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# LINEAR-EXPONENTIAL-QUADRATIC GAUSSIAN CONTROL FOR STOCHASTIC EQUATIONS IN A HILBERT SPACE

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**ABSTRACT.** In this paper a control problem for a controlled linear stochastic equation in a Hilbert space and an exponential quadratic cost functional of the state and the control is formulated and solved. The stochastic equation can model a variety of stochastic partial differential equations with the control restricted to the boundary or to discrete points in the domain. The solution method does not require solving a Hamilton-Jacobi-Bellman equation and the method provides an explanation for an additional term in the Riccati equation as compared to the Riccati equation for a control problem with the corresponding quadratic cost functional. The optimal cost is also given explicitly. Some examples of controlled stochastic partial differential equations are given.

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## 1. INTRODUCTION

An important generalization of the linear-quadratic Gaussian (LQG) control problem is the linear-exponential-quadratic Gaussian (LEQG) control problem particularly for its application in risk sensitive control and its relation to differential games. An LEQG problem is similar to an LQG control problem except that the cost is an exponential of a quadratic functional of the state and the control. The LEQG problem for finite dimensional linear systems is solved in [15] by determining a solution to the Hamilton-Jacobi-Bellman (HJB) equation associated with this stochastic control problem. A different approach to the solution of this finite dimensional problem is given in [6] where a combination of the methods of completion of squares and absolute continuity of measures is used for the solution. This latter approach provides an explanation for the additional term of the Riccati equation for the LEQG problem as compared with the Riccati equation for the associated LQG problem and this approach is more elementary and direct than solving the HJB equation for the LEQG problem.

A natural generalization of this LEGQ control problem for systems in finite dimensional spaces is to linear stochastic equations in an infinite dimensional Hilbert

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space that can model various types of controlled linear stochastic partial differential equations. In this paper such a problem is formulated and solved. A semigroup approach is used where the semigroups are analytic (e.g. [19]). The control is restricted to discrete points in the domain or to the boundary of the domain to describe a typical controlled physical system and is the primary reason for restriction to analytic semigroups. Thus in addition to the infinitesimal generator acting on the state, the linear transformation acting on the control is also an unbounded operator so that properties of the solution of the Riccati equation require more refinement than for distributed control to ensure that the optimal control in the system equation is well defined.

## 2. PRELIMINARIES

The controlled linear stochastic system is described by the following stochastic differential equation

$$\begin{aligned} dX(t) &= AX(t)dt + BU(t)dt + \Phi dW(t) \\ X(0) &= X_0 \end{aligned} \tag{2.1}$$

where $X(t) \in H$ for $t \in [0,T]$, $X_0 \in H$, $H$ is a real, separable, infinite dimensional Hilbert space, and $(W(t), t \in [0,T])$ is a standard cylindrical Wiener process in $H$. The probability space is denoted $(\Omega, \mathcal{F}, \mathbb{P})$ where $\mathbb{P}$ is induced from the standard cylindrical measure for the Wiener process and $\mathcal{F}$ is the $\mathbb{P}$-completion of the Borel $\sigma$-algebra on $\Omega$. Let $(\mathcal{F}(t), t \in [0,T])$ be an increasing $\mathbb{P}$-complete family of sub-$\sigma$-algebras of $\mathcal{F}$ such that $X(t)$ is $\mathcal{F}(t)$ measurable for each $t \in [0,T]$ and $(< l, W(t) >, \mathcal{F}(t), t \in [0,T])$ is a real-valued Brownian martingale with local variance $|l|_H^2$ for each nonzero $l \in H$. The linear operator $A$ is the infinitesimal generator of an analytic semigroup on $H$ (e.g. [19]). Thus for some $\beta > 0$ the operator $-A + \beta I$ is strictly positive so that the fractional powers $(-A+\beta I)^\gamma$ and $(-A^* + \beta I)^\gamma$ and the spaces $D_A^\gamma = D((-A+\beta I)^\gamma)$ and $D_{A^*}^\gamma = D((-A^*+\beta I)^\gamma)$ with the graph norm topology for $\gamma \in \mathbb{R}$ can be defined. The linear space $D(\cdot)$ denotes the domain of $\cdot$. It is assumed that $B \in \mathcal{L}(H_1, D_A^{\epsilon-1})$ where $H_1$ is a real, separable Hilbert space and $\epsilon \in (0,1)$. The linear operator $\Phi$ is assumed to be Hilbert-Schmidt. It is assumed that for each $x \in H$ there is a $u_x \in L^2([0,T], H_1)$ such that

$$y(\cdot) = S(\cdot)x + \int_0^{\cdot} S(\cdot - r)Bu_x(r)dr \in L^2([0,T], H) \tag{2.2}$$

