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NONLINEAR RETARDED EVOLUTION EQUATIONS WITH NONLOCAL INITIAL CONDITIONS

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ABSTRACT. In this paper we prove some sufficient condition for the existence and global uniform asymptotic stability of C^0-solutions for a class of nonlinear retarded differential evolution equation of the form

$$\begin{cases} u'(t) \in Au(t) + f(t, u(t), u(t-\tau_1), \dots, u(t-\tau_n)), & t \in \mathbb{R}_+, \\ u(s) = g(u)(s), & s \in [-\tau, 0], \end{cases}$$

where X is a real Banach space, A is the generator of a nonlinear compact semigroup, $f : \mathbb{R}_+ \times [\overline{D(A)}]^{n+1} \to X$ is continuous and $g : C_b([-\tau, +\infty); \overline{D(A)}) \to C([-\tau, 0]; \overline{D(A)})$ is nonexpansive.

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1. INTRODUCTION

Let X be a real Banach space, $A : D(A) \subseteq X \rightsquigarrow X$ the generator of a nonlinear semigroup of nonexpansive mappings $\{S(t) : \overline{D(A)} \to \overline{D(A)};\ t \geq 0\}$. If $a \in \mathbb{R}$, $C_b([a, +\infty); X)$ denotes the space of all continuous and bounded functions from $[a, +\infty)$, endowed with the sup-norm $\|\cdot\|_{C_b([a,+\infty);X)}$. Further, $C_b([a, +\infty); \overline{D(A)})$ denotes the closed subset in $C_b([a, +\infty); X)$ consisting of all functions u with $u(t) \in \overline{D(A)}$ for each $t \in [a, +\infty)$. As usual, $C([a, b]; X)$ is the space of all continuous functions from $[a, b]$ to X endowed with the sup-norm $\|\cdot\|_{C([a,b];X}$ and $C([a, b]; \overline{D(A)})$ is the closed subset of $C([a, b]; X)$ containing all $u \in C([a, b]; X)$ with $u(t) \in \overline{D(A)}$ for each $t \in [a, b]$.

Let $0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \tau_n = \tau$. Let $f : \mathbb{R}_+ \times \left[\overline{D(A)}\right]^{n+1} \to X$ be continuous and $g : C_b([-\tau, +\infty); \overline{D(A)}) \to C([-\tau, 0]; \overline{D(A)})$ nonexpansive.

In the present paper we prove some existence and uniform asymptotic stability results for C^0-solution to the following class of nonlinear retarded differential evolution

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equation with nonlocal initial data

$$\begin{cases} u'(t) \in Au(t) + f(t, u(t), u(t-\tau_1), \ldots, u(t-\tau_n)), & t \in \mathbb{R}_+, \\ u(s) = g(u)(s), \quad s \in [\,-\tau, 0\,]. \end{cases} \tag{1.1}$$

Usually, in (1.1), A is a partial differential operator, linear or not, which drives the system in the absence of any external reaction, while the perturbed term f represents the instantaneous response or feedback, i.e. the manner in which the system reacts at each time t in order to self-correct its behavior according to both the instantaneous state, $u(t)$, and the previous states, $u(t-\tau_1), u(t-\tau_2), \ldots, u(t-\tau_n)$.

As function $g : C_b([\,-\tau, +\infty); \overline{D(A)}) \to C([\,-\tau, 0\,]; \overline{D(A)})$ we can take:

(*i*) $g(u)(s) = u(2\pi + s)$, $s \in [\,-\tau, 0\,]$ (2π-periodicity condition);
(*ii*) $g(u)(s) = -u(2\pi + s)$, $s \in [\,-\tau, 0\,]$ (2π-antiperiodicity condition);
(*iii*) $g(u)(s) = \displaystyle\int_\tau^{+\infty} k(\theta)u(s+\theta)\,d\theta$, $s \in [\,-\tau, 0\,]$, where the $k \in L^1([\,\tau, +\infty), ; \mathbb{R})$ and $\displaystyle\int_\tau^{+\infty} |k(\theta)|\,d\theta = 1$ (mean condition).

We notice that (*i*) $\sim$ (*iii*) correspond to particular instances of the general choice of g as

$$g(u)(s) = \int_\tau^{+\infty} \mathcal{N}(u(s+\theta))\,d\mu(\theta), \tag{1.2}$$

where $\mathcal{N} : X \to X$ is a (possible nonlinear) nonexpansive operator and μ is a σ-finite and complete measure on $[\,\tau, +\infty)$ with $\mu([\,\tau, +\infty)) = 1$ and $\lim_{\delta\downarrow 0} \mu([\,\tau, \delta + \tau\,]) = 0$. We emphasize that g given by (1.2) falls also into our general framework. Furthermore, the case in which $\mu : \Sigma \to \mathcal{L}(X)$, Σ being the σ-field of Lebesgue measurable subsets in $[\,\tau, +\infty)$, is an operator-valued measure with $\lim_{\delta\downarrow 0} \|\mu([\,\tau, \delta + \tau\,])\|_{\mathcal{L}(X)} = 0$, is also covered by our main results, Theorems 3.2$\sim$3.3.

Problems like (1.1) have been intensively studied in the last years. We begin by mentioning the paper of Yongxiang Li [25] which is the starting point of our analysis. More precisely, Yongxiang Li *loc. cit.* extends some previous results in Yongxiang Li [24], by proving some existence, uniqueness, global uniform asymptotic stability as well as regularity results for a particular problem of type (1.1), by assuming that A is the infinitesimal generator of an analytic compact semigroup in a Hilbert space H, while $g : C_b([\,-\tau, +\infty); X) \to C([\,-\tau, 0\,]; X)$ has the simple form $g(u)(s) = u(\omega + s)$ for some $\omega > 0$ and each $s \in [\,-\tau, 0\,]$, which corresponds to an ω-periodicity condition. For other existence results concerning periodic problems see S. Aizicovici, N. S. Papageorgiou, V. Staicu [3], R. Caşcaval, I. I. Vrabie [13], N. Hirano [18], N. Hirano, N. Shioji [19], A. Paicu [27], I. I. Vrabie [31] – for f single-valued and depending only on u–, and C. Castaing, D. P. Monteiro-Marques [14], V. Lakshmikantham, N.

S. Papageorgiou [23], A. Paicu [26], N. S. Papageorgiou [29], Shuchuan Hu, N. S. Papageorgiou [20], [21] – for f multi-valued and depending only on u. For anti-periodic problems, which are also covered by our main results, see S. Aizicovici, N. H.Ṗavel, I. I. Vrabie [4] and the references therein. As concern the case of differential equations or inclusions subjected to nonlocal initial data without delay, we mention here the pioneering work of L. Byszewski [12]. Some other results in this topic were obtained by S. Aizicovici, H. Lee [1], S. Aizicovici, M. McKibben [2], J. García-Falset [16] and J. García-Falset, S. Reich [17] – for f single-valued – and S. Aizicovici, V. Staicu [5] and A. Paicu, I. I. Vrabie [28] – for F multi-valued. All these studies are motivated by the fact that such kind of problems represent mathematical models for the evolution of various phenomena. For a model describing the gas flow through a thin transparent tube expressed as a problem with nonlocal initial conditions, see K. Deng [15].

The paper is divided into 6 sections. Section 2 contains some background material, intended to make the paper self-contained. In Section 3 we formulate the main results, i.e. Theorems 3.2 $\sim$ 3.4 which are extensions, to the fully nonlinear case and in a general Banach space frame, of the main results of Yongxiang Li [25]. Moreover, in the case of periodic conditions and under Lipschitz conditions on f, we show that there is no need to assume the compactness of the semigroup. Compare our Theorem 3.4 with Theorem 1.2 in Yongxiang Li *loc. cit.* It should be mentioned that if, in addition, we assume that X is Hilbert and A is a subdifferential, we can recover most of the regularity properties of the C^0-solution proved by Yongxiang Li *loc. cit.* via the analyticity of the generated semigroup. In Section 4 we include six preliminary lemmas, while in Section 5 we give the complete proofs of the main results. It should be noticed that the proofs are based on an interplay between compactness arguments and metric fixed point techniques used in both Paicu, Vrabie [28] and Yongxiang Li *loc. cit.* In the last Section 6 we analyze two illustrating examples referring to the porous medium equation.

2. **PRELIMINARIES**

We assume familiarity with the theory of m-dissipative operators and nonlinear evolution equations in Banach spaces, and we refer to Barbu [8], [9], Lakshmikantham–Leela [22] and Vrabie [31] for details. However, we recall for easy references some basic concepts and results we will use in the sequel.

Let X be a real Banach space with norm $\|\cdot\|$ and let $r > 0$. We denote by $B(0, r)$ the closed ball with center 0 and radius r. Let $x, y \in X$ and $h \in \mathbb{R} \setminus \{0\}$. We denote by

$$[x, y]_h := \frac{1}{h}(\|x + hy\| - \|x\|),$$

and we recall that there exist the limit

$$[x, y]_+ = \lim_{h \downarrow 0} [x, y]_h.$$

Remark 2.1. For each $x, y \in X$ and $\alpha > 0$, we have

(*i*) $[\alpha x, y]_+ = [x, y]_+$
(*ii*) $|[x, y]_+| \leq \|y\|$.

For other properties of the mapping $(x, y) \mapsto [x, y]_+$, see Lakshmikantham–Leela [22].

An operator $A : D(A) \subseteq X \rightsquigarrow X$ is called *dissipative* if for each $x_i \in D(A)$ and $y_i \in Ax_i$, $i = 1, 2$, we have

$$[x_1 - x_2, y_2 - y_1]_+ \geq 0.$$

It is called *m-dissipative* if it is dissipative, and, in addition, $R(I - \lambda A) = X$, for each $\lambda > 0$.

Let $f \in L^1(a, b; X)$ and let us consider the evolution equation

$$u'(t) \in Au(t) + f(t). \tag{2.1}$$

A function $u : [a, b] \to \overline{D(A)}$ is called *a C^0-solution*, or *integral solution* of (2.1) on $[a, b]$, if $u \in C([a, b]; X)$ and satisfies:

$$\|u(t) - x\| \leq \|u(s) - x\| + \int_s^t [u(\tau) - x, f(\tau) + y]_+ d\tau$$

for each $x \in D(A)$, $y \in Ax$ and $a \leq s \leq t \leq b$.

