Dynamic Systems and Applications 21 (2012) 441-456

ON EXISTENCE OF SOLUTIONS TO NONLINEAR OPTIMAL CONTROL SYSTEMS

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ABSTRACT. In the paper we study an optimal control problem governed by the system of second order ordinary differential equations in potential form. By means of variational methods we prove the theorem on the existence of global solutions to a control problem. Moreover, the theorem on the existence of optimal solution to Bolza problem is presented.

Keywords. Bolza problem, saddle point, existence of optimal solutions, Γ-convergence, epi/hypo-convergence

AMS (MOS) Subject Classification. 49J15, 34K10, 34K35, 93C15

1. INTRODUCTION

A classical control problem can be described by the following system

$$\dot{x}(t) = f(t, x(t), u(t)), \ t \in [0,T] \tag{1.1}$$

where $f : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^n$, U is a fixed subset of $\mathbb{R}^m$, x is an absolutely continuous function, $u \in L^\infty([0,T], U) = \mathcal{U}$, and $T > 0$ is a fixed terminal time. We seek the control u that steers a solution of system (1.1) from a state $x(0) \in Z_0$ to a state $x(T) \in Z_T$ and minimizes a cost functional $I(x,u)$ where Z_0, Z_T are some fixed subsets of $\mathbb{R}^n$.

The cost functional may have, for example, the following form

$$I(x,u) = g^0(x(0), x(T)) + \int_0^T f^0(t, x(t), u(t))\, dt \tag{1.2}$$

where $f^0 : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}$ and $g^0 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. In the theory of optimal control problem (1.1)–(1.2) is known as an optimal control problem of Bolza type.

Let us denote by $\bar{\mathcal{X}}$ the set of all trajectories of system (1.1) such that there exist a control $u \in \mathcal{U}$ for which a corresponding trajectory x_u is a global solution on the interval $[0,T]$ satisfying the conditions

$$x_u(0) \in Z_0 \text{ and } x_u(T) \in Z_T. \tag{1.3}$$

Received November 15, 2011

Obviously, only if the set $\bar{\mathcal{X}}$ is nonempty can we look for an optimal control. Moreover, the assumption on boundness of the set $\bar{\mathcal{X}}$ plays an essential role in the theorem on the existence of optimal processes to the problem (1.1)–(1.3). However, for nonlinear systems this kind of assumption cannot be easily verified. There is one exception to this rule, i.e. the situation where the function f satisfies the condition

$$\langle x, f(t,x,u)\rangle = \sum_{i=1}^{n} x^i f^i(t,x,u) \leq \alpha\left(1+|x|^2\right) \tag{1.4}$$

for some constant $\alpha > 0$ (cf. [9, Chapter 9.4]). The condition (1.4) appears in many papers concerning the problem of the existence of optimal processes (see for example [4, 5, 6] and references therein).

In the present paper we consider a control problem governed by the system of second order ordinary differential equations with some potential vector field f. It should be emphasized that such systems are of Newton type. By means of variational methods, applying Ky-Fan's theorem on the existence of saddle points of an appropriate functional (cf. [16]) and some results from Γ-convergence theory (see Section 4 for details), we prove under some appropriate assumptions that

- for an admissible control u there is the unique trajectory x_u of the control problem defined on the whole interval $[0,T]$ (see Theorem 3.1, Section 3),

- the set of all trajectories of the control problem are commonly bounded, i.e. there exist a constant $c > 0$ such that $|x_u(t)| \leq c$ for $t \in [0,T]$ and an admissible control u (see Theorem 3.1, Section 3),

- a trajectory x_u depends continuously on the control u with respect to strong and weak topologies in the set of controls (see Theorems 4.1 and 4.2, Section 4).

It should be stressed that in all aforementioned propositions we have not assumed that the function on the right hand side of the control problem satisfies the condition (1.4). Finally, in the last section, we prove the theorem on the existence of optimal solution to a problem of Bolza type, where a cost functional has the form defined in (1.2). Moreover, some example illustrating the main result of the paper is provided.

2. FORMULATION OF THE PROBLEM AND SOME BASIC ASSUMPTIONS

We consider the following system of second order equations

$$\begin{cases} \ddot{x}^1(t) &= \varphi^1\left(t, x^1(t), x^2(t), u(t)\right), \\ \ddot{x}^2(t) &= -\varphi^2\left(t, x^1(t), x^2(t), u(t)\right), \end{cases} \tag{2.1}$$

with the boundary conditions

$$\begin{aligned} x^1(0) = x_0^1,\ x^1(T) = x_T^1, \\ x^2(0) = x_0^2,\ x^2(T) = x_T^2, \end{aligned} \tag{2.2}$$

where $x_0^1, x_T^1 \in \mathbb{R}^{n_1}$, $x_0^2, x_T^2 \in \mathbb{R}^{n_2}$ are fixed and $x^1(t) \in \mathbb{R}^{n_1}$, $x^2(t) \in \mathbb{R}^{n_2}$, $\varphi^1 : [0,T] \times \mathbb{R}^{n_1+n_2} \times M \to \mathbb{R}^{n_1}$, $\varphi^2 : [0,T] \times \mathbb{R}^{n_1+n_2} \times M \to \mathbb{R}^{n_2}$, $M \subset \mathbb{R}^m$ is compact, $u \in \mathcal{U} := \{u \in L^\infty([0,T], \mathbb{R}^m) : u(t) \in M \text{ for a.e. } t \in [0,T]\}$.