The cost functional $J$ is an exponential of a quadratic functional of $X$ and $U$ that is given by

$$J(U) = \mathbb{E}\exp\Big[\frac{\mu}{2}\int_0^T \langle QX(s), X(s)\rangle + \langle RU(s), U(s)\rangle ds$$

$$+\frac{\mu}{2}\langle MX(T),X(T)\rangle\Big] \tag{2.3}$$

where $T>0$ is fixed, $\mu>0$ is fixed, and $Q$ and $R$ are strictly positive, self-adjoint operators on $H$ and $H_1$ respectively.

The Riccati equation to solve the LQG problem with the linear stochastic system (2.1) and the quadratic cost that appears in the exponential function (2.3) is the following formal equation

$$-\frac{dP}{dt} = A^*P+PA-PBR^{-1}B^*P+Q \tag{2.4}$$

$$P(T) = M \tag{2.5}$$

The equation (2.4) can be modified to a mathematically meaningful inner product equation as

$$\begin{aligned}-\frac{d}{dt}\langle Px,y\rangle &= \langle Ax,Py\rangle+\langle Px,Ay\rangle-\langle R^{-1}B^*Px,B^*y\rangle\\ &+ \langle Qx,y\rangle\end{aligned} \tag{2.6}$$

for $x,y\in D(A)$. It is known that there is a unique, nonnegative self-adjoint solution of (2.6) (cf. [5], [12], [13], [17]).

The family of admissible controls, $\mathcal{U}$, is

$$\mathcal{U}=\{U:[0,T]\times\Omega\to H_1|U \text{ is adapted to } (\mathcal{F}(t),t\in[0,T]) \text{ and } \int_0^T|U(t)|^p dt<\infty \text{ a.s.}\}$$

where $p>\max\{2,1/\epsilon\}$ is fixed.

## 3. MAIN RESULT

In this section an optimal control is explicitly given for the control problem for the linear system (2.1) and the cost (2.3). The authors are not aware of any previous results for an explicit optimal control for an exponential quadratic cost with a linear stochastic system with boundary or point control in a general Hilbert space.

**Theorem 3.1.** *The optimal control problem given by* (2.1) *and* (2.3) *has an optimal control,* $(U^*(t),t\in[0,T])$*, in* $\mathcal{U}$ *that is given by*

$$U^*(t)=-R^{-1}B^*P(t)X(t) \tag{3.1}$$

*where* $(P(t),t\in[0,T])$ *is assumed to be the unique, symmetric, positive* $\mathcal{L}(H,D_{A^*}^{1-\epsilon})$*-valued solution of the following Riccati equation*

$$\begin{aligned}-\frac{d}{dt}\langle Px,y\rangle &= \langle Ax,Py\rangle+\langle Px,Ay\rangle-\langle R^{-1}B^*Px,B^*Py\rangle\\ &- \mu\langle\Phi^*Px,\Phi^*Py\rangle+\langle Qx,y\rangle\\ \langle P(T)x,y\rangle &= \langle Mx,y\rangle\end{aligned} \tag{3.2}$$

*for $x, y \in D(A)$ and the optimal cost is*

$$J(U^*) = G(0)\exp\left[\frac{\mu}{2}\langle P(0)X_0, X_0\rangle\right] \tag{3.3}$$

*and $(G(t), t \in [0,T])$ satisfies*

$$\begin{aligned} -\frac{dG}{dt} &= \left(\frac{\mu}{2}tr(P\Phi\Phi^*)\right)\ G \\ G(T) &= 1 \end{aligned} \tag{3.4}$$