Remark 2.2. If $u : [a, b] \to \overline{D(A)}$ is a C^0-*solution* of (2.1) on $[a, b]$ then, in view of (*ii*) in Remark 2.1, it follows that

$$\|u(t) - x\| \leq \|u(s) - x\| + \int_s^t \|f(\theta) + y\| d\theta$$

for each $x \in D(A)$, $y \in Ax$ and $a \leq s \leq t \leq b$.

Theorem 2.3. *Let $\omega > 0$ and let $A : D(A) \subseteq X \rightsquigarrow X$ be an m-dissipative operator such that $A + \omega I$ is dissipative. Then, for each $\xi \in \overline{D(A)}$ and $f \in L^1(a, b; X)$, there exists a unique C^0-solution of (2.1) on $[a, b]$ which satisfies $u(a) = \xi$. If $f, g \in L^1(a, b; X)$ and u, v are two C^0-solutions of (2.1) corresponding to f and g respectively, then:*

$$\|u(t) - v(t)\| \leq e^{-\omega(t-s)} \|u(s) - v(s)\| + \int_s^t e^{-\omega(t-\theta)} \|f(\theta) - g(\theta)\| d\theta \tag{2.2}$$

for each $a \leq s \leq t \leq b$.

In particular, if $x \in D(A)$ and $y \in Ax$, we have

$$\|u(t) - x\| \leq e^{-\omega(t-s)} \|u(s) - x\| + \int_s^t e^{-\omega(t-\theta)} \|f(\theta) + y\| d\theta \tag{2.3}$$

for each $a \le s \le t \le b$.

See Barbu [9] Theorem 4.1, p. 128.

Let $\xi \in X$, $\tau \in [\,a,b)$ and $f \in L^1(a,b\,;X)$. We denote by $u(\cdot,\tau,\xi,f)$ the unique C^0-solution $v : [\,\tau,b\,] \to \overline{D(A)}$, of the problem (2.1) which satisfies $v(\tau) = \xi$. We denote by $\{S(t) : \overline{D(A)} \to \overline{D(A)},\ t \ge 0\}$ the semigroup generated by A on $\overline{D(A)}$, i.e., $S(t)\xi = u(t,0,\xi,0)$ for each $\xi \in X$ and $t \ge 0$. We say that the semigroup generated by A on $\overline{D(A)}$ is *compact* if, for each $t > 0$, $S(t)$ is a compact operator.

A subset $\mathcal{F}$ in $L^1(a,b\,;X)$ is called *uniformly integrable* if, for each $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that, for each measurable subset E in $[\,a,b\,]$ whose Lebesgue measure $\lambda(E) < \delta(\varepsilon)$, we have

$$\int_E \|f(s)\|\,ds \le \varepsilon,$$

uniformly for $f \in \mathcal{F}$.

Remark 2.4. Let $\mathcal{F} \subseteq L^1(a,b;X)$. It is easy to see that:

(i) if $\mathcal{F}$ is uniformly integrable then it is norm bounded in $L^1(a,b;X)$;
(ii) if $\mathcal{F}$ is bounded in $L^p(a,b\,;X)$ for some $p > 1$, then it is uniformly integrable;
(iii) if there exists $k \in L^1(a,b;\mathbb{R}_+)$ such that

$$\|f(t)\| \le k(t)$$

for each $f \in \mathcal{F}$ and a.e. $t \in (a,b)$, then $\mathcal{F}$ is uniformly integrable.

The following compactness result will be useful in that follows.

Theorem 2.5. *Let* $A : D(A) \subseteq X \rightsquigarrow X$ *be m-dissipative and such that* A *generates a compact semigroup. Let* $B \subseteq \overline{D(A)}$ *be bounded and let* $\mathcal{F}$ *be uniformly integrable in* $L^1(a,b;X)$*. Then, for each* $c \in (a,b)$*, the* C^0*-solutions set*

$$\{u(\cdot,a,\xi,f)\,;\ \xi \in B,\ f \in \mathcal{F}\}$$

is relatively compact in $C([\,c,b\,];X)$*. If, in addition,* B *is relatively compact in* X*, then the* C^0*-solutions set is relatively compact even in* $C([\,a,b\,];X)$*.*

See Baras [7] or Theorem 2.3.3, p. 47, in Vrabie [32].

3. THE MAIN RESULTS

Let $a \in (-\infty,0\,]$. On the linear space $C_b([\,a,+\infty);X)$, let us consider the family of seminorms $\{\|\cdot\|_k;\ k = 1,2,\dots\}$, defined by

$$\|u\|_k = \sup\{\|u(t)\|;\ t \in [\,a,k\,]\}$$

for each $k = 1, 2, \ldots$, and each $u \in C_b([\,a, +\infty); X)$. Endowed with this family of seminorms, $C_b([\,a, +\infty); X)$ is a separated locally convex space, denoted by $\widetilde{C}_b([\,a, +\infty); X)$.

The assumptions we need in that follows are listed below.

(H_A) the operator $A : D(A) \subseteq X \rightsquigarrow X$ satisfies:

(a_1) A is m-dissipative, $0 \in D(A)$, $0 \in A0$ and there exists $\omega > 0$ such that $A + \omega I$ is dissipative;

(a_2) the semigroup generated by A on $\overline{D(A)}$ is compact;

(H_f) the function $f : \mathbb{R}_+ \times \left[\overline{D(A)}\right]^{n+1} \to X$ is continuous and:

(f_1) there exist $\ell_k > 0$, $k = 0, 1, \ldots, n$, and $m > 0$ such that

$$\|f(t, u)\| \leq \sum_{k=0}^{n} \ell_k \|u_k\| + m$$

for each $t \in \mathbb{R}_+$ and $u \in \left[\overline{D(A)}\right]^{n+1}$, $u = (u_0, u_1, \ldots, u_n)$;

(f_2) there exist $\ell_k > 0$, $k = 0, 1, \ldots, n$, and $m > 0$ such that

$$\|f(t, u) - f(t, v)\| \leq \sum_{k=0}^{n} \ell_k \|u_k - v_k\|$$

for each $t \in \mathbb{R}_+$ and $u, v \in \left[\overline{D(A)}\right]^{n+1}$, $u = (u_0, u_1, \ldots, u_n)$, $v = (v_0, v_1, \ldots, v_n)$ and

$$\|f(t, 0, 0, \ldots, 0)\| \leq m$$

for each $t \in [\,0, +\infty)$;

(H_c) the constants ℓ_k, τ_k, $k = 0, 1, \ldots, n$, and $\omega > 0$ satisfy the so-called nonresonance conditions:

(c_1) $\ell = \sum_{k=0}^{n} \ell_k < \omega$;

(c_2) $b = \sum_{k=0}^{n} e^{\omega \tau_k} \ell_k < \omega$;

(H_g) the function $g : C_b([\,-\tau, +\infty); \overline{D(A)}) \to C([\,-\tau, 0\,]; \overline{D(A)})$ satisfies:

(g_1) for each $u, v \in C_b([\,-\tau, +\infty); \overline{D(A)})$, we have

$$\|g(u) - g(v)\|_{C([\,-\tau,0\,];X)} \leq \|u - v\|_{C_b([\,0,+\infty);X)};$$

(g_2) for each $u \in C_b([\,-\tau, +\infty); \overline{D(A)})$, we have

$$\|g(u)\|_{C([\,-\tau,0\,];X)} \leq \|u\|_{C_b([\,0,+\infty);X)};$$

(g_3) for each bounded set $\mathfrak{U}$ in $C_b([\,-\tau, +\infty); \overline{D(A)})$ which is relatively compact in $\widetilde{C}_b([\,\delta, +\infty); X)$ for each $\delta \in (0, +\infty)$, the set $g(\mathfrak{U})$ is relatively compact in $C([\,-\tau, 0\,]; X)$;

(g_4) there exists $a > 0$ such that for each $u, v \in C_b([\,-\tau, +\infty); \overline{D(A)})$, we have

$$\|g(u) - g(v)\|_{C([\,-\tau,0\,];X)} \leq \|u - v\|_{C_b([\,a,+\infty);X)}.$$

Remark 3.1. We emphasize that whenever the function g is defined as in $(i) \sim (ii)$ in Introduction and $\tau < 2\pi$, then g satisfies $(g_1) \sim (g_4)$. If g is of the form (iii) it satisfies $(g_1) \sim (g_3)$, and if the support of the measure μ is in $(\tau, +\infty)$ it satisfies (g_4) as well.

Now we may proceed to the statements of our main results.

Theorem 3.2. *If* (a_1), (a_2), (f_1), (c_1), *and* $(g_1) \sim (g_3)$ *are satisfied, then the problem* (1.1) *has at least one* C^0*-solution,* $u \in C_b([\,-\tau, +\infty); \overline{D(A)})$, *satisfying*

$$\|u\|_{C_b([\,-\tau,+\infty);X)} \leq \frac{m}{\omega - \ell}. \tag{3.1}$$

Theorem 3.3. *If* (a_1), (a_2), (f_2), (c_2), *and* $(g_1) \sim (g_3)$ *are satisfied, then the problem* (1.1) *has at least one* C^0*-solution,* $u \in C_b([\,-\tau, +\infty); \overline{D(A)})$, *satisfying* (3.1). *In addition, each solution of* (1.1) *is globally asymptotically stable.*

In the case of periodic conditions, i.e. $g(u)(t) = u(2\pi + t)$ for $t \in [\,-\tau, 0\,]$, Theorem 3.3 can be substantially improved in the sense that, in the absence of (a_2), we obtain a unique C^0-solution. More precisely, let us consider the periodic problem

$$\begin{cases} u'(t) \in Au(t) + f(t, u(t), u(t - \tau_1), \ldots, u(t - \tau_n))), & t \in \mathbb{R}, \\ u(t) = u(t + 2\pi), \quad t \in \mathbb{R}. \end{cases} \tag{3.2}$$

Theorem 3.4. *If* (a_1), (f_2), (c_1) *are satisfied and* f *is* 2π*-periodic with respect to its first argument, then the problem* (3.2) *has unique* 2π*-periodic* C^0*-solution,* $u : \mathbb{R} \to \overline{D(A)}$. *If, instead of* (c_1), *the stronger condition* (c_2) *is satisfied, then* u *is globally asymptotically stable.*

Another existence, uniqueness and stability result is stated below.