In what follows, we assume that the vector field $\varphi := (\varphi^1, \varphi^2)$ is potential, i.e. there exists a function $\Phi : [0,T] \times \mathbb{R}^{n_1+n_2} \times M \to \mathbb{R}$ such that $\Phi_{x^1} = \varphi^1$ and $\Phi_{x^2} = \varphi^2$. For convenience we shall sometimes write x, x_0, x_T instead of (x^1, x^2), (x_0^1, x_0^2), (x_T^1, x_T^2), respectively. We require that the functions φ and Φ meet the following assumptions:

(A1): φ, Φ are Carathéodory functions, i.e. they are measurable with respect to t for any $(x,u) \in \mathbb{R}^{n_1+n_2} \times \mathbb{R}^m$ and continuous with respect to (x,u) for a.e. $t \in [0,T]$;

(A2): there are functions $a_0, a_1 \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $b_0, b_1 \in L^2([0,T], \mathbb{R}^+)$ such that

$$|\varphi(t,x,u)| \le a_0(|x|)\, b_0(t)$$
$$|\Phi(t,x,u)| \le a_1(|x|)\, b_1(t)$$

for a.e. $t \in [0,T]$, and every $x \in \mathbb{R}^{n_1+n_2}$, $u \in M$;

(A3):

(a) for any $x^2 \in H^1([0,T], \mathbb{R}^{n_2})$, there is a constant $\alpha_1 \in \left(0, \frac{1}{2}\frac{\pi^2}{T^2}\right)$ and functions $\beta_1 \in L^2([0,T], \mathbb{R}^{n_1})$, $\gamma_1 \in L^1([0,T], \mathbb{R})$, such that

$$\Phi\left(t, x^1, x^2(t), u\right) \ge -\alpha_1 \left|x^1\right|^2 + \left(\beta_1(t), x^1\right) + \gamma_1(t)$$

for a.e. $t \in [0,T]$, and every $x^1 \in \mathbb{R}^{n_1}$, $u \in M$,

(b) for any $x^1 \in H^1([0,T], \mathbb{R}^{n_1})$, there is a constant $\alpha_2 \in \left(0, \frac{1}{2}\frac{\pi^2}{T^2}\right)$ and functions $\beta_2 \in L^2([0,T], \mathbb{R}^{n_2})$, $\gamma_2 \in L^1([0,T], \mathbb{R})$, such that

$$\Phi\left(t, x^1(t), x^2, u\right) \le \alpha_2 \left|x^2\right|^2 - \left(\beta_2(t), x^2\right) + \gamma_2(t)$$

for a.e. $t \in [0,T]$, and every $x^2 \in \mathbb{R}^{n_2}$, $u \in M$;

(A4):

(a) for any $x^2 \in H^1([0,T], \mathbb{R}^{n_2})$, the function

$$\psi_1(x^1) := \Phi\left(t, x^1, x^2(t), u\right) + \alpha_1 \left|x^1\right|^2, \ x^1 \in \mathbb{R}^{n_1}$$

is convex for a.e. $t \in [0,T]$, and every $u \in M$, where α_1 is defined in (A3),

(b) for any $x^1 \in H^1([0,T], \mathbb{R}^{n_1})$, the function

$$\psi_2(x^2) := \Phi\left(t, x^1(t), x^2, u\right) - \alpha_2 \left|x^2\right|^2, \ x^2 \in \mathbb{R}^{n_2}$$

is concave for a.e. $t \in [0,T]$, and every $u \in M$, where α_2 is defined in (A3).

3. CONTROL PROBLEM

We first show that there exists a solution to the system (2.1)–(2.2) in the space $H^2([0,T],\mathbb{R}^{n_1}) \times H^2([0,T],\mathbb{R}^{n_2})$ such that

$$x^1 = y + \lambda_1,$$
$$x^2 = z + \lambda_2,$$

where $y \in H_0^1([0,T],\mathbb{R}^{n_1})$, $z \in H_0^1([0,T],\mathbb{R}^{n_2})$ and the functions λ_1, λ_2 are given by

$$\lambda_1(t) := \frac{x_T^1 - x_0^1}{T} t + x_0^1,$$
$$\lambda_2(t) := \frac{x_T^2 - x_0^2}{T} t + x_0^2.$$

Moreover, denote by H the space $H_0^1([0,T],\mathbb{R}^{n_1}) \times H_0^1([0,T],\mathbb{R}^{n_2})$ with the norm

$$\|x\| = \left\|\left(x^1, x^2\right)\right\| = \sqrt{\|x^1\|^2 + \|x^2\|^2}.$$

The problem of the existence of a solution to the system (2.1) satisfying conditions (2.2) and some its properties are determined by the following theorem.

Theorem 3.1. *Let the functions φ, Φ satisfy* (A1)–(A4). *Then for any $u \in \mathcal{U}$ and any $x_0, x_T \in \mathbb{R}^{n_1+n_2}$ there exists the unique solution $x_{x_0,x_T,u}$ to the system (2.1)–(2.2). Moreover, for any $c > 0$ there is a constant $\bar{c} > 0$ such that if $|x_0^i| \le c$, $|x_T^i| \le c$, $i = 1, 2$, and $u \in \mathcal{U}$, then the unique solution $x_{x_0,x_T,u} \in H^2([0,T],\mathbb{R}^n)$ satisfies $|x_{x_0,x_T,u}(t)| \le \bar{c}$ for $t \in [0,T]$.*

Proof. Fix $u \in \mathcal{U}$, $x_0^1, x_T^1 \in \mathbb{R}^{n_1}$, $x^2 \in H^1([0,T],\mathbb{R}^{n_2})$. Let

$$B_u^1 : H_0^1([0,T],\mathbb{R}^{n_1}) \to \mathbb{R}$$

be a functional of the form

$$B_u^1(y) := \int_0^T \left(\frac{1}{2}\left|\dot{y}(t) + \dot{\lambda}_1(t)\right|^2 + \Phi\left(t, y(t) + \lambda_1(t), x^2(t), u(t)\right)\right) dt.$$

Let us observe that the functional B_u^1 is coercive. To demonstrate this, we use (A3), the Poincaré inequality

$$\int_0^T |y(t)|^2\, dt \le \left(\frac{T}{\pi}\right)^2 \int_0^T |\dot{y}(t)|^2\, dt$$

and the Schwarz inequality to obtain the estimate

$$B_u^1(y) \ge \left(\frac{1}{2} - \alpha_1 \left(\frac{T}{\pi}\right)^2\right) \|y\|^2 + \delta_1 \|y\| + \delta_2, \tag{3.1}$$

where δ_1, δ_2 are some constants depending on x_0^1, x_T^1 and x^2. Since

$$\frac{1}{2} - \alpha_1 \left(\frac{T}{\pi}\right)^2 > 0,$$

the functional B_u^1 is coercive.