*Proof.* To determine an optimal control a combination of a completion of squares and a Radon-Nikodym derivative for Wiener measure is used. Using the Riccati equation (3.2) and an Itô formula (Lemma 3.3 [9]) it follows that

$$\begin{aligned} &\frac{1}{2}\langle P(T)X(T), X(T)\rangle - \frac{1}{2}\langle P(0)X_0, X_0\rangle \\ = &\ \frac{1}{2}\int_0^T (\langle (P(t)BR^{-1}B^*P(t) - Q)X(t), X(t)\rangle \\ + &\ 2\langle B^*P(t)X(t), U(t)\rangle)dt + \int_0^T \langle P(t)\Phi dW(t), X(t)\rangle \\ - &\ \frac{\mu}{2}\int_0^T (\langle \Phi^*P(t)X(t), \Phi^*P(t)X(t)\rangle + \frac{1}{2}tr(P(t)\Phi\Phi^*))dt \end{aligned} \tag{3.5}$$

Let $L$ be the quadratic functional given by

$$L(U) = \frac{\mu}{2}(\int_0^T (\langle QX, X\rangle + \langle RU, U\rangle)dt + \langle MX(T), X(T)\rangle) \tag{3.6}$$

so that the cost functional, $J$, can be expressed as

$$J(U) = \mathbb{E}\exp[L(U)] \tag{3.7}$$

where $\mathbb{E}$ is expectation for the measure $\mathbb{P}$. Using (3.5) it follows that

$$\begin{aligned} &L(U) - \frac{\mu}{2}\langle P(0)X_0, X_0\rangle \\ &= \frac{\mu}{2}[\int_0^T (\langle RU, U\rangle + \langle R^{-1}B^*PX, B^*PX\rangle + 2\langle B^*PX, U\rangle)dt \\ &+2\int_0^T \langle PX, \Phi dW\rangle - \mu\int_0^T \langle \Phi^*PX, \Phi^*PX\rangle dt + \int_0^T tr(P\Phi\Phi^*)dt] \\ &= \frac{\mu}{2}\int_0^T |R^{-\frac{1}{2}}[RU + B^*PX]|^2 dt \\ &+\mu\int_0^T \langle PX, \Phi dW\rangle - \frac{\mu^2}{2}\int_0^T \langle \Phi^*PX, \Phi^*PX\rangle dt \\ &+\frac{\mu}{2}\int_0^T tr(P\Phi\Phi^*)dt \end{aligned} \tag{3.8}$$

Thus

$$\mathbb{E}\exp[L(U)] \tag{3.9}$$

$$= \widetilde{\mathbb{E}}\exp[\frac{\mu}{2}\langle P(0)X_0, X_0\rangle + \frac{\mu}{2}\int_0^T (|R^{-\frac{1}{2}}[RU + B^*PX]|^2 dt$$

$$+\frac{\mu}{2}\int_0^T tr(P\Phi\Phi^*)dt]$$

where $\widetilde{\mathbb{E}}$ is the (local) expectation with respect to $\widetilde{\mathbb{P}}$ given by

$$d\widetilde{\mathbb{P}} = \exp[\mu\int_0^T \langle PX, \Phi dW\rangle - \frac{\mu^2}{2}\int_0^T \langle P\Phi\Phi^* X, X\rangle dt]d\mathbb{P} \tag{3.10}$$

For an arbitrary admissible control $U$, the exponential functional in (3.10) may not be a Radon-Nikodym derivative on $[0,T]$, but it is a local martingale which suffices to determine the optimal control. While the Radon-Nikodym derivative depends on the solution $(X(t),\ t \in [0,T])$, the minimization determines a control $(U^*(t),\ t \in [0,T])$ that is the absolute minimum. The equation (3.9) is minimized by choosing the (optimal) control

$$U^*(t) = -R^{-1}B^*P(t)X(t) \tag{3.11}$$

For the optimal control the exponential in (3.10) is a Radon-Nikodym derivative from the absolute continuity results for Gaussian measures. The other terms in (3.9) give the optimal cost (3.3). $\square$

The difference between the Riccati equation (2.6) for the LQG problem and the Riccati equation (3.2) for the LEQG problem is the term $-\mu\langle\Phi^* Px, \Phi^* Py\rangle$ that arises from the quadratic term in the exponential function for the Radon-Nikodym derivative that transforms the Wiener measure for $(\Phi W(t),\ t \in [0,T])$ by adding a drift term that appears in a stochastic integral. For the completion of squares for the LQG problem the stochastic integral term has expectation zero, so it disappears with the operation of expectation. For the completion of squares for the LEQG problem there is an exponential of the stochastic integral term so it does not have expectation zero. The Radon-Nikodym derivative (exponential martingale) is the natural way to eliminate this exponential of a stochastic integral.