Theorem 3.5. *If* (a_1), (f_2), (c_2) *and* (g_4) *are satisfied, then the problem* (1.1) *has unique* C^0*-solution,* $u \in C_b([\,-\tau, +\infty); \overline{D(A)})$, *which satisfies* (3.1) *and is globally asymptotically stable.*

As concerns the regularity of the C^0-solutions, we have the following theorem essentially based on a fundamental regularity result due to Brezis [10]. See Theorem 4.11, p. 156 in Barbu [9].

Theorem 3.6. *If, in addition to the hypotheses of Theorem* 3.2, *we assume that* X *is a Hilbert space and* $A = \partial\varphi$ *is the subdifferential of a l.s.c., proper and convex function* $\varphi : H \to [\,0, +\infty\,]$, *then the* C^0*-solution of the problem* (1.1) *satisfies:*

(i) $u(t) \in D(A)$ *a.e. for* $t \in [\,0, +\infty)\,;$

(ii) $t \mapsto t^{1/2}u'(t)$ *belongs to* $L^2(0,a;H)$ *for each* $a>0$;

(iii) $t \mapsto \varphi(u(t))$ *belongs to* $L^1(0,a) \cap AC([\,\delta, a\,])$ *for each* $0<\delta<a$;

(iv) If, in addition, $g(u)(0) \in D(\varphi)$*, then* $t \mapsto u'(t)$ *belongs to* $L^2(0,a;H)$ *and* $t \mapsto \varphi(u(t))$ *belongs to* $AC([\,0,a\,])$ *for each* $a>0$.

In the specific case: $g(u)(s) = u(2\pi + s)$ *for each* $u \in C_b([\,-\tau,+\infty);\overline{D(A)})$ *and* $s \in [\,-\tau, 0\,]$*, we have* $g(u)(0) \in D(\varphi)$ *and thus (iv) holds.*

We briefly explain the idea of proof.

First, we show that, for each starting history $\varphi \in C([\,-\tau,0\,];\overline{D(A)})$, the problem

$$\begin{cases} u'(t) \in Au(t) + f(t,u(t),u(t-\tau_1),\dots,u(t-\tau_n)), & t \in \mathbb{R}_+ \\ u(s) = \varphi(s), & s \in [\,-\tau,0\,] \end{cases} \tag{3.3}$$

has at least one C^0-solution $u_\varphi \in C([\,0,+\infty);\overline{D(A)})$. In the hypotheses of Theorem 3.3 this solution is unique.

Second, we consider $\varepsilon \in (0,1)$ and we prove that the problem

$$\begin{cases} u'(t) \in Au(t) + f(t,u(t),u(t-\tau_1),\dots,u(t-\tau_n)), & t \in \mathbb{R}_+ \\ u(s) = (1-\varepsilon)g(u)(s), & s \in [\,-\tau,0\,] \end{cases} \tag{3.4}$$

has at least one C^0-solution $u_\varepsilon \in C([\,0,+\infty);\overline{D(A)})$. In the hypotheses of Theorem 3.3 this solution is unique.

In order to do this – in the case of Theorem 3.2 –, we show that for each fixed $\varepsilon \in (0,1)$, and $h \in C([\,0,+\infty);X)$ the problem

$$\begin{cases} u'(t) \in Au(t) + h(t), & t \in \mathbb{R}_+ \\ u(s) = (1-\varepsilon)g(u)(s), & s \in [\,-\tau,0\,] \end{cases}$$

has a unique C^0-solution, denoted by u_ε^h. Then, we prove that $F : C([\,0,+\infty);X) \to C([\,0,+\infty);X)$, defined by

$$F(h)(t) = f(t,u_\varepsilon^h(t),u_\varepsilon^h(t-\tau_1),\dots,u_\varepsilon^h(t-\tau_n))$$

for each $h \in C([\,0,+\infty);X)$ and $t \in [\,0,+\infty)$, where u_ε^h is the unique C^0-solution of the problem above, maps a suitably defined nonempty, closed and convex subset, $\mathcal{K}$, in $\widetilde{C}_b([\,0,+\infty);X)$ – the space $C([\,0,+\infty);X)$ endowed with the uniform convergence topology on compact intervals – into itself, is continuous and $F(\mathcal{K})$ is relatively compact. Alternatively, under the hypotheses of Theorem 3.3, we show that the operator $Q : v \mapsto u_\varepsilon^v$, which associates to $v \in C_b([\,-\tau,+\infty);\overline{D(A)})$ the unique C^0-solution u_ε^v of the problem

$$\begin{cases} u'(t) \in Au(t) + f(t,u(t),u(t-\tau_1),\dots,u(t-\tau_n)) \quad \text{for} \quad t \in \mathbb{R}_+ \\ u(s) = (1-\varepsilon)g(v)(s), \quad s \in [\,-\tau,0\,] \end{cases} \tag{3.5}$$

is a strict contraction on $C_b([-\tau,+\infty);\overline{D(A)})$. So, either F has at least one fixed point – in the hypotheses of Theorem 3.2 – or Q has a unique fixed point – in the hypotheses of Theorem 3.3. In both cases, the fixed point defines a C^0-solution of the problem (3.4).

Third, we prove that the set of C^0-solutions of the problems (3.5) as $\varepsilon \in (0,1)$, i.e. $\{u_\varepsilon;\ \varepsilon \in (0,1)\}$ is relatively compact in $C_b([-\tau,+\infty);X)$.

Fourth, we show that there exists $u \in C_b([-\tau,+\infty);\overline{D(A)})$ such that, on a sequence $\varepsilon \downarrow 0$, we have $u = \lim_{\varepsilon\downarrow 0} u_\varepsilon$ in $C_b([-\tau,+\infty);X)$ and u is C^0-solution of the problem (1.1) on $[-\tau,+\infty)$.

Fifth, we prove that, in the case of Theorem 3.3, under the additional nonresonance condition (c_2), each solution of (1.1) is globally asymptotically stable.

4. PRELIMINARY LEMMAS

For the sake of convenience and clarity, we divided the proofs of our main results into several steps. The next Gronwall-type inequality is Lemma 4.1 in Yongxiang Li [25].

Lemma 4.1. *Let $0=\tau_0<\tau_1<\cdots<\tau_n=\tau$, and let $x:[-\tau,+\infty)\to\mathbb{R}$ be a continuous function satisfying*

$$x(t) \le x(0) + \int_0^t \sum_{k=0}^n b_k x(s-\tau_k)\,ds$$

for each $t\in[0,\infty)$, where $b_k>0$, $k=0,1,\ldots,n$ are given constants. Then

$$x(t) \le \|x\|_{C([-\tau,0];\mathbb{R})}\, e^{bt}$$

for each $t \ge 0$, where $b=\sum_{k=0}^n b_k$.

Lemma 4.2. *If (a_1), (a_2), (f_1) and (c_1) are satisfied then, for each starting history $\varphi \in C([-\tau,0];\overline{D(A)})$, the problem (3.3) has at least one C^0-solution $u \in C([-\tau,\infty);\overline{D(A)})$. If, instead of (c_1), the stronger nonresonance condition (c_2) is satisfied, then*

$$\|u(t)\| \le \left[\|\varphi\|_{C([-\tau,0];X)} + \frac{m}{\omega-\ell}\right] e^{(b-\omega)t} + \frac{m}{\omega-\ell} \tag{4.1}$$

for each $t\in[0,+\infty)$ and thus $u \in C_b([-\tau,+\infty);\overline{D(A)})$.

Proof. Let $0<\lambda=\tau_1=\min\{\tau_1,\tau_2,\ldots,\tau_n\}$, $\tau=\tau_n=\max\{\tau_1,\tau_2,\ldots,\tau_n\}$, let $\varphi \in C([-\tau,0];\overline{D(A)})$ and let $g_0:[0,\lambda]\times\overline{D(A)}\to X$ be given by

$$g_0(t,v) = f(t,v,\varphi(t-\tau_1),\ldots,\varphi(t-\tau_n))$$

for each $(t,v)\in[\,0,\lambda\,]\times\overline{D(A)}$. Now, let us consider the Cauchy problem

$$\begin{cases} v_1'(t)\in Av_1(t)+g_0(t,v_1(t)), & t\in[\,0,\lambda\,]\\ v_1(0)=\varphi(0), \end{cases}$$

Since f is continuous and has linear growth, it follows that g_0 is continuous and has linear growth. In addition, A generates a compact semigroup and consequently, by virtue of Theorem 2.1 in Vrabie [30] – see also Theorem 3.8.1, p. 131 in Vrabie [32] – the Cauchy problem above has at least one C^0-solution $v_1:[\,0,\lambda\,]\to\overline{D(A)}$. Next, fix such a C^0-solution v_1, let

$$u_1(t)=\begin{cases}\varphi(t) & \text{if } t\in[\,-\tau,0\,]\\ v_1(t) & \text{if } t\in[\,0,\lambda\,]\end{cases}$$

and let us define the function $g_1:[\,\lambda,2\lambda\,]\times\overline{D(A)}\to X$ by

$$g_1(t,v)=f(t,v,u_1(t-\tau_1),\ldots,u_1(t-\tau_n))$$

for $t\in[\,\lambda,2\lambda\,]$. Let us consider the Cauchy problem

$$\begin{cases} v_2'(t)\in Av_2(t)+g_1(t,v_2(t)), & t\in[\,\lambda,2\lambda\,]\\ v_2(\lambda)=u_1(\lambda). \end{cases}$$

Using the very same arguments as before, we conclude that this problem has at least one C^0-solution $v_2:[\,\lambda,2\lambda\,]\to\overline{D(A)}$. So, we can define the function $u_2:[\,-\lambda,2\lambda\,]\to\overline{D(A)}$ by

$$u_2(t)=\begin{cases}u_1(t) & \text{if } t\in[\,-\tau,\lambda\,]\\ v_2(t) & \text{if } t\in[\,\lambda,2\lambda\,].\end{cases}$$

Inductively repeating this procedure on $[\,k\lambda,(k+1)\lambda\,]$ for $k=2,3,\ldots$, we can define a function $u\in C([\,-\tau,+\infty);\overline{D(A)})$ which turns out to be a C^0-solution of the problem (3.3).