The weak convergence in $H_0^1([0,T],\mathbb{R}^n)$ implies the uniform convergence (see [14, Lemma 1.2]), and therefore from (A2) and the Lebesgue dominated convergence theorem we deduce that B_u^1 is weakly lower semicontinous. Furthermore, the functional

$$H_0^1([0,T],\mathbb{R}^{n_1}) \ni y \mapsto \int_0^T \left(\frac{1}{2}|\dot{y}(t)|^2 - \alpha_1 |y(t)|^2\right) dt$$

is strictly convex (see [14] for details), and naturally, the functional

$$H_0^1([0,T],\mathbb{R}^{n_1}) \ni y \mapsto \int_0^T \left(\frac{1}{2}\left|\dot{y}(t) + \dot{\lambda}_1(t)\right|^2 - \alpha_1 |y(t) + \lambda_1(t)|^2\right) dt \in \mathbb{R}$$

is strictly convex since its second derivative is a positive definite operator (see [18] for details). Consequently, by the assumption (A4), the functional

$$\begin{aligned} B_u^1(y) &= \int_0^T \left(\Phi\left(t, y(t) + \lambda_1(t), x^2(t), u(t)\right) + \alpha_1 |y(t) + \lambda_1(t)|^2\right) dt \\ &\quad + \int_0^T \left(\frac{1}{2}\left|\dot{y}(t) + \dot{\lambda}_1(t)\right|^2 - \alpha_1 |y(t) + \lambda_1(t)|^2\right) dt \end{aligned}$$

is strictly convex. Similarly, for fixed $u \in \mathcal{U}$, $x_0^2, x_T^2 \in \mathbb{R}^{n_2}$ and $x^1 \in H^1([0,T],\mathbb{R}^{n_1})$ the functional $B_u^2 : H_0^1([0,T],\mathbb{R}^{n_2}) \to \mathbb{R}$ defined by

$$B_u^2(z) := \int_0^T \left(\frac{1}{2}\left|\dot{z}(t) + \dot{\lambda}_2(t)\right|^2 - \Phi\left(t, x^1(t), z(t) + \lambda_2(t), u(t)\right)\right) dt,$$

is weakly lower semicontinuous, strictly convex and coercive. Thus, for the functional

$$\begin{aligned} A_u(y,z) &= \int_0^T \left(\frac{1}{2}\left|\dot{y}(t) + \dot{\lambda}_1(t)\right|^2 - \frac{1}{2}\left|\dot{z}(t) + \dot{\lambda}_2(t)\right|^2\right) dt \\ &\quad + \int_0^T \Phi(t, y(t) + \lambda_1(t), z(t) + \lambda_2(t), u(t))\, dt \end{aligned} \tag{3.2}$$

there exists a saddle point $(y_0, z_0) \in H$ (see [16] and [19, Theorem 6]). Additionally, $A_u(\cdot, z)$ is strictly convex for each $z \in H_0^1([0,T],\mathbb{R}^{n_2})$ and $A_u(y,\cdot)$ is strictly concave for each $y \in H_0^1([0,T],\mathbb{R}^{n_1})$, and therefore the saddle point is unique. To sum up, there exists exactly one point $(y_0, z_0) \in H$ such that

$$A_u(y, z_0) \leq A_u(y_0, z_0) \leq A_u(y_0, z)$$

for all $y \in H_0^1([0,T],\mathbb{R}^{n_1})$ and $z \in H_0^1([0,T],\mathbb{R}^{n_2})$. It is easily seen that the system

$$\begin{cases} \ddot{y}^1(t) &= \varphi^1(t, y(t) + \lambda_1(t), z(t) + \lambda_2(t), u(t)), \\ \ddot{z}^2(t) &= -\varphi^2(t, y(t) + \lambda_1(t), z(t) + \lambda_2(t), u(t)), \end{cases} \tag{3.3}$$

is the Euler-Lagrange equation for the functional (3.2). By the assumption (A4), the only solution to the above system is the saddle point of the functional (3.2). Next, it is easy to check that A_u is Gâteaux differentiable and the equation

$$D_G A_u(y_0, z_0) = 0$$

implies

$$\int_0^T \left(\dot{y}_0(t) + \dot{\lambda}_1(t), \dot{h}(t)\right) dt =$$

$$-\int_0^T \left(\Phi_{x^1}(t, y_0(t) + \lambda_1(t), z_0(t) + \lambda_2(t), u(t)), h(t)\right) dt$$

for all $h \in H_0^1([0,T], \mathbb{R}^{n_1})$. By the du Bois-Reymond lemma (cf. [14]), we get that there is a constant vector $\hat{c} \in \mathbb{R}^{n_1}$ such that

$$\dot{y}_0(t) + \dot{\lambda}_1(t) = \int_0^t \Phi_{x^1}(s, y_0(t) + \lambda_1(t), z_0(t) + \lambda_2(t), u(s))\, ds + \hat{c}$$

for a.e. $t \in [0,T]$. Applying the same reasoning to the function z_0, we get that $x^1_{x^1_0, x^1_T, u} := y_0 + \lambda_1 \in H^2([0,T], \mathbb{R}^{n_1})$, $x^2_{x^2_0, x^2_T, u} := z_0 + \lambda_2 \in H^2([0,T], \mathbb{R}^{n_2})$ satisfy the system (2.1)–(2.2) and $(x^1_{x^1_0, x^1_T, u}, x^2_{x^2_0, x^2_T, u})$ is the unique solution to this system.