## 4. **SOME EXAMPLES**

Some examples are given that indicate the range of applicability of the optimal control result. Initially some other work is noted that is related to the equations in the examples. In [1] a general stochastic initial and boundary value problem for second order evolution equations is formulated and solved. Existence, uniqueness and regularity results are verified. In Chapter 3 of [2] abstract boundary value problems in a Banach space are formulated and solved which allow for noise and inputs to be distributed or on the boundary. This abstract setting can include the equations in the following examples. However specific regularity conditions must be verified for an application to the optimal control problem in the theorem here. Typically this verification requires some special features of a particular equation. The specific regularity conditions to satisfy the conditions of the theorem here are discussed in the examples.

**Example 1.** This is a family of examples from elliptic differential operators that is discussed in more detail in [9]. Let $G$ be a bounded, open domain in $\mathbb{R}^n$ with $C^\infty$-boundary $\partial G$ with $G$ locally on one side of $\partial G$ and let $L(x,D)$ be an elliptic differential operator of the form

$$L(x,D)f = \Sigma_{i,j=1}^n D_i a_{ij}(x) D_j f + \Sigma_{i=1}^n [b_i(x) D_i f + D_i(d_i(x) f)] + c(x) f$$

where the coefficients $a_{ij}, b_i, d_i, c$ are elements of $C^\infty(G)$

$$\Sigma a_{ij}(x)\xi_i\xi_j \geq \hat{\nu}|\xi|^2 \tag{4.1}$$

where $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$, $x \in G$, $\hat{\nu} > 0$ is a constant, and $\{a_{ij}\}$ is symmetric. Consider a stochastic parabolic control problem formally described by the equations

$$\frac{\partial y}{\partial t} = L(x,D)y(t) + \eta(t,x) \tag{4.2}$$

for $(t,x) \in \mathbb{R}_+ \times G$ and

$$\frac{\partial y}{\partial \nu} + h(x)y(t,x) = u(t,x) \tag{4.3}$$

for $(t,x) \in \mathbb{R}_+ \times \partial G$ and $y(0,x) = y_0(x)$ where $\frac{\partial}{\partial \nu} = \Sigma_{i,j=1}^n a_{ij}\nu_j D_i$ is the outward normal derivative, $\nu = (\nu_1, \ldots, \nu_n)$ is the unit outward normal to $\partial G$, the process $(\eta(t,x), (t,x) \in \mathbb{R}_+ \times G)$ formally denotes a space dependent white noise, $u \in L^2([0,T], L^2(\partial G))$, $h \in C^\infty(\partial G)$, and $h \geq 0$.

To give a mathematical description to (4.2) and (4.3), a semigroup approach (e.g. [19]) is used. Let $H = L^2(G)$, $H_1 = L^2(\partial G)$ and define the infinitesimal generator as $Af = L(x,D)f$ so that $A : D(A) \to H$ and $D(A) = \{f \in H^2(G) : \frac{\partial f}{\partial \nu} = 0 \text{ on } \partial G\}$. It is well known that A generates an analytic semigroup (e.g. [19]) and the linear operator $(A - \beta I)$ is strictly negative for some $\beta \geq 0$.

To define the control operator in the stochastic equation, consider the elliptic problem

$$(L(x,D) - \beta)z = 0 \qquad \text{on } G \tag{4.4}$$

$$\frac{\partial z}{\partial \nu} + hz = -g \qquad \text{on } \partial G \tag{4.5}$$