Finally, we will show that, under the additional condition (c_2), we conclude that u belongs to $C_b([\,-\tau,+\infty);\overline{D(A)})$.

Taking $x=y=0$ in (2.3), and using (f_1), we deduce

$$\|u(t)\|\le e^{-\omega t}\|u(0)\|+\int_0^t e^{-\omega(t-s)}\left[\sum_{k=0}^n \ell_k\|u(s-\tau_k)\|+m\right]ds.$$

Set $x(t)=e^{\omega t}\left[\|u(t)\|-\frac{m}{\omega-\ell}\right]$, $b_k=e^{\omega\tau_k}\ell_k$, $k=0,1,\ldots,n$, and let us observe that

$$x(t)\le x(0)+\int_0^t\sum_{k=0}^n b_k x(s-\tau_k)\,ds$$

for each $t\in[\,0,+\infty)$. Thus, by Lemma 4.1, we conclude that

$$x(t)\le\|x\|_{C([\,-\tau,0\,];\mathbb{R})}\,e^{bt}$$

for each $t \in [0, +\infty)$. Then

$$e^{\omega t}\left[\|u(t)\| - \frac{m}{\omega - \ell}\right] \le \|x\|_{C([-\tau,0];\mathbb{R})}\, e^{bt}$$

which shows that

$$\|u(t)\| \le \|x\|_{C([-\tau,0];\mathbb{R})}\, e^{(b-\omega)t} + \frac{m}{\omega - \ell}$$

for each $t \in [0, +\infty)$.

Since, for $t \in [-\tau, 0]$, $x(t) = e^{\omega t}\left[\|\varphi(t)\| - \frac{m}{\omega-\ell}\right]$, it follows that u satisfies (4.1) and so $u \in C_b([-\tau, +\infty); \overline{D(A)})$. This completes the proof. □

Lemma 4.3. *If* (a_1), (f_2) *and* (c_1) *are satisfied then, for each starting history* $\varphi \in C([-\tau, 0]; \overline{D(A)})$, *the problem* (3.3) *has a unique* C^0*-solution* $u : [-\tau, \infty) \to \overline{D(A)})$. *If, instead of* (c_1), *the stronger nonresonance condition* (c_2) *is satisfied, then* u *satisfies* (4.1) *and thus* $u \in C_b([-\tau, \infty); \overline{D(A)})$.

Proof. The proof repeats the same routine as that in Lemma 4.2, with the special mention that the compactness argument should be replaced by a standard Lipschitz technique. Since the uniqueness is obvious, the proof is complete. □

Lemma 4.4. *If* (a_1), (f_2), (c_2) *and* (g_1) *are satisfied then, for each* $\varepsilon \in (0, 1)$ *the problem* (3.4) *has a unique* C^0*-solution* u_ε.

Proof. Let $\varepsilon \in (0, 1)$. In view of Lemma 4.3, for each $v \in C_b([-\tau, +\infty); \overline{D(A)})$, the Cauchy problem (3.5) has a unique solution $u_\varepsilon^v = Q_\varepsilon(v) \in C_b([-\tau, +\infty); \overline{D(A)})$.

To prove that Q_ε is a strict contraction, we distinguish between two cases: (1) $t \in [-\tau, 0]$ and (2) $t \in (0, +\infty)$. If $t \in [-\tau, 0]$, by (g_1), we have

$$\|Q_\varepsilon(v)(t) - Q_\varepsilon(w)(t)\| = (1 - \varepsilon)\|g(v)(t) - g(w)(t)\|$$

$$\le (1 - \varepsilon)\|v - w\|_{C_b([0,+\infty);X)} \le (1 - \varepsilon)\|v - w\|_{C_b([-\tau,+\infty);X)}.$$

Hence

$$\|Q_\varepsilon(v)(t) - Q_\varepsilon(w)(t)\| \le (1 - \varepsilon)\|v - w\|_{C_b([-\tau,+\infty);X)}. \tag{4.2}$$

for each $t \in [-\tau, 0]$.

If $t \in (0, +\infty)$, by (2.2) in Theorem 2.3 and (f_2), we get

$$\|Q_\varepsilon(v)(t) - Q_\varepsilon(w)(t)\| \le (1 - \varepsilon)e^{-t\omega}\|g(v)(0) - g(w)(0)\|$$

$$+ \int_0^t \sum_{k=0}^{n} e^{\omega\tau_k}\ell_k e^{-\omega t} e^{\omega(s-\tau_k)}\|Q_\varepsilon(v)(s - \tau_k) - Q_\varepsilon(w)(s - \tau_k)\|\, ds.$$

Denoting $x(t) = e^{t\omega}\|Q_\varepsilon(v)(t) - Q_\varepsilon(w)(t)\|$ and $b_k = e^{\omega\tau_k}\ell_k$, $k = 0, 1, \ldots, n$, and observing that

$$x(0) = (1 - \varepsilon)\|g(v)(0) - g(w)(0)\|),$$

we get

$$x(t) \le x(0) + \int_0^t \sum_{k=0}^n b_k x(s-\tau_k)\,ds$$

for each $t \in [\,0,+\infty)$. By Lemma 4.1, (f_2), (c_2) and (g_1), we conclude that

$$\|Q_\varepsilon(v)(t) - Q_\varepsilon(w)(t)\| \le (1-\varepsilon)e^{(b-\omega)t}\|v-w\|_{C_b([\,0,+\infty)\,;X)}.$$

Thus

$$\|Q_\varepsilon(v)(t) - Q_\varepsilon(w)(t)\| \le (1-\varepsilon)\|v-w\|_{C_b([\,-\tau,+\infty);X)} \tag{4.3}$$

for each $t \in [\,0,+\infty)$. From, (4.2) and (4.3), we conclude that

$$\|Q_\varepsilon(v) - Q_\varepsilon(w)\|_{C_b([\,-\tau,+\infty);X)} \le (1-\varepsilon)\|v-w\|_{C_b([\,-\tau,+\infty);X)},$$

which shows that Q_ε is a strict contraction. By virtue of Banach Fixed Point Theorem, we conclude that Q_ε has a unique fixed point which is the unique solution of the problem (3.4). □

Lemma 4.5. *Let us assume that* (a_1), (a_2) (f_1), (c_1) *and* $(g_1) \sim (g_3)$ *are satisfied. Then, for each* $\varepsilon \in (0,1)$, *the problem* (3.4) *has at least one* C^0*-solution* $u_\varepsilon \in C_b([\,-\tau,+\infty);\overline{D(A)})$.

In addition, under the assumptions (a_1), (f_1), (c_1) *and* (g_1) *each solution* $u_\varepsilon \in C_b([\,-\tau,+\infty);\overline{D(A)})$ *of the problem* (3.4) *satisfies*

$$\|u_\varepsilon\|_{C_b([\,-\tau,+\infty);X)} \le \frac{m}{\omega-\ell}. \tag{4.4}$$

Proof. Let $h \in C_b([\,0,+\infty);X)$ and let us consider the problem

$$\begin{cases} u'(t) \in Au(t) + h(t), & t \in [\,0,+\infty) \\ u(s) = (1-\varepsilon)g(u)(s), & s \in [\,-\tau,0\,]. \end{cases} \tag{4.5}$$

We begin by noticing that this problem has a unique C^0-solution $u_\varepsilon^h \in C_b([\,-\tau,+\infty);\overline{D(A)})$. Indeed, let $h \in C_b([\,0,+\infty);X)$ be arbitrary but fixed and let us observe that, for each v in $C_b([\,-\tau,+\infty);\overline{D(A)})$, in view of Theorem 2.3, and of (g_1), the problem

$$\begin{cases} u'(t) \in Au(t) + h(t), & t \in [\,0,+\infty) \\ u(s) = (1-\varepsilon)g(v)(s), & s \in [\,-\tau,0\,] \end{cases}$$

has a unique C^0-solution $u = S(v) \in C_b([\,-\tau,+\infty);\overline{D(A)})$. By (g_1), we have

$$\begin{aligned}\|S(v)(t) - S(w)(t)\| &\le (1-\varepsilon)\|v-w\|_{C_b([\,0,+\infty);X)} \\ &\le (1-\varepsilon)\|v-w\|_{C_b([\,-\tau,+\infty);X)}\end{aligned}$$

for each $v, w \in C([\,-\tau,+\infty);\overline{D(A)})$ and each $t \in [\,-\tau,0\,]$. Next, by (2.2) and (g_1), we get

$$\begin{aligned}\|S(v)(t) - S(w)(t)\| &\le (1-\varepsilon)e^{-\omega t}\|g(v)-g(w)\|_{C([\,-\tau,0\,];X)} \\ &\le (1-\varepsilon)\|v-w\|_{C_b([\,0,+\infty);X)} \le (1-\varepsilon)\|v-w\|_{C_b([\,-\tau,+\infty);X)}\end{aligned}$$

for each $v, w \in C_b([-\tau, +\infty); \overline{D(A)})$ and each $t \in [0, +\infty)$. So, the operator S is a strict contraction and thus it has a unique fixed point u_ε^h. Since u_ε^h is a C^0-solution of (4.5) if and only if it is a fixed point of S, it follows that u_ε^h is the unique C^0-solution of (4.5).

Let us observe that

$$\|u_\varepsilon^h\|_{C_b([0,+\infty);X)} \leq \frac{1}{\omega}\|h\|_{C_b([0,+\infty);X)}. \tag{4.6}$$

Indeed, we have

$$\|u_\varepsilon^h(t)\| \leq (1-\varepsilon)e^{-\omega t}\|u_\varepsilon^h\|_{C_b([0,+\infty);X)} + \frac{1-e^{-\omega t}}{\omega}\|h\|_{C_b([0,+\infty);X)}, \tag{4.7}$$

for each $t \in [0, +\infty)$. We distinguish between three possible cases.