Furthermore, choose some $c > 0$ such that x^i_0 and x^i_T are bounded as follows $|x^i_0| \leq c$ and $|x^i_T| \leq c$, $i = 1, 2$. Then for each solution $(x^1_{x^1_0, x^1_T, u}, x^2_{x^2_0, x^2_T, u})$ to the system (2.1)–(2.2), the point $(x^1_{x^1_0, x^1_T, u} - \lambda_1, x^2_{x^2_0, x^2_T, u} - \lambda_2)$ is the saddle point of the functional (3.2). Defining $y_0 = x^1_{x^1_0, x^1_T, u} - \lambda_1$, $z_0 = x^2_{x^2_0, x^2_T, u} - \lambda_2$ and using the estimate in (3.1), we obtain

$$\begin{aligned} A_u(y_0, z_0) &= \max_z A_u(y_0, z) \geq A_u(y_0, 0) \\ &\geq \left(\frac{1}{2} - \alpha_1 \left(\frac{T}{\pi}\right)^2\right) \|y_0\|^2 + \delta_1 \|y_0\| + \delta_2 =: g_1(y_0), \end{aligned}$$

where now δ_1 and δ_2 depend on the choice of the constant vector c. Moreover,

$$\begin{aligned} A_u(y_0, z_0) &= \min_y A_u(y, z_0) \leq A_u(0, z_0) \\ &\leq \left(\alpha_2 \left(\frac{T}{\pi}\right)^2 - \frac{1}{2}\right) \|z_0\|^2 + \gamma_1 \|z_0\| + \gamma_2 =: g_2(z_0) \end{aligned}$$

and again γ_1 and γ_2 depend on the choice of the constant vector c. The functions $g_1 : H_0^1([0,T], \mathbb{R}^{n_1}) \to \mathbb{R}$, $-g_2 : H_0^1([0,T], \mathbb{R}^{n_2}) \to \mathbb{R}$ are coercive and bounded below, and therefore there exists a constant $\tilde{c} > 0$ independent of $x^1_0, x^1_T, x^2_0, x^2_T$ and u such that

$$\begin{aligned} \|y_0\| &\leq \tilde{c}, \\ \|z_0\| &\leq \tilde{c}. \end{aligned}$$

Finally, we have

$$\begin{aligned} \left|x^1_{x^1_0, x^1_T, u}(t)\right| &\leq |y_0(t)| + |\lambda_1(t)| \leq \int_0^t |\dot{y}_0(s)|\, ds + 2c \\ &\leq \int_0^T |\dot{y}_0(s)|\, ds + 2c \leq \sqrt{T}\, \|y_0\| + 2c \leq \sqrt{T}\tilde{c} + 2c =: \bar{c} \end{aligned}$$

for $t \in [0,T]$. The same conclusion can be drawn for $x^2_{x^2_0, x^2_T, u}$, which completes the proof. □

From the previous theorem and its proof it follows that instead of solving the system (2.1)–(2.2), we can look for (unique) saddle points of the functional (3.2) and therefore we can state the following corollary.

Corollary 3.2. *A pair* (y_0, z_0) *is a saddle point of the functional* (3.2) *if and only if the functions*

$$
\begin{aligned}
x^1 &= y_0 + \lambda_1, \\
x^2 &= z_0 + \lambda_2
\end{aligned}
$$

form a solution to the system (2.1)–(2.2). *Moreover, if* $|x^i_0| \le c$ *and* $|x^i_T| \le c$, $i = 1, 2$, *for some* $c > 0$, *there exist balls*

$$
\begin{aligned}
B_1(r_1) &= \left\{ y \in H^1_0([0,T], \mathbb{R}^{n_1}) : \|y\| \le r_1 \right\}, \\
B_2(r_2) &= \left\{ z \in H^1_0([0,T], \mathbb{R}^{n_2}) : \|z\| \le r_2 \right\},
\end{aligned}
$$

such that $(y_0, z_0) \in B_1(r_1) \times B_2(r_2) \subset H$ *for all* $u \in \mathcal{U}$.

4. CONTINUOUS DEPENDENCE ON CONTROLS

Throughout this section we assume that (A1)–(A4) are satisfied. Moreover, in this part we use the notion of epi/hypo-convergence. This type of convergence for bivariate functions extends the theory of epi-convergence (Γ-convergence), originally developed by Wijsman, Mosco and De Giorgi. For more information about epi-convergence we refer the interested readers to the book of Dal Maso [10]. The definition of epi/hypo-convergence was introduced by H. Attouch and R. Wets in [1]. A similar notion of convergence for saddle function was initially presented by E. Cavazzuti under the name of multiple Γ-convergence in [7, 8] and by G. Greco in [13]. In our paper we use epi/hypo-convergence theory developed by D. Aze, H. Attouch and R. Wets in [2, 3]. For extensions on applications of Γ-convergence to parabolic and hyperbolic systems one can see for example [12] and references therein, while for evolution inclusions [11] and [15].

Now, let us recall the definition of the epi/hypo-convergence. Let X and Y be topological spaces and let $\{F_\nu\}_{\nu \in \mathbb{N}}$ be a sequence of functions defined on the product space $X \times Y$ with values in $\bar{\mathbb{R}}$. The topology on X be denoted by τ, whereas the symbol σ be reserved for the topology on Y. The *epi/hypo-limit superior,* denoted by $e_\tau/h_\sigma - \operatorname{ls} F_\nu$, is defined by

$$
e_\tau/h_\sigma - \operatorname{ls} F_\nu(x,y) = \sup_{V \in \mathcal{N}_\sigma(y)} \inf_{U \in \mathcal{N}_\tau(x)} \overline{\lim_{\nu \to \infty}} \sup_{u \in U} \inf_{v \in V} F_\nu(u,v),
$$

and the *hypo/epi limit inferior*, denoted by $h_\sigma/e_\tau - \operatorname{li} F_\nu$, is defined by

$$h_\sigma/e_\tau - \operatorname{li} F_\nu(x,y) = \inf_{U\in\mathcal{N}_\tau(x)} \sup_{V\in\mathcal{N}_\sigma(y)} \varliminf_{\nu\to\infty} \inf_{v\in V} \sup_{u\in U} F_\nu(u,v),$$

where $\mathcal{N}_\tau(x)$ stands for the set of all neighbourhoods of x in (X,τ) and analogously $\mathcal{N}_\sigma(y)$ denotes the set of all neighbourhoods of y in (Y,σ). A function $F : X\times Y \to \overline{\mathbb{R}}$ is an *epi/hypo-limit* of the sequence $\{F_\nu\}_{\nu\in\mathbb{N}}$ if

$$e_\tau/h_\sigma - \operatorname{ls} F_\nu \le F \le h_\sigma/e_\tau - \operatorname{li} F_\nu.$$