For $g \in L^2(\partial G)$, there is a unique solution $z \in H^{\frac{3}{2}}(G)$ [18]. Define $\hat{B} \in \mathcal{L}(H_1, H^{\frac{3}{2}}(G))$ by the equation, $\hat{B}g = -z$. For $\epsilon < \frac{3}{4}$, $\hat{B} \in \mathcal{L}(H_1, D_A^\epsilon)$ because $D_A^{\frac{3}{4}-\gamma} = H^{\frac{3}{2}-2\gamma}(G)$ for a sufficiently small $\gamma > 0$ [14]. Let $y_\beta(t,x) = e^{-\beta t} y(t,x)$ and $\eta(t,x)dt = \Phi dW(t)$ for some $\Phi \in \mathcal{L}(H)$ and a standard cylindrical Wiener process $(W(t), t \in [0,T])$ in $H$. From (4.4), (4.5) it follows that

$$dy_\beta = (L(x,D) - \beta)y_\beta dt + e^{-\beta t}\Phi dW(t) \tag{4.6}$$

$$\frac{\partial y_\beta}{\partial \nu} + hy_\beta = e^{-\beta t}u = u_\beta(t) \qquad \text{on } \partial G \tag{4.7}$$

$$y_\beta(0) = y(0) \tag{4.8}$$

Formally performing the differentiation $(\frac{\partial}{\partial t})\hat{B}u_\beta(t)$, it follows that

$$d\omega_\beta(t) = ((L(x,D) - \beta)y_\beta(t) - \hat{B}v_\beta(t))dt + e^{-\beta t}\Phi dW(t) \tag{4.9}$$

$$\frac{\partial \omega_\beta}{\partial \nu} + h\omega_\beta = 0 \qquad \text{on } \mathbb{R}_+ \times \partial G \tag{4.10}$$

where $v_\beta$ is the formal time derivative of $u_\beta$ and $\omega_\beta(t) = y_\beta(t) - \hat{B}u_\beta(t)$. For (4.6) the mild solution is

$$\begin{aligned}\omega_\beta(t) = S_\beta(t)(y(0) + \hat{B}u(0)) + \int_0^t S_\beta(t-r)\Phi e^{\beta r}dW(r) \\ - \int_0^t S_\beta(t-r)\hat{B}v_\beta(r)dr\end{aligned} \tag{4.11}$$

where $S_\beta(t) = e^{t(A-\beta I)}$. Formally integrating by parts in the Lebesgue integral in (4.11) and canceling the common term $e^{-\beta t}$ the equality that results is

$$y(t) = S(t)y(0) + \int_0^t S(t-r)Bu(r)dr + \int_0^t S(t-r)\Phi dW(r) \tag{4.12}$$

which is a mild solution to a stochastic equation of the form (2.1) where $B = \Psi^*$ and $\Psi^* \in \mathcal{L}(D_{A^*}^{1-\epsilon}, H_1)$ extends the linear operator $\hat{B}^*(A^* - \beta I)$.

**Example 2.** For this family of examples the stochastic equation (2.1) is modified as follows.

$$dX(t) = (A_0 + A_1 + A_0BC)X(t)dt + A_0BDU(t)dt + \Phi dW(t) \tag{4.13}$$

$$X(0) = x \tag{4.14}$$

where $A_0$ is the infinitesimal generator of analytic semigroup $(S_0(t), t \in [0,T]), A_0 = A_0^*, B \in \mathcal{L}(H_1, D_A^\epsilon)$ for some $\epsilon \in (0,1), A_1^* \in \mathcal{L}(D_A^\eta, H)$ for some $\eta \in [0,1), C \in \mathcal{L}(H, H_1)$ and $D \in \mathcal{L}(H_2, H_1)$ where $H_1$ and $H_2$ are separable Hilbert spaces.

Let $A(x,D)$ be a 2m-order differential operator of the form

$$A(x,D)y = \Sigma_{|p|,|q| \le m}(-1)^{|p|}D^p(a_{pq}(x)D^q y) \tag{4.15}$$

where $x \in \mathcal{O}$ and $\mathcal{O}$ is a bounded domain in $\mathbb{R}^n$ whose boundary $\partial\mathcal{O}$ is infinitely smooth with $x \in \mathcal{O}$ on one side of the boundary. The coefficients $a_{pq}(\cdot)$ are in $C^\infty(\bar{\mathcal{O}})$ for all values of the multi-indices $p, q$ with the lengths $|p| \le m, |q| \le m$. Let