Case 1. If $\|u_\varepsilon^h\|_{C_b([0,+\infty);X)} = \|u_\varepsilon^h(0)\|$, then from (4.7), we deduce

$$\|u_\varepsilon^h\|_{C_b([0,+\infty);X)} \leq (1-\varepsilon)\|u_\varepsilon^h\|_{C_b([0,+\infty);X)}$$

and so $\|u_\varepsilon^h\|_{C_b([0,+\infty);X)} = 0$. Clearly, in this case, (4.6) holds.

Case 2. If $\|u_\varepsilon^h\|_{C_b([0,+\infty);X)} = \|u_\varepsilon^h(t)\|$ for some $t \in (0, +\infty)$, then, again from (4.7) and the obvious inequality $1-\varepsilon < 1$, we get

$$\|u_\varepsilon^h\|_{C_b([0,+\infty);X)} \leq e^{-\omega t}\|u_\varepsilon^h\|_{C_b([0,+\infty);X)} + \frac{1-e^{-\omega t}}{\omega}\|h\|_{C_b([0,+\infty);X)},$$

and thus, since $1-e^{-\omega t} > 0$, we obtain (4.6).

Case 3. If there is no $t \in [0, +\infty)$ such that $\|u_\varepsilon^h\|_{C_b([0,+\infty);X)} = \|u_\varepsilon^h(t)\|$, then, there exists at least one sequence $(t_k)_k$ of positive real numbers, with $\lim_k t_k = +\infty$ and $\lim_k \|u_\varepsilon^h(t_k)\| = \|u_\varepsilon^h\|_{C_b([0,+\infty);X)}$. Then, taking $t = t_k$ in (4.7) and letting $k \to \infty$, we complete the proof of (4.6).

From (4.6) and (g_2), we deduce

$$\|u_\varepsilon^h(t)\| = (1-\varepsilon)\|g(u_\varepsilon^h)(t)\| \leq (1-\varepsilon)\|u_\varepsilon^h\|_{C_b([0,+\infty);X)} \leq \frac{1}{\omega}\|h\|_{C_b([0,+\infty);X)}$$

for each $t \in [-\tau, 0]$ and hence

$$\|u_\varepsilon^h\|_{C_b([-\tau,+\infty);X)} \leq \frac{1}{\omega}\|h\|_{C_b([0,+\infty);X)}. \tag{4.8}$$

Now, let us define the operator $F : C_b([0, +\infty); X) \to C_b([0, +\infty); X)$ by

$$F(h)(t) = f(t, u_\varepsilon^h(t), u_\varepsilon^h(t-\tau_1), \ldots, u_\varepsilon^h(t-\tau_n))$$

for each $h \in C_b([0, +\infty); X)$ and each $t \in [0, +\infty)$, $u_\varepsilon^h \in C_b([-\tau, +\infty); \overline{D(A)})$ being the unique C^0-solution of (4.5). At this point let us observe that, by (f_1), we deduce that $\|F(h)(t)\| \leq \sum_{k=0}^n \ell_k\|u_\varepsilon^h(t-\tau_k)\| + m$, while from (4.8), we get

$$\|F(h)\|_{C_b([0,+\infty);X)} \leq \frac{\ell}{\omega}\|h\|_{C_b([0,+\infty);X)} + m.$$

Now, let

$$r=\frac{\omega}{\omega-\ell}m,$$

and let $\mathcal{K}=\{u\in C_b([\,0,+\infty);X);\ \|u\|_{C_b([\,0,+\infty);X)}\leq r\}$ which is convex and closed in $\widetilde{C}_b([\,0,+\infty);X)$. Let us observe that the operator F maps $\mathcal{K}$ into itself. Indeed, if $\|h\|_{C_b([\,0,+\infty);X)}\leq r$, from the last inequality we obtain

$$\|F(h)\|_{C_b([\,0,+\infty);X)}\leq\frac{\ell}{\omega}r+m\leq r.$$

We will show next that F is continuous and compact with respect to the locally convex space structure. To check the continuity of F let us observe that

$$\|u_\varepsilon^h(t)-u_\varepsilon^{\widetilde{h}}(t)\|\leq(1-\varepsilon)e^{-\omega t}\|u_\varepsilon^h-u_\varepsilon^{\widetilde{h}}\|_{C_b([\,-\tau,+\infty);X)}$$
$$+\frac{1-e^{-\omega t}}{\omega}\|h-\widetilde{h}\|_{C_b([\,0,+\infty);X)},$$

for each $t\in[\,0,+\infty)$. Reasoning as in the case of (4.8), we deduce

$$\|u_\varepsilon^h-u_\varepsilon^{\widetilde{h}}\|_{C_b([\,-\tau,+\infty);X)}\leq\frac{1}{\omega}\|h-\widetilde{h}\|_{C_b([\,0,+\infty);X)}$$

which shows that $h\mapsto u_\varepsilon^h$ is Lipschitz continuous. Thus F is continuous from $\mathcal{K}$ into itself with respect to the Banach space structure of $C_b([\,0,+\infty);X)$ and consequently, it is continuous from $\mathcal{K}$ with respect to the locally convex space structure.

To prove the compactness, let us observe that, in view of Theorem 2.5, for each $k=1,2\ldots$, the set $\{u_\varepsilon^h;\ \|h\|_{C_b([\,0,+\infty);X)}\leq r\}$ is relatively compact in $C_b([\,\delta,k\,]);\overline{D(A)})$ for each $\delta\in(0,k)$.

Now, (g_3) implies that $\{(1-\varepsilon)g(u_\varepsilon^h);\ \|h\|_{C_b([\,0,+\infty);X}\leq r\}$ is relatively compact in $C([\,-\tau,0\,];X)$.

Thus, $\{u_\varepsilon^h(0);\ \|h\|_{C_b([\,0,+\infty);X)}\leq r\}=\{g(u_\varepsilon^h)(0);\ \ \|h\|_{C(_b[\,0,+\infty);X)}\leq r\}$ is relatively compact in $\overline{D(A)}$. Again by Theorem 2.5, we conclude that $\{u_\varepsilon^h;\ \|h\|_{C([\,0,+\infty);X}\leq r\}$ is relatively compact in $C([\,0,k\,];\overline{D(A)})$. So, F is compact in $\widetilde{C}_b([\,0,+\infty);X)$ and hence, by Tychonoff Fixed Point Theorem 1.2.8, p. 6 in Vrabie [32], it has at least one fixed point h. Obviously $u_\varepsilon^h=u_\varepsilon$ is a C^0-solution of the problem (3.4). By (4.8) and the fact that every C^0-solution of the problem (3.4) is a fixed point of F and thus belongs to $\mathcal{K}$, it follows that it satisfies (4.4) and the proof is complete. □

Lemma 4.6. *Let us assume that* (a_1), (a_2), (f_1), (c_2) *and* $(g_1)\sim(g_3)$ *are satisfied. Then,* $\{u_\varepsilon;\ \varepsilon\in(0,1)\}$, *whose existence is ensured either by Lemma* 4.4 *– if the stronger condition* (f_2) *is satisfied – or by Lemma* 4.5, *is relatively compact in* $\widetilde{C}_b([\,-\tau,+\infty);X)$.

Proof. Let $k=1,2,\ldots,$ be arbitrary. By (4.4), it follows that $\{u_\varepsilon;\ \varepsilon\in(0,1)\}$ is uniformly bounded on $[\,0,k\,]$. From (f_1), we get that $\{f(\cdot,u_\varepsilon(\cdot),u_\varepsilon(\cdot-\tau_1),\ldots,u_\varepsilon(\cdot-\tau_n));\ \varepsilon\in(0,1)\}$ is uniformly bounded and thus uniformly integrable on $[\,0,k\,]$. By virtue of Theorem 2.5, we deduce that $\{u_\varepsilon;\ \varepsilon \in (0,1)\}$ is relatively compact in $C([\,\delta, k\,]; X)$ for each $\delta \in (0,k)$. Now, (g_3) implies that $\{g(u_\varepsilon);\ \varepsilon \in (0,1)\}$ is relatively compact in $C([\,-\tau, 0\,]; X)$. Thus, $\{u_\varepsilon(0);\ \varepsilon \in (0,1)\} = \{g(u_\varepsilon)(0);\ \varepsilon \in (0,1)\}$ is relatively compact in $\overline{D(A)}$. Using once again Theorem 2.5, we conclude that $\{u_\varepsilon;\ \varepsilon \in (0,1)\}$ is relatively compact in $C([\,0,k\,]; X)$. As we already have shown, this set is relatively compact in $C([\,-\tau, 0\,]; X)$ and consequently it is relatively compact in $C([\,-\tau, k\,]; X)$, as claimed. But k was arbitrary, and this completes the proof. □

5. PROOFS OF THE MAIN RESULTS

5.1. Proof of Theorem 3.2.

Proof. By Lemma 4.5, we know that for each $\varepsilon \in (0,1)$ the problem (3.4) has at least one solution $u_\varepsilon \in C_b([\,-\tau, +\infty); \overline{D(A)})$. By Lemma 4.6, the set $\{u_\varepsilon;\ \varepsilon \in (0,1)\}$ is relatively compact in $\widetilde{C}_b([\,-\tau, +\infty); \overline{D(A)})$. So, at least on a sequence of ε's there exists

$$\lim_{\varepsilon \downarrow 0} u_\varepsilon = u$$

uniformly on $[\,-\tau, k\,]$ for $k = 1, 2, \ldots$. Passing to the limit on that sequence for $\varepsilon \downarrow 0$ in both (3.4) and (4.4), we conclude that u is a C^0-solution of the problem (1.1) satisfying (3.1) and this completes the proof. □

5.2. Proof of Theorem 3.3.

Proof. In view of Lemma 4.4, we know that for each $\varepsilon \in (0,1)$ the problem (3.4) has a unique C^0-solution $u_\varepsilon \in C_b([\,\tau, +\infty); \overline{D(A)})$. By Lemma 4.6, the set $\{u_\varepsilon;\ \varepsilon \in (0,1)\}$ is relatively compact in $\widetilde{C}_b([\,-\tau, +\infty); \overline{D(A)})$. Therefore, at least on a sequence of ε's there exists

$$\lim_{\varepsilon \downarrow 0} u_\varepsilon = u$$

uniformly on $[\,-\tau, k\,]$ for $k = 1, 2, \ldots,$. Passing to the limit on that sequence for $\varepsilon \downarrow 0$ in both (3.4) and (4.4), we conclude that u is a C^0-solution of the problem (1.1) satisfying (3.1) and this proves the existence part.