Then we write that $F = e_\tau/h_\sigma - \lim F_\nu$. If (X,τ) and (Y,σ) are metrizable, it is possible to give a sequential characterization of the epi/hypo-limit (see [3, Corollary 2.2]). Namely, we say that the sequence F_ν epi/hypo-converges to F at (x,y) if

(i): to every $y_\nu \xrightarrow{\sigma} y$, there corresponds $x_\nu \xrightarrow{\tau} x$ such that

$$F(x,y) \le \varliminf_{\nu\to\infty} F_\nu(x_\nu,y_\nu),$$

(ii): to every $x_\nu \xrightarrow{\tau} x$, there corresponds $y_\nu \xrightarrow{\sigma} y$ such that

$$\varlimsup_{k\to\infty} F_\nu(x_\nu,y_\nu) \le F(x,y).$$

Moreover, the sequence F_ν is said to be epi/hypo-convergent to F if conditions **(i)** and **(ii)** are satisfied for all (x,y).

Furthermore, a point $(x_0^\varepsilon, y_0^\varepsilon)$ is called an ε-saddle point of F_ν if

$$\sup_y F_\nu(x_0^\varepsilon, y) - \varepsilon \le F_\nu(x_0^\varepsilon, y_0^\varepsilon) \le \inf_x F_\nu(x, y_0^\varepsilon) + \varepsilon.$$

The set of all ε-saddle points of F_ν will be denoted as $V_{F_\nu}^\varepsilon$. In the theorem on continuous dependence of solutions on controls we use the topological set convergence in the sense of Painlevé-Kuratowski. Symbols $\tau - \underline{\operatorname{Lim}}_{\nu\to\infty} V_\nu$ and $\tau - \overline{\operatorname{Lim}}_{\nu\to\infty} V_\nu$ stand for lower and upper limit, respectively, of the sequence $\{V_\nu\}_{\nu\in\mathbb{N}}$ in the τ topology. If w and s denote weak and strong topology in H, respectively, then it is evident (see [10, Remark 4.11 and Remark 6.4]) that the following inclusions hold

$$s - \underline{\operatorname{Lim}}_{\nu\to\infty} V_\nu \subset s - \overline{\operatorname{Lim}}_{\nu\to\infty} V_\nu, \quad w - \underline{\operatorname{Lim}}_{\nu\to\infty} V_\nu \subset w - \overline{\operatorname{Lim}}_{\nu\to\infty} V_\nu$$

and

$$s - \overline{\operatorname{Lim}}_{\nu\to\infty} V_\nu \subset w - \overline{\operatorname{Lim}}_{\nu\to\infty} V_\nu, \quad s - \underline{\operatorname{Lim}}_{\nu\to\infty} V_\nu \subset w - \underline{\operatorname{Lim}}_{\nu\to\infty} V_\nu.$$

Next, for any sequence of admissible controls $\{u_k\}_{k\in\mathbb{N}_0}$, let us denote by

$$A_k = A_{u_k}, \quad k = 0,1,2,\ldots,$$

the sequence of corresponding functionals, where A_u is described by (3.2). Moreover, the symbol x_k will be used to denote the unique saddle point of A_k, $k = 0,1,2,\ldots$.

Using this notations, we can now state and prove the theorem on continuous dependence of solutions on controls concerning the strong topology in the set of controls.

Theorem 4.1. *If $\{u_k\}_{k\in\mathbb{N}} \subset \mathcal{U}$ converges to $u_0 \in \mathcal{U}$ in $L^2([0,T],\mathbb{R}^m)$, then the sequence of solutions $\{x_k\}_{k\in\mathbb{N}}$ to the system (2.1)–(2.2) converges uniformly to the solution x_0.*

Proof. It suffices to show that the corresponding sequence $(y_k, z_k) = (x_k^1 - \lambda_1, x_k^2 - \lambda_2)$ of the saddle points of the functional (3.2) converges uniformly to $(y_0, z_0) = (x_0^1 - \lambda_1, x_0^2 - \lambda_2)$. It is well-known that on the norm bounded set $B_i(r_i)$ defined in Corollary 3.2, the weak topology of $H_0^1(\Omega, \mathbb{R}^{n_i})$, $i = 1, 2$ is induced by the metric of $L^2(\Omega, \mathbb{R}^{n_i})$ (see [10, Corollary 8.8 and Example 8.9]). By Proposition 5.2 of [10], Proposition 3.8, Theorem 3.10 and Proposition 3.12 of [2], if follows that if for every $z \in H_0^1([0,T], \mathbb{R}^{n_2})$ the sequence $\{A_k(\cdot, z)\}_{k\in\mathbb{N}}$ converges to $A_0(\cdot, z)$ uniformly on the ball $B_1(r_1) \subset H_0^1([0,T], \mathbb{R}^{n_1})$, and for every $y \in H_0^1([0,T], \mathbb{R}^{n_1})$ the sequence $\{A_k(y, \cdot)\}_{k\in\mathbb{N}}$ converges to $A_0(y, \cdot)$ uniformly on the ball $B_2(r_2) \subset H_0^1([0,T], \mathbb{R}^{n_2})$, then

$$\emptyset \neq w - \overline{\mathrm{Lim}}_{k\to\infty} V_k \subset V_0 \subset \bigcap_{\varepsilon>0} s - \underline{\mathrm{Lim}}_{k\to\infty} V_k^{\varepsilon}, \tag{4.1}$$

where V_k is the set of all saddle points of the functional A_k and V_k^{ε} is the set of its all ε-saddle points. In our case all sets V_k are singletons, i.e. $V_k = \{(y_k, z_k)\} \in B_1(r_1) \times B_2(r_2)$. Therefore, we can write the first inclusion of (4.1) as follows

$$w - \lim_{k\to\infty} (y_k, z_k) = (y_0, z_0). \tag{4.2}$$

The weak convergence of the subsequence (y_k, z_k) in H implies uniform convergence of this subsequence.

