfying

- $F : [0, T] \times I\!\!R^n \to cl\, conv(I\!\!R^n)$ is a Carathéodory-type multifunction,
- for any $x \in S^2([0, T])$ a set-valued process $F \circ x_-$ is μ-integrably bounded.

When we assume that the solution set $\mathcal{T}(\xi, Z, F)$ is non-empty and the multifunction F satisfies the Assumption 4.9 we get also closedness of the set $\mathcal{T}(\xi, Z, F)$ in $S^2([0, T])$. It is true for any $\mathcal{H}^2$-semimartingale Z which satisfies Assumption 4.10 below.

Assumption 4.10. We assume that Z is an $\mathcal{H}^2$-semimartingale such that the measure μ_Z is absolutely continuous with respect to $\lambda \otimes P$ on $\mathcal{P}$, (λ means the Lebesgue measure on $[0, T]$), i.e. $\mu_Z \ll \lambda \otimes P$.

Theorem 4.11. *Let Z be an $\mathcal{H}^2$-semimartingale, $Z_0 = 0$ and satisfies the Assumption* 4.10. *Let $F : [0,T] \times I\!R^n \to cl\ conv(I\!R^n)$ satisfies the Assumption* 4.9. *Then for any $\xi \in L^2(\Omega, \mathcal{F}_0, P; I\!R^n)$ the set $\mathcal{T}(\xi, Z, F)$ is closed in $S^2([0,T])$.*

Proof. The proof is similar to the proof of Theorem 4.7 until the inequality (4.3)

$$(I_2)^2 \leq 2 \int_{(s,t]\times\Omega} H^2(F(\tau, x^k_{\tau-}), F(\tau, x_{\tau-}))d\mu_Z$$

Now, we show that $H(F(t, x^k_{t-}), F(t, x_{t-})) \to 0$ μ_Z-a.e. Since $\|x^k - x\|_{S^2} \to 0$ when $k \to \infty$ then for any $t \in [0,T]$, $\|x^k_t - x_t\|_{L^2(\Omega)} \to 0$ when $k \to \infty$. We also get

$$\begin{aligned} &\|x^k - x\|^2_{L^2([0,T]\times\Omega; I\!R^n)} \\ &= \int_\Omega \int_0^T |x^k_\tau - x_\tau|^2 d\tau P(d\omega) = \int_0^T \int_\Omega |x^k_\tau - x_\tau|^2 P(d\omega) d\tau \\ &= \int_0^T \|x^k_\tau - x_\tau\|^2_{L^2(\Omega)} d\tau \to 0 \quad d\tau \otimes dP\text{-a.e.} \end{aligned}$$

By the Assumption 4.10 we obtain $\|x^k - x\| \to 0$ μ_Z-a.e. Since x^k and x are cádlág processes we have the same reasoning as above for the processes x^k_- and x_-. Moreover, $\|x^k_- - x_-\|_{S^2} \leq \|x^k - x\|_{S^2}$.

By the Assumption 4.10 $F(t, \cdot)$ is continuous for any $t \in [0,T]$. So we get that $\|x^k_- - x_-\| \to 0$ μ_Z-a.e. implies $H(F(t, x^n_{t-}), F(t, x_{t-})) \to 0$ μ_Z-a.e. and we have the result. □

ACKNOWLEDGEMENTS

The author is grateful to prof. M. Michta for his valuable comments and suggestions improving the paper.

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