$$\bar{A}(x,D) = \Sigma_{|p|=m,|q|=m}(-1)^m D^p(a_{pq}(x)D^q y) \tag{4.16}$$

for $x \in \mathcal{O}$. Assume that $\bar{A}(x, D)$ is uniformly elliptic, that is,

$$|\Sigma_{|p|=|q|=m}(-1)^m a_{pq}(x)\lambda^{p+q}| \geq \bar{\nu}|\lambda|^{2m} \tag{4.17}$$

for some $\bar{\nu} > 0$ where $\lambda^{p+q} = \lambda_1^{p_1+q_1} \cdots \lambda_n^{p_n+q_n}$. Furthermore, let $\widetilde{\mathcal{B}} = (\widetilde{\mathcal{B}}_0, \ldots, \widetilde{\mathcal{B}}_{n-1})$ be a system of boundary operators.

$$\widetilde{\mathcal{B}}_j\phi = \Sigma_{|h|\leq m_j} b_{jh}(x)D^h\phi \qquad x \in \partial\mathcal{O} \tag{4.18}$$

where $j = 0, 1, \ldots, m-1$, $0 \leq m_0 < m_1 < \cdots < m_{m-1} \leq 2m-1$, $b_{jh}$, $\phi \in C^\infty(\partial\mathcal{O})$. Assume that $a_{pq} = a_{qp}$ for $|p| = |q| = m$, so the system $(\bar{A}(x, D), \mathcal{B}_j, j = 0, 1, \ldots, m-1)$ is formally self-adjoint, the system $(\mathcal{B}_j)$ is normal and covers $\bar{A}(x, D)$ and there is a Green function for the problem $\frac{\partial y}{\partial t} = \bar{A}(x, D)y, \mathcal{B}y = 0$ ([3], [18]). For example Dirichlet or Neumann boundary conditions may be considered. With the above assumptions there is a $\beta > 0$ such that $-(\bar{A}(x, D) - \beta I)$ defined on $\{y \in C^\infty(\bar{\mathcal{B}}), \mathcal{B}y = 0\}$ extends to an operator $A_0, \mathcal{D}(A_0) = \{y \in H^{2m}(\mathcal{O}), \widetilde{\mathcal{B}}y = 0\}$ that is strictly negative and self-adjoint in $H = L^2(\mathcal{O})$. Consider the elliptic problem

$$\bar{A}(x, D)y + \beta y = 0 \qquad \text{on } \mathcal{O} \tag{4.19}$$

$$\widetilde{\mathcal{B}}y = g \qquad \text{on } \partial\mathcal{O} \tag{4.20}$$

Let $H_1 = H^{\sigma_0}(\partial\mathcal{O}) \times \cdots \times H^{\sigma_{m-1}}(\partial\mathcal{O})$ where $\sigma_j > -(m_j + \frac{1}{2})$ for $j = 0, \ldots, m-1$ and choose $0 < s < \min_j(\frac{1}{2}, \sigma_j + m_j + \frac{1}{2}), \epsilon = \frac{s}{2m}$ (cf. [18] for the spaces $H^\sigma(\partial\mathcal{O})$). It is known that $H^s(\mathcal{O}) = D_A^\epsilon$ [20]. Denote by $\mathcal{B} : g \mapsto -y$ the Green mapping for the elliptic problem (4.19). It is known that $\mathcal{B} \in \mathcal{L}(H_1, H^s(\mathcal{O}))$ [18].

Consider a stochastic parabolic problem given heuristically by the equation

$$\frac{\partial y}{\partial t} = -A(x, D)y + N(t, x) \quad (t, x) \in \mathbb{R}_+ \times \mathcal{O} \quad y(0, x) = y_0(x) \tag{4.21}$$

$$\widetilde{\mathcal{B}}y = g(t, x) \qquad (t, x) \in \mathbb{R}_+ \times \partial\mathcal{O} \tag{4.22}$$

where $g(t, \cdot) \in H_1$ and $N$ represents a space-dependent Gaussian white noise. Let $N(t, \cdot)dt = \Phi dW(t)$ where $W(t)$ is a standard cylindrical Wiener process in $H = L^2(\mathcal{O})$. Formally there is the equation

$$\begin{aligned}
\frac{\partial y(t, x)}{\partial t} &= (-\bar{A}(x, D) - \beta)y(t, x) + (\bar{A}(x, D) - A(x, D) + \beta)y(t, x) \\
&\quad + \Phi\frac{dW}{dt} \qquad (t, x) \in \mathbb{R}_+ \times \mathcal{O} \\
y(0, x) &= y_0(x) \qquad x \in \mathcal{O} \\
\widetilde{\mathcal{B}}y(t, x) &= g(t, x) \qquad (t, x) \in \mathbb{R}_+ \times \partial\mathcal{O}
\end{aligned}$$