Let $z \in H_0^1([0,T], \mathbb{R}^{n_2})$. We start with the proof of the uniform convergence of the sequence $\{A_k(\cdot, z)\}_{k\in\mathbb{N}}$. Suppose, on the contrary, that $\{A_k(\cdot, z)\}_{k\in\mathbb{N}}$ does not converge to $A_0(\cdot, z)$ uniformly on $B_1(r_1)$. This means that there exists a sequence $\{y_k\}_{k\in\mathbb{N}} \subset B_1(r_1)$ and a positive constant ε such that

$$|A_k(y_k, z) - A_0(y_k, z)| > \varepsilon$$

for infinitely many indices k. Passing to a subsequence, we may assume that $y_k \rightharpoonup y_0$ in $H_0^1([0,T], \mathbb{R}^{n_1})$ and $u_k(t) \to u_0(t)$ for a.e. $t \in [0,T]$.

Let $\delta := |A_k(y_k, z) - A_0(y_k, z)|$, it is easily seen that

$$\begin{aligned} \delta \ &\leq \ \int_0^T |\Phi(t, y_k(t) + \lambda_1(t), z(t) + \lambda_2(t), u_k(t))|\, dt \\ &- \ \int_0^T |\Phi(t, y_k(t) + \lambda_1(t), z(t) + \lambda_2(t), u_0(t))|\, dt \end{aligned}$$

$$\begin{aligned} &\leq \int_0^T |\Phi(t, y_k(t) + \lambda_1(t), z(t) + \lambda_2(t), u_k(t)) \\ &\quad -\Phi(t, y_0(t) + \lambda_1(t), z(t) + \lambda_2(t), u_0(t))|\, dt \\ &+ \int_0^T |\Phi(t, y_k(t) + \lambda_1(t), z(t) + \lambda_2(t), u_0(t)) \\ &\quad -\Phi(t, y_0(t) + \lambda_1(t), z(t) + \lambda_2(t), u_0(t))|\, dt \end{aligned}$$

for $k \in \mathbb{N}$. From (A2) and the Lebesgue dominated convergence theorem, we conclude that the above integrals tend to zero, contrary to our supposition. The same reasoning applies to the uniform convergence of the sequence $\{A_k(y, \cdot)\}_{k\in\mathbb{N}}$, which proves the theorem. □

Now, we consider the weak convergence in the set of the admissible controls. If we assume that the function Φ is linear with respect to the control u, i.e. Φ has the form

$$\Phi(t, x, u) = \Phi_1(t, x) + \Phi_2(t, x)\, u,$$

it is possible to draw the same conclusion as in Theorem 4.1. Actually, we arrive at the following statement.

Theorem 4.2. *If $\{u_k\}_{k\in\mathbb{N}} \subset \mathcal{U}$ converges to $u_0 \in \mathcal{U}$ in the weak topology of $L^2([0, T], \mathbb{R}^m)$, then the sequence of solutions $\{x_k\}_{k\in\mathbb{N}}$ to the system* (2.1)–(2.2) *converges uniformly to the solution x_0.*

Proof. By the same kind of reasoning as in the proof of the previous theorem it is enough to demonstrate that for every $z \in H_0^1([0, T], \mathbb{R}^{n_2})$ the sequence $\{A_k(\cdot, z)\}_{k\in\mathbb{N}}$ converges to $A_0(\cdot, z)$ uniformly on the ball $B_1(r_1) \subset H_0^1([0, T], \mathbb{R}^{n_1})$ and, for every $y \in H_0^1([0, T], \mathbb{R}^{n_1})$ the sequence $\{A_k(y, \cdot)\}_{k\in\mathbb{N}}$ converges to $A_0(y, \cdot)$ uniformly on the ball $B_2(r_2) \subset H_0^1([0, T], \mathbb{R}^{n_2})$. Similarly, as in the proof of Theorem 4.1, we focus only on the first aforementioned convergence. Let $z \in H_0^1([0, T], \mathbb{R}^{n_2})$ and suppose, on the contrary, that $\{A_k(\cdot, z)\}_{k\in\mathbb{N}}$ does not converge to $A_0(\cdot, z)$ uniformly on $B_1(r_1)$. Thus, there exists a sequence $\{y_k\}_{k\in\mathbb{N}} \subset B_1(r_1)$ and a positive constant ε such that

$$|A_k(y_k, z) - A_0(y_k, z)| > \varepsilon$$

for infinitely many indices k. Passing to a subsequence, we may assume that $y_k \rightharpoonup y_0$ in $H_0^1([0, T], \mathbb{R}^{n_1})$. Let $\delta := |A_k(y_k, z) - A_0(y_k, z)|$. It is easy to notice that

$$\begin{aligned} \delta &= \left|\int_0^T \Phi_2(t, y_k(t) + \lambda_1(t), z(t) + \lambda_2(t))\, u_k(t) \right. \\ &\quad \left. -\Phi_2(t, y_k(t) + \lambda_1(t), z(t) + \lambda_2(t))\, u_0(t)\, dt\right| \\ &\leq \int_0^T |\Phi_2(t, y_k(t) + \lambda_1(t), z(t) + \lambda_2(t)) \\ &\quad -\Phi_2(t, y_0(t) + \lambda_1(t), z(t) + \lambda_2(t))|\, |u_k(t) - u_0(t)|\, dt \end{aligned}$$

$$+ \quad \left| \int_0^T \left(\Phi_2\left(t, y_0\left(t\right) + \lambda_1\left(t\right), z\left(t\right) + \lambda_2\left(t\right)\right)\right) \left(u_k\left(t\right) - u_0\left(t\right)\right) dt \right|$$

for $k \in \mathbb{N}$. From (A2) and the Lebesgue dominated convergence theorem, we deduce that the first integral on the right hand side of the above inequality tends to zero (the sequence$\{\|u_k\|_{L^2}\}_{k\in\mathbb{N}}$ is bounded). The weak convergence in $L^2\left([0,T], \mathbb{R}^m\right)$ implies that the second integral from the above inequality also converges to zero. □