Finally, if $v : [\,-\tau, +\infty) \to \overline{D(A)}$ is an arbitrary C^0-solution of the evolution equation

$$\begin{cases} v'(t) \in Av(t) + f(t, v(t), v(t-\tau_1), \ldots, v(t-\tau_n)), & t \in [\,0, +\infty) \\ v(t) = \varphi(t), & t \in [\,-\tau, 0\,], \end{cases}$$

and $u : [\,-\tau, +\infty) \to \overline{D(A)}$ is a C^0-solution of the problem (1.1), from (f_2) and (2.2), we get

$$e^{\omega t}\|u(t) - v(t)\| \le \|u(0) - v(0)\| + \int_0^t e^{\omega s} \sum_{k=0}^{n} \ell_k \|u(s-\tau_k) - v(s-\tau_k)\|\, ds \tag{5.1}$$

for each $t \geq 0$. On the other hand, from the initial condition and (g_2), we have

$$\int_0^t e^{\omega s} \sum_{k=0}^n \ell_k \|u(s-\tau_k) - v(s-\tau_k)\| \, ds = \sum_{k=0}^n \int_0^{\tau_k} e^{\omega s} \ell_k \|u(s-\tau_k) - v(s-\tau_k)\| \, ds$$

$$+ \sum_{k=0}^n \int_{\tau_k}^t e^{\omega s} \ell_k \|u(s-\tau_k) - v(s-\tau_k)\| \, ds \leq \sum_{k=0}^n \int_{-\tau}^0 e^{\omega s} \ell_k \|g(u)(s) - \varphi(s)\| \, ds$$

$$+ \sum_{k=0}^n \int_0^{t-\tau_k} e^{\omega s} \ell_k e^{\omega \tau_k} \|u(s) - v(s)\| \, ds$$

$$\leq \ell \frac{1 - e^{-\omega\tau}}{\omega} \left[\|u\|_{C_b([-\tau,+\infty);X)} + \|\varphi\|_{C([-\tau,0];X)} \right] + \int_0^t e^{\omega s} b \|u(s) - v(s)\| \, ds,$$

where $b = \sum_{k=0}^n \ell_k e^{\omega \tau_k}$. By (3.1) and the last inequalities, we obtain

$$\int_0^t e^{\omega s} \sum_{k=0}^n \ell_k \|u(s-\tau_k) - v(s-\tau_k)\| \, ds \leq \ell \frac{1 - e^{-\omega\tau}}{\omega} \left[\frac{m}{\omega - \ell} + \|\varphi\|_{C([-\tau,0];X)} \right]$$

$$+ \int_0^t e^{\omega s} b \|u(s) - v(s)\| \, ds. \tag{5.2}$$

Finally, from (5.1) and (5.2), we deduce

$$e^{\omega t} \|u(t) - v(t)\| \leq M + \int_0^t b e^{\omega s} \|u(s) - v(s)\| \, ds,$$

for each $t \in [0, +\infty)$, where

$$M = \|u(0) - v(0)\| + \ell \frac{1 - e^{-\omega\tau}}{\omega} \left[\frac{m}{\omega - \ell} + \|\varphi\|_{C([-\tau,0];X)} \right].$$

From Gronwall Lemma 1.5.2, p. 44 in Vrabie [34], we get

$$\|u(t) - v(t)\| \leq e^{(b-\omega)t} M$$

for each $t \geq 0$. Since, by (c_2), $b - \omega < 0$, it follows that u is asymptotically stable, as claimed. The proof is complete. □

5.3. **Proof of Theorem 3.4.**

Proof. Let $v \in C_{2\pi}(\mathbb{R}; \overline{D(A)})$ – the space of all continuous and 2π-periodic functions from $\mathbb{R}$ to $\overline{D(A)}$ and let us consider the problem

$$\begin{cases} u'(t) \in Au(t) + f(t, v(t), v(t-\tau_1), \ldots, v(t-\tau_n)), \ t \in \mathbb{R} \\ u(0) = u(2\pi). \end{cases} \tag{5.3}$$

Since A is m-dissipative and (f_2), (c_1) are satisfied, it readily follows that (5.3) has a unique solution $u = Q(v)$. Let $v, w \in C_{2\pi}(\mathbb{R}; \overline{D(A)})$. Since, by the periodicity condition, we have

$$\|v(\cdot - \tau_k) - w(\cdot - \tau_k)\|_{C([0,2\pi];X)} = \|v - w\|_{C([0,2\pi];X)}$$

for $k = 1, 2, \dots, n$, recalling that $A + \omega I$ is dissipative, from Theorem 2.3, it follows that

$$\|Q(v)(t) - Q(w)(t)\| \leq e^{-\omega t}\|Q(v)(0) - Q(w)(0)\|$$
$$+e^{-\omega t}\int_0^t e^{\omega s}\sum_{k=0}^{n}\ell_k\|v(s-\tau_k) - w(s-\tau_k)\|\,ds$$
$$\leq e^{-\omega t}\|Q(v) - Q(w)\|_{C([\,0,2\pi\,];X)} + \frac{\ell}{\omega}\left(1 - e^{-\omega t}\right)\|v - w\|_{C([\,0,2\pi\,];X)}.$$

Let us observe that, thanks to the periodicity condition, we may assume with no loss of generality that

$$\|Q(v) - Q(w)\|_{C([\,0,2\pi\,];X)} = \|Q(v)(t) - Q(w)(t)\|$$

with $t > 0$. So, from the last inequality we obtain

$$\left(1 - e^{-\omega t}\right)\|Q(v) - Q(w)\|_{C([\,0,2\pi\,];X)} \leq \frac{\ell}{\omega}\left(1 - e^{-\omega t}\right)\|v - w\|_{C([\,0,2\pi\,];X)}.$$

Taking into account that $t > 0$, we get

$$\|Q(v) - Q(w)\|_{C([\,0,2\pi\,];X)} \leq \frac{\ell}{\omega}\|v - w\|_{C([\,0,2\pi\,];X)}.$$

Since by (c_1) it follows that $0 < \frac{\ell}{\omega} < 1$, we conclude that Q is a strict contraction. By Banach Fixed Point Theorem, it follows that Q has a unique fixed point which is the unique 2π-periodic C^0-solution. Since the global asymptotic stability of the solution under the additional hypotheses (c_2) follows exactly as in the case of Theorem 3.3, the proof is complete. $\square$

5.4. **Proof of Theorem 3.5.**

Proof. We will use two fixed point arguments. First, let $v, w \in C_b([\,-\tau, +\infty); \overline{D(A)})$ be arbitrary but fixed and let us consider the delay equation

$$\begin{cases} u'(t) \in Au(t) + f(t, v(t), v(t-\tau_1), \dots, v(t-\tau_n)), & t \in \mathbb{R}_+ \\ u(t) = g(w)(t), \quad t \in [\,-\tau, 0\,]. \end{cases} \tag{5.4}$$

which, for any fixed v, has a unique C^0-solution $u = T(w)$. We will show that T^2 is a strict contraction. Let $w, \widetilde{w} \in C_b([\,-\tau, +\infty); \overline{D(A)})$, let $t \geq 0$. By (g_4) in (H_g), we deduce

$$\|T(w)(t) - T(\widetilde{w})(t)\| \leq e^{-\omega t}\|g(w) - g(\widetilde{w})\|_{C([\,-\tau,0\,];X)} \leq e^{-\omega t}\|w - \widetilde{w}\|_{C([a,+\infty);X)}.$$

So, for $t \geq 0$, we have

$$\|T^2(w)(t) - T^2(\widetilde{w})(t)\| \leq e^{-\omega t}\|g(T(w)) - g(T(\widetilde{w}))\|_{C([\,-\tau,0\,];X)}$$
$$\leq e^{-\omega t}\|T(w) - T(\widetilde{w})\|_{C([a,+\infty);X)} \leq \sup_{t\in[\,a,+\infty)}\|T(w)(t) - T(\widetilde{w})(t)\|$$
$$\leq e^{-\omega a}\|w - \widetilde{w}\|_{C([a,+\infty);X)} \leq e^{-\omega a}\|w - \widetilde{w}\|_{C([-\tau,+\infty);X)}.$$

If $t \in [-\tau, 0)$, again by (g_4) in (H_g) and the preceding inequality, we have

$$\|T^2(w)(t) - T^2(\widetilde{w})(t)\| = \|g(T(w)(t) - g(T(\widetilde{w}(t))\|$$

$$\leq \|T(w) - T(w)\|_{C([a,+\infty);X)} \leq e^{-\omega a}\|w - \widetilde{w}\|_{C([-\tau,+\infty);X)}.$$

Hence T^2 is a strict contraction and accordingly the problem

$$\begin{cases} u'(t) \in Au(t) + f(t, v(t), v(t-\tau_1), \dots, v(t-\tau_n)), & t \in \mathbb{R}_+ \\ u(t) = g(u)(t), & t \in [-\tau, 0] \end{cases} \tag{5.5}$$

has a unique solution u which is the unique fixed point of T, i.e. $T(u) = u$.

Second, let $Q : C_b([-\tau, +\infty); \overline{D(A)}) \to C_b([-\tau, +\infty); \overline{D(A)})$ be defined by $Q(v) = u$, where $u \in C_b([-\tau, +\infty); \overline{D(A)})$ is the unique C^0-solution of the problem (5.5). Let $v, w \in C_b([-\tau, +\infty); \overline{D(A)})$ and $t \geq 0$ be arbitrary. From (2.3) and (g_4), we deduce

$$\|Q(v)(t) - Q(w)(t)\| \leq e^{-\omega t}\|Q(v) - Q(w)\|_{C_b([-\tau,+\infty);X)}$$

$$+\frac{\ell}{\omega}(1 - e^{-\omega t})\|v - w\|_{C_b([-\tau,+\infty);X)}. \tag{5.6}$$

We distinguish between three possible cases.