Now let $v = y + Bg$ so that

$$\frac{\partial v}{\partial t} = A_0 v + A_1 y + \Phi\frac{dW}{dt} + B\frac{\partial g}{\partial t} \qquad \text{on } \mathbb{R}_+ \times \mathcal{O} \tag{4.23}$$

$$\widetilde{\mathcal{B}}v = 0 \qquad \text{on } \mathbb{R}_+ \times \partial\mathcal{O} \tag{4.24}$$

where $A_1$ is the closure of the operator $\bar{A}(x,D) - A(x,D) + \beta I$ and $A_1^* \in \mathcal{L}(D_A^\eta, H)$ for $\eta = (2m-1)/2m$. Formally integrating by parts in the mild solution of (4.23) the following equality is satisfied

$$\begin{aligned} y(t) &= S_0(t)y_0 + \int_0^t S_0(t-r)A_1 y(r)dr = \int_0^t A_0 S_0(t-r)Bg(r)dr \\ &+ \int_0^t S_0(t-r)\Phi dW(r) \end{aligned} \tag{4.25}$$

where $S_0$ is the strongly continuous semigroup on $H$ generated by $A_0$. Let $H_2$ be a Hilbert space and let $C \in \mathcal{L}(H, H_1)$ and $D \in \mathcal{L}(H_2, H_1)$. The function $g$ in (4.22) is assumed to have the form

$$g(t,x) = Cy(t) + Du(t) \tag{4.26}$$

where $u \in \mathcal{U}$. Thus the equation (4.25) is the mild solution of an equation of the form (4.13). Additional information about this example is given in [10].

**Example 3.** A third example is a structurally damped plate with random loading and point control (cf. [9] for more details). Consider the following model of a plate in the deflection $\omega$

$$\begin{aligned} \omega_{tt}(t,x) + \Delta^2\omega(t,x) - \alpha\Delta\omega(t,x) = \delta(x-x_0)u(t) + \eta(t,x) \\ \text{for } (t,x) \in \mathbb{R}_+ \times G \end{aligned} \tag{4.27}$$

$$\omega(0,\cdot) = \omega_0 \qquad \omega_t(0,\cdot) = \omega_1 \tag{4.28}$$

$$\omega|_{\mathbb{R}_+\times\partial G} = \Delta\omega|_{\mathbb{R}_+\times\partial G} = 0 \tag{4.29}$$

where $\alpha > 0$ is a constant, $\eta(t,x)$ formally represents a space-dependent Gaussian white noise on the open, bounded, smooth domain $G \subset \mathbb{R}^n$ for $n \leq 3$, and $\delta(x-x_0)$ is the Dirac distribution at $x_0 \in G$. The cost functional is

$$J(\omega_0, \omega_1, u, T) = \int_0^T (|\omega(t)|^2_{H^2(G)} + |\omega_t(t)|^2_{L^2(G)} + |u(t)|^2)dt \tag{4.30}$$

The deterministic version of this control problem, that is $\eta \equiv 0$, is given in [4], [16]. Define the linear operator $\mathcal{A}$ by the equation $\mathcal{A}h = \Delta^2 h$ where $D(\mathcal{A}) = \{h \in H^4(G) : h|_{\partial G} = \Delta h|_{\partial G} = 0\}$. The plate deflection equations are rewritten where $H = D(\mathcal{A}^{1/2}) \times L^2(G) = (H^2(G) \cap H_0^1(G)) \times L^2(G), H_1 = \mathbb{R}$, and

$$\begin{pmatrix} 0 & I \\ -\mathcal{A} & -\mathcal{A}^{1/2} \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ \delta(x-x_0)u \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & \Phi_1 \end{pmatrix}$$

These three arrays represent $\mathcal{A}, Bu, \Phi$ respectively.

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