5. OPTIMAL CONTROL PROBLEM

In the present section, some sufficient conditions for the existence of solutions to optimal control problem are formulated. Moreover, we assume that an optimal control problem is governed by the system of second order ordinary differential equations with boundary conditions (2.1)–(2.2) where the function on the right hand side of the system is linear with respect to control. Namely, the system has the form

$$\text{(5.1)} \qquad \begin{cases} \ddot{x}^1\left(t\right) = \varphi^{11}\left(t, x^1\left(t\right), x^2\left(t\right)\right) + \varphi^{12}\left(t, x^1\left(t\right), x^2\left(t\right)\right) u\left(t\right), \\ \ddot{x}^2\left(t\right) = -\varphi^{21}\left(t, x^1\left(t\right), x^2\left(t\right)\right) - \varphi^{22}\left(t, x^1\left(t\right), x^2\left(t\right)\right) u\left(t\right), \\ x^1\left(0\right) = x_0^1, \ x^1\left(T\right) = x_T^1, \\ x^2\left(0\right) = x_0^2, \ x^2\left(T\right) = x_T^2, \end{cases}$$

for a.e. $t \in [0,T]$, where $x_0^1, x_T^1 \in \mathbb{R}^{n_1}$, $x_0^2, x_T^2 \in \mathbb{R}^{n_2}$ are fixed, $\varphi^{11} : [0,T] \times \mathbb{R}^{n_1+n_2} \to \mathbb{R}^{n_1}$, $\varphi^{12} : [0,T] \times \mathbb{R}^{n_1+n_2} \to \mathbb{R}^{m\times n_1}$ $\varphi^{21} : [0,T] \times \mathbb{R}^{n_1+n_2} \times \mathbb{R}^m \to \mathbb{R}^{n_2}$, $\varphi^{22} : [0,T] \times \mathbb{R}^{n_1+n_2} \to \mathbb{R}^{m\times n_2}$, $x \in H^2\left([0,T], \mathbb{R}^{n_1+n_2}\right)$ and

$$\text{(5.2)} \qquad (\dot{x}(0), \dot{x}(T)) := \left(\dot{x}^1\left(0\right), \ \dot{x}^2\left(0\right), \ \dot{x}^1\left(T\right), \ \dot{x}^2\left(T\right)\right) \in \mathbb{R}^{2(n_1+n_2)},$$

$$\text{(5.3)} \qquad u \in \mathcal{U} := \left\{u \in L^\infty\left([0,T], \mathbb{R}^m\right) : u\left(t\right) \in M \text{ for a.e. } t \in [0,T]\right\},$$

where M is a compact subset of $\mathbb{R}^m$, and $T > 0$, $m, n_1, n_2 \geq 1$. We consider problem (5.1) with the following cost indicator

$$\text{(5.4)} \qquad I\left(x,u\right) = g^0\left(\dot{x}\left(0\right), \dot{x}\left(T\right)\right) + \int_0^T f^0\left(t, x\left(t\right), \dot{x}\left(t\right), u\left(t\right)\right) dt$$

where $g^0 : \mathbb{R}^{2(n_1+n_2)} \to \mathbb{R}$ and $f^0 : [0,T] \times \mathbb{R}^{2(n_1+n_2)} \times \mathbb{R}^m \to \mathbb{R}$.

On the functions g^0 and f^0 we impose the following assumptions:

(A5): the function g^0 is lower semicontinuous and coercive;

(A6):

(a) the function f^0 is continuous with respect to $(x, \dot{x}, u)$, convex with respect to u and measurable with respect to t,

(b) for each $C > 0$, there is a function $h_C \in L^1([0,T], \mathbb{R}^+)$ such that

$$f^0(t, x, \dot{x}, u) \le h_C(t)$$

for any $|x| \le C$, $|\dot{x}| \le C$, $u \in M \subset \mathbb{R}^m$ and a.e. $t \in [0,T]$,

(c) there exists a function $h^0 \in L^1([0,T], \mathbb{R})$ such that

$$f^0(t, x, \dot{x}, u) \ge h^0(t)$$

for $x \in \mathbb{R}^{n_1+n_2}$, $\dot{x} \in \mathbb{R}^{n_1+n_2}$, $u \in M$ and a.e. $t \in [0,T]$.

Let $\mathcal{D}$ be the set of all admissible pairs, i.e.

$$\mathcal{D} := \left\{(x,u) \in H^2\left([0,T], \mathbb{R}^{n_1+n_2}\right) \times \mathcal{U} : \ x \text{ satisfies (5.1) for } u \in \mathcal{U}\right\}.$$

Remark 5.1. Let $(x_{\tilde{u}}, \tilde{u})$ be an admissible control, i.e. $(x_{\tilde{u}}, \tilde{u}) \in \mathcal{D}$. If

$$\tilde{l} := I(x_{\tilde{u}}, \tilde{u}) + 1,$$

it is easy to observe that the set

$$A = \left\{(x_u, u) \in \mathcal{D} : I(x_u, u) \le \tilde{l}\right\}$$

is nonempty and we have the following inclusions

$$\begin{aligned} A &\subset \left\{(x_u, u) \in \mathcal{D} : g^0(\dot{x}_u(0), \dot{x}_u(T)) + \int_0^T h^0(t)\, dt \le \tilde{l}\right\} \\ &= \left\{(x_u, u) \in \mathcal{D} : g^0(\dot{x}_u(0), \dot{x}_u(T)) \le l\right\} \end{aligned}$$

where $l := \tilde{l} - \int_0^T h^0(t)\, dt$. Since g^0 is coercive, there exists a constant $c > 0$ such that for every x_u

$$|\dot{x}_u(0)| \le c \tag{5.5}$$

where $(x_u, u) \in A$.