Case 1. If $\|Q(v) - Q(w)\|_{C_b([-\tau,+\infty);X)} = \|Q(v)(t_0) - Q(w)(t_0)\|$ for a certain $t_0 \in [-\tau, 0]$ then, by the initial condition and (g_4), we get

$$\|Q(v) - Q(w)\|_{C_b([-\tau,+\infty);X)} \leq \|Q(v) - Q(w)\|_{C_b([a,+\infty);X)}.$$

So, either for some $t_k \to +\infty$, $\|Q(v) - Q(w)\|_{C_b([-\tau,+\infty);X)} = \lim_k \|Q(v)(t_k) - Q(w)(t_k)\|$, situation analyzed in **Case 3.** below, or there exists $t_1 \geq a > 0$ such that

$$\|Q(v) - Q(w)\|_{C_b([-\tau,+\infty);X)} = \|Q(v)(t_1) - Q(w)(t_1)\| = \|Q(v) - Q(w)\|_{C_b([a,+\infty);X)}$$

$$\leq e^{-\omega t_1}\|Q(v) - Q(w)\|_{C_b([-\tau,+\infty);X)} + \frac{\ell}{\omega}(1 - e^{-\omega t_1})\|v - w\|_{C_b([-\tau,+\infty);X)}.$$

Since $t_1 > 0$, this yields

$$\|Q(v) - Q(w)\|_{C_b([-\tau,+\infty);X)} \leq \frac{\ell}{\omega}\|v - w\|_{C_b([-\tau,+\infty);X)}. \tag{5.7}$$

But by (c_1) $0 < \frac{\ell}{\omega} < 1$, and this proves that Q is a strict contraction.

Case 2. There exists $t_1 > 0$ such that

$$\|Q(v) - Q(w)\|_{C_b([-\tau,+\infty);X)} = \|Q(v)(t_1) - Q(w)(t_1)\|.$$

Reasoning as before, we conclude again that Q is a strict contraction.

Case 3. If for each $t \in [-\tau, +\infty)$

$$\|Q(v)(t) - Q(w)(t)\| < \|Q(v) - Q(w)\|_{C_b([-\tau,+\infty);X)},$$

then there exists $(t_k)_k$ with $\lim_k t_k = +\infty$ and such that

$$\lim_k \|Q(v)(t_k) - Q(w)(t_k)\| = \|Q(v) - Q(w)\|_{C_b([-\tau,+\infty);X)}.$$

Setting $t = t_k$ in (5.6) and letting $k \to +\infty$, we get (5.7). Since the global uniform asymptotic stability follows as in the case of Theorem 3.3, this completes the proof. □

6. EXAMPLES

Example 6.1. Let Ω be a nonempty, bounded and open subset in $\mathbb{R}^d$, $d \geq 1$, with C^1 boundary Γ and let let $\varphi : D(\varphi) \subseteq \mathbb{R} \rightsquigarrow \mathbb{R}$ be maximal monotone with $0 \in \varphi(0)$. Let us consider the porous medium equation subjected to nonlocal initial conditions

$$\tag{6.1} \begin{cases} \dfrac{\partial u}{\partial t}(t,x) \in \Delta\varphi(u(t,x)) - \omega u(t,x) + h(t,u)(x) \ \text{ in } \ \mathbb{R}_+ \times \Omega \\ h(t,u)(x) = f(t,x,u(t,x),u(t-\tau_1,x),\dots,u(t-\tau_n,x)) \ \text{ in } \ \mathbb{R}_+ \times \Omega \\ \varphi(u(t,x)) = 0 \ \text{ on } \ \mathbb{R}_+ \times \Gamma \\ u(t,x) = \displaystyle\int_\tau^{+\infty} \mathcal{N}(u(\theta+t,\cdot))(x)\, d\mu(\theta) \ \text{ in } \ [-\tau, 0\,] \times \Omega, \end{cases}$$

where $\tau = \max\{\tau_1, \tau_2, \dots, \tau_n\}$.

Let Δ be the Laplace operator in the sense of distributions over Ω. If $\varphi : D(\varphi) \subseteq \mathbb{R} \rightsquigarrow \mathbb{R}$, and $u : \Omega \to D(\varphi)$, we denote by

$$\mathcal{S}_\varphi(u) = \{v \in L^1(\Omega);\ v(x) \in \varphi(u(x)),\ \text{a.e. for } x \in \Omega\}.$$

The (i) part in the next result is essentially due to Brezis, Strauss [11] while the (ii) part to Badii, Diaz, Tesei [6].

Theorem 6.2. *Let Ω be a nonempty, bounded and open subset in $\mathbb{R}^d$ with C^1 boundary Γ and let $\varphi : D(\varphi) \subseteq \mathbb{R} \rightsquigarrow \mathbb{R}$ be maximal monotone with $0 \in \varphi(0)$.*

(i) *Then the operator $\Delta\varphi : D(\Delta\varphi) \subseteq L^1(\Omega) \rightsquigarrow L^1(\Omega)$, defined by*

$$\begin{cases} D(\Delta\varphi) = \{u \in L^1(\Omega);\ \exists v \in \mathcal{S}_\varphi(u) \cap W_0^{1,1}(\Omega),\ \Delta v \in L^1(\Omega)\} \\ \Delta\varphi(u) = \{\Delta v;\ v \in \mathcal{S}_\varphi(u) \cap W_0^{1,1}(\Omega)\} \cap L^1(\Omega) \text{ for } u \in D(\Delta\varphi), \end{cases}$$

is m-dissipative on $L^1(\Omega)$.

(ii) *If, in addition, $\varphi : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and C^1 on $\mathbb{R} \setminus \{0\}$ and there exist two constants $C > 0$ and $a > 0$ if $d \leq 2$ and $a > (d-2)/d$ if $d \geq 3$ such that*

$$\varphi'(r) \geq C|r|^{a-1}$$

for each $r \in \mathbb{R} \setminus \{0\}$, then $\Delta\varphi$ generates a compact semigroup.

Theorem 6.3. *Let Ω be a nonempty, bounded and open subset in $\mathbb{R}^d$ with C^1 boundary Γ, let $\omega > 0$ and let $\varphi : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$ and C^1 on $\mathbb{R} \setminus \{0\}$ and for which there exist two constants $C > 0$ and $a > 0$ if $d \leq 2$ and $a > (d-2)/d$ if $d \geq 3$ such that*

$$\varphi'(r) \geq C|r|^{a-1}$$

for each $r \in \mathbb{R} \setminus \{0\}$. Let $f : \mathbb{R}_+ \times \Omega \times \mathbb{R}^{n+1} \to \mathbb{R}$ be such that:

(F_1) *the function $(t, x, u) \mapsto f(t, x, u)$ is continuous on $\mathbb{R}_+ \times \Omega \times \mathbb{R}^{n+1}$;*
(F_2) *there exist $\ell_k > 0$, $k = 0, 1, \ldots, n$ and $m > 0$ such that*

$$|f(t, x, u)| \leq \sum_{k=0}^{n} \ell_k |u_k| + m$$

for each $(t, x, u) \in \mathbb{R}_+ \times \Omega \times \mathbb{R}^{n+1}$;
(F_4) $\sum_{k=0}^{n} \ell_k < \omega$.

Let $\mathcal{N} : L^1(\Omega) \to L^1(\Omega)$ be a nonexpansive operator with $\mathcal{N}(0) = 0$ and let μ be a finite and complete measure on $[\tau, +\infty)$ with $\lim_{\delta \downarrow 0} \mu([\tau, \delta + \tau]) = 0$ and $\mu([0, +\infty)) = 1$. Then the problem (6.1) has at least one C^0-solution $u \in C_b([-\tau, +\infty); L^1(\Omega))$.

Proof. Let $X = L^1(\Omega)$ and let $A : D(A) \subseteq X \rightsquigarrow X$ be defined as $A = \Delta\varphi$ where $\Delta\varphi$ is defined as in Theorem 6.2. Let $f : \mathbb{R}_+ \times L^1(\Omega) \to L^1(\Omega)$ be defined as $f(t, u)(x) = f(t, x, u(x))$ for each $t \in \mathbb{R}_+$ each $u \in L^1(\Omega)$ and a.e. for $x \in \Omega$ and $g : C([0, 2\pi]; L^1(\Omega) \to C([-\tau, 0]; L^1(\Omega)$ be defined by

$$[g(u)(t)](x) = \int_0^{+\infty} \mathcal{N}(u(\theta + t, \cdot))(x)\, d\mu(\theta)$$

for each $t \in [-\tau, 0]$, each $u \in L^1(\Omega)$ and a.e. for $x \in \Omega$. With the notations above the problem (6.1) can be rewritten in the abstract form (1.1). In view of Theorem 6.2 A satisfies (a_1) and (a_2). Moreover from (F_1) and (F_2), we deduce that f satisfies (f_1) while from the fact that $\mathcal{N}$ is nonexpansive, $\mathcal{N}(0) = 0$, $\lim_{\delta \downarrow 0} \mu([\tau, \tau + \delta]) = 0$, we conclude that g satisfies (g_1), (g_2) and (g_3). So Theorem 3.2 applies and this completes the proof. □

Theorem 6.4. *Let Ω be a nonempty, bounded and open subset in $\mathbb{R}^d$ with C^1 boundary Γ, let $\omega > 0$ and let $\varphi : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$ and C^1 on $\mathbb{R} \setminus \{0\}$ and for which there exist two constants $C > 0$ and $a > 0$ if $d \leq 2$ and $a > (d-2)/d$ if $d \geq 3$ such that*

$$\varphi'(r) \geq C|r|^{a-1}$$

for each $r \in \mathbb{R} \setminus \{0\}$. Let $f : \mathbb{R}_+ \times \Omega \times \mathbb{R}^{n+1} \to \mathbb{R}$ be such that:

(F_1) *the function $(t, x, u) \mapsto f(t, x, u)$ is continuous on $\mathbb{R}_+ \times \Omega \times \mathbb{R}^{n+1}$;*

$(\widetilde{F}_2)$ *there exist* $\ell_k > 0$, $k = 0, 1, \ldots, n$ *such that*

$$|f(t,x,u) - f(t,x,v)| \le \sum_{k=0}^{n} \ell_k |u_k - v_k|$$

for each $(t,x,u), (t,x,v) \in \mathbb{R}_+ \times \Omega \times \mathbb{R}^{n+1}$;

(F_3) $t \mapsto \int_\Omega |f(t,x,0)|\,dx$ *is bounded on* $\mathbb{R}_+$;

(F_4) $\sum_{k=0}^{n} \ell_k < \omega$.

Let $\mathcal{N} : L^1(\Omega) \to L^1(\Omega)$ *be a nonexpansive operator with* $\mathcal{N}(0) = 0$ *and let* μ *be a finite and complete measure on* $[\tau, +\infty)$ *with* $\lim_{\delta \downarrow 0} \mu([\tau, \delta + \tau]) = 0$ *and* $\mu([0, +\infty)) = 1$. *Then the problem* (6.1) *has at least one* C^0*-solution* $u \in C_b([-\tau, +\infty); L^1(\Omega))$. *If, instead of* (F_4) *the stronger nonresonance condition*

(F_5) $\sum_{k=0}^{n} \ell_k e^{\omega \tau_k} < \omega$;

is satisfied, then each C^0*-solution of* (6.1) *is globally asymptotically stable.*

Proof. The proof follows the same lines as the preceding one except that instead of Theorem 3.2, we have to use Theorem 3.3. □

Example 6.5. Finally, let us consider the nonlinear parabolic equation subjected to periodic conditions

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) \in \Delta\varphi(u(t,x)) - \omega u(t,x) + h(t,u)(x) & \text{in } \mathbb{R} \times \Omega \\ h(t,u)(x) = f(t,x,u(t,x),u(t-\tau_1,x),\ldots,u(t-\tau_n,x)) & \text{in } \mathbb{R} \times \Omega \\ \varphi(u(t,x)) = 0 & \text{on } \mathbb{R} \times \Gamma \\ u(t,x) = u(t+2\pi,x) & \text{in } \mathbb{R} \times \Omega. \end{cases} \tag{6.2}$$

Here, the equation could be degenerate since $\varphi : D(\varphi) \subseteq \mathbb{R} \rightsquigarrow \mathbb{R}$ is allowed to be non-strictly increasing. Moreover, if this is the case, the semigroup generated by $A = \Delta\varphi$ is not compact. See Remark 2.7.1, p. 72 in Vrabie [32].

Theorem 6.6. *Let* Ω *be a nonempty, bounded and open subset in* $\mathbb{R}^d$ *with* C^1 *boundary* Γ, *let* $\omega > 0$ *and let* $\varphi : D(\varphi) \subseteq \mathbb{R} \rightsquigarrow \mathbb{R}$ *be maximal monotone with* $0 \in \varphi(0)$. *Let* $f : \mathbb{R} \times \Omega \times \mathbb{R}^{n+1} \to \mathbb{R}$ *be* 2π*-periodic with respect to its first argument and such that*:

(F_1) *the function* $(t,x,u) \mapsto f(t,x,u)$ *is continuous on* $\mathbb{R} \times \Omega \times \mathbb{R}^{n+1}$;

(F_2) *there exist* $\ell_k > 0$, $k = 0, 1, \ldots, n$ *such that*

$$|f(t,x,u) - f(t,x,v)| \le \sum_{k=0}^{n} \ell_k |u_k - v_k|$$

for each $(t,x,u), (t,x,v) \in \mathbb{R}_+ \times \Omega \times \mathbb{R}^{n+1}$;

(F_3) $t \mapsto \int_\Omega |f(t,x,0)|\,dx$ *is bounded on* $\mathbb{R}_+$;

(F_4) $\sum_{k=0}^{n} \ell_k < \omega$.

Then the problem (6.2) *has unique* 2π*-periodic* C^0*-solution* $u \in C(\mathbb{R}; L^1(\Omega))$. *If, instead of* (F_4) *the stronger nonresonance condition*

(F_5) $\sum_{k=0}^{n} \ell_k e^{\omega \tau_k} < \omega$;

is satisfied, then each C^0*-solution of* (6.2) *is globally asymptotically stable.*

Proof. The conclusion follows from Theorem 3.4. □

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REFERENCES

[1] S. Aizicovici and H. Lee, Nonlinear nonlocal Cauchy problems in Banach spaces, *Applied Math. Letters*, **18**(2005), 401–407.

[2] S. Aizicovici and M. McKibben, Existence results for a class of abstract nonlocal Cauchy problems, *Nonlinear Anal.*, **39**(2000), 649–668.

[3] S. Aizicovici, N. S. Papageorgiou and V. Staicu, Periodic solutions of nonlinear evolution inclusions in Banach spaces, *J. Nonlin. & Convex. Anal.*, **7**(2006), 163–177.

[4] S. Aizicovici, N. H. Pavel and I. I. Vrabie, Anti-periodic solutions to strongly nonlinear evolution equations in Hilbert spaces, *An. Ştiinţ. Univ. "Al. I. Cuza" Iaşi, Secţ. I a Mat.*, **XLIV**, (1998), 227–234.

[5] S. Aizicovici and V. Staicu, Multivalued evolution equations with nonlocal initial conditions in Banach spaces, *Nonlinear Differential Equations Appl. NoDEA*, **14**(2007), 361–376.

[6] M. Badii, J. I. Diaz and A. Tesei, Existence and attractivity for a class of degenerate functional parabolic problems, *Rend. Sem. Mat. Univ. Padova*,**78**(1987), 109–124.

[7] P. Baras, Compacité de l'opérateur definissant la solution d'une équation d'évolution non linéaire $(du/dt) + Au \ni f$, *C. R. Acad. Sci. Sér. I Math.*, **286**(1978), 1113-1116.

[8] V. Barbu, *Nonlinear semigroups and differential equation in Banach spaces*, Editura Academiei, Bucureşti, Noordhoff, 1976.

[9] V. Barbu, *Nonlinear Differential Equations of Monotone Type in Banach Spaces*, Springer Monographs in Mathematics, Springer Verlag, 2010.

[10] H. Brezis, Propriétés régularisantes de certains semi-groupes non linéaires, *Israel J. Math.*, **9** (1971), 513–34.

[11] H. Brezis and W. Strauss, Semilinear elliptic equations in L^1, *J. Math. Soc. Japan*, **25**(1973), 565-590.

[12] L. Byszewski, Theorems about the existence and uniqueness of solutions of semilinear evolution nonlocal Cauchy problems, *J. Math. Anal. Appl.*, **162**(1991), 494–505.

[13] R. Caşcaval and I. I. Vrabie, Existence of periodic solutions for a class of nonlinear evolution equations, *Rev. Mat. Univ. Complut. Madrid*, **7**(1994), 325–338.

[14] C. Castaing and D. P. Monteiro-Marques, Periodic solutions of evolution problems associated with a moving convex set, *C. R. Acad. Sci. Paris, Série A*, **321**(1995), 531–536.

[15] K. Deng, exponential decay of solutions of semilinear parabolic equations with initial boundary conditions, *J. Math. Anal. Appl.*, **179**(1993), 630–637.

[16] J. García-Falset, Existence results and asymptotic behaviour for nonlocal abstract Cauchy problems, *J. Math. Anal. Appl.*, **338**(2008), 639–652.

[17] J. García-Falset and S. Reich, Integral solutions to a class of nonlocal evolution equations, *Communications in Contemporary Mathematics*, **12**(2010), 1032–1054.

[18] N. Hirano, Existence of periodic solutions for nonlinear evolution equations in Hilbert spaces, *Proc. Amer. Math. Soc.*, **120**(1994), 185–192.

[19] N. Hirano and N. Shioji, Invariant sets for nonlinear evolution equations, Cauchy problems and periodic problems, *Abstract and Applied Analysis*, **3** (2004), 183–203.

[20] Shuchuan Hu and N. S. Papageorgiou, *On the existence of periodic solutions for a class of nonlinear inclusions*, Bollettino UMI, (71 7-B)(1993), 591–605.

[21] Shuchuan Hu and N. S. Papageorgiou, *Handbook of multivalued analysis*, Mathematics and Its Applications, Kluwer Academic Publishers, 2000.

[22] V. Lakshmikantham and S. Leela, *Nonlinear differential equations in abstract spaces*, International Series in Nonlinear Mathematics, **2**, Pergamon Press, 1981.

[23] V. Lakshmikantham and N. S. Papageorgiou, Periodic solutions of nonlinear evolution inclusions, *Computational and Applied Math.*, **52**(1994), 277–286.

[24] Yongxiang Li, Existence and uniqueness of periodic solution for a class of semilinear evolution equations, *J. Math. Anal. Appl.* **349**(2009) 226-234.

[25] Yongxiang Li, Existence and asymptotic stability of periodic solution for evolution equations with delays, *Journal of Functional Analysis*, in print.

[26] A. Paicu, Periodic solutions for a class of differential inclusions in general Banach spaces, *J. Math. Anal. Appl.* **337**(2008), 1238–1248.

[27] A. Paicu, Periodic solutions for a class of nonlinear evolution equations in Banach spaces, *An. Ştiinţ. "Al. I. Cuza", Iaşi, Ser. Nouă Mat.*, **LV**(2009), p. 107–118.

[28] A. Paicu and I. I. Vrabie, A class of nonlinear evolution equations subjected to nonlocal initial conditions, *Nonlinear Anal.*, **72**(2010), 4091–4100.

[29] N. S. Papageorgiou, Periodic trajectories for evolution inclusions associated with time-dependent subdifferentials, *Ann. Univ. Sci. Budapest*, **37**(1994), 139–155.

[30] I. I. Vrabie, The nonlinear version of Pazy's local existence theorem, *Israel J. Math.*, **32** (1979), 221–235.

[31] I. I. Vrabie, Periodic solutions for nonlinear evolution equations in a Banach space, *Proc. Amer. Math. Soc.*, **109(3)**(1990), 653-661.

[32] I. I. Vrabie, *Compactness methods for nonlinear evolutions*, Pitman Monographs and Surveys in Pure and Applied Mathematics, Second Edition **75**, Longman and John Wiley & Sons, 1995.

[33] I. I. Vrabie, C_0*-semigroups and applications*, North-Holland Mathematics Studies **191**, Elsevier, 2003.

[34] I. I. Vrabie, *Differential Equations. Basic Concepts, Results and Applications*, Second Edition, World Scientific, 2011.