Moreover, we denote by $\mathcal{X}$ a family of all functions x_u such that $(x_u, u) \in A$ and by $\dot{\mathcal{X}}$ a family of derivatives of functions from $\mathcal{X}$, i.e. $\dot{\mathcal{X}} := \{\dot{x}_u : x_u \in \mathcal{X}\}$.

Lemma 5.2. *Let us assume that* (A1)–(A2) *are satisfied, then the families of functions belonging to sets $\mathcal{X}$ and $\dot{\mathcal{X}}$ are equibounded and equicontinuous.*

Proof. From the second part of Theorem 3.1, it follows that there is a constant $d > 0$ such that for any $x \in \mathcal{X}$

$$|x(t)| \le d \qquad \text{for } t \in [0,T] \tag{5.6}$$

hence the family $\mathcal{X}$ is equibounded. Let $x \in \mathcal{X}$ be a solution to (5.1) corresponding to a control $u \in \mathcal{U}$. Then, by (A2), (A5), (A6) and (5.5)

$$|\dot{x}(t)| \le |\dot{x}(0)| + \int_0^T |\varphi(t, x(t), u(t))|\, dt \le c + \int_0^T a_0(|x(t)|)\, b_0(t)\, dt \le d_1,$$

where $d_1 \geq 0$ and

$$\varphi = (\varphi^1, \varphi^2) := (\varphi^{11} + \varphi^{12}u, \varphi^{21} + \varphi^{22}u). \tag{5.7}$$

Since $\mathcal{X}$ is equibounded, a_0 is continuous and d_1 is independent of x, the set $\dot{\mathcal{X}}$ is equibounded. Furthermore, for each $t_1, t_2 \in [0, T]$ such that $t_1 < t_2$ we have

$$\begin{aligned} |x(t_2) - x(t_1)| &\leq \int_{t_1}^{t_2} |\dot{x}(t)|\, dt \leq d_1 |t_2 - t_1|, \\ |\dot{x}(t_2) - \dot{x}(t_1)| &\leq \int_{t_1}^{t_2} |\varphi(t, x(t), u(t))|\, dt \\ &\leq \int_{t_1}^{t_2} a_0(|x(t)|)\, b_0(t)\, dt \leq d_2 \sqrt{t_2 - t_1}, \end{aligned}$$

where $d_2 > 0$ is independent of x, and consequently $\mathcal{X}$ and $\dot{\mathcal{X}}$ are equicontinuous. □

Let us recall that a pair $(x_{u^*}, u^*) \in H^2([0, T], \mathbb{R}^{n_1+n_2}) \times \mathcal{U}$ is an optimal process if it satisfies (5.1) and the inequality $I(x_{u^*}, u^*) \leq I(x_u, u)$ holds for any pair $(x_u, u) \in \mathcal{D}$. Using the results from Sections 3 and 4, we now prove a theorem on the existence of optimal processes to our optimal control problem.

Theorem 5.3. *If* (A1)–(A6) *are satisfied, the function φ has the form* (5.7) *and the set $M \subset \mathbb{R}^m$ is compact and convex, then the optimal control problem* (5.1)–(5.3) *possesses at least one solution.*

Proof. From (A6) and classical theorems on semicontinuity of integral functionals (cf. [9, 17]), we deduce that the second part of the cost indicator is lower semicontinuous with respect to the strong topology of $H^1([0, T], \mathbb{R}^{n_1+n_2})$ in x and the weak topology of $L^2([0, T], \mathbb{R}^m)$ in u. From $(A5)$, we infer that the functional $I(x, u)$ is lower semicontinuous with respect to the strong topology of $H^1([0, T], \mathbb{R}^{n_1+n_2})$ in x and the weak topology of $L^2([0, T], \mathbb{R}^m)$ in u. Let $\{(x^k, u^k)\}_{k\in\mathbb{N}} \subset \mathcal{D}$ be a minimizing sequence to the problem (5.1)–(5.3), i.e.

$$\begin{aligned} \lim_{k\to\infty} I(x^k, u^k) &= \inf_{(x,u)\in\mathcal{D}} \left(g^0(\dot{x}(0), \dot{x}(T)) + \int_0^T f^0(t, x(t), \dot{x}(t), u(t))\, dt \right) \\ &= \inf_{(x,u)\in A} \left(g^0(\dot{x}(0), \dot{x}(T)) + \int_0^T f^0(t, x(t), \dot{x}(t), u(t))\, dt \right). \end{aligned}$$

Since the set $M \subset \mathbb{R}^m$ is compact and convex, the sequence $\{u^k\}_{k\in\mathbb{N}}$ is compact in the weak topology of $L^2([0, T], \mathbb{R}^m)$. Passing, if necessary to a subsequence, we may assume that $u^k \rightharpoonup u^0$ in $L^2([0, T], \mathbb{R}^m)$. Thus, Theorem 3.1 and Theorem 4.2 ensure the existence of the unique solution x^k for any u^k and the uniform convergence of $\{x^k\}_{k\in\mathbb{N}}$ to x^0 corresponding to u^0. By Lemma 5.2 and the Arzelà-Ascoli theorem, we have the compactness of the sequence $\{(x^k, \dot{x}^k)\}_{k\in\mathbb{N}}$ in the topology of uniform convergence. Thus we may assume, without loss of generality, that

$$x^k \rightrightarrows x^0, \text{ and } \dot{x}^k \rightrightarrows v \tag{5.8}$$

to some continuous function v. Moreover, by the theorem on differentiation of the uniformly convergent, with derivatives, sequence term by term, we have $\dot{x}^0 = v$. Let us note that, for any measurable set $E \subset [0,T]$ from (A2), we have

$$\left|\int_E^k \ddot{x}^k(t)\,dt\right| \le \int_E \left|\varphi\left(t, x^k(t), u^k(t)\right)\right| dt \le \int_E a_0\left(\left|x^k(t)\right|\right) b_0(t)\,dt$$
$$\le d_3 \int_E b_0(t)\,dt \le d_3\sqrt{\mu(E)}\,\|b_0\|_{L^2},$$

for some $d_3 > 0$. Hence the sequence $\{\ddot{x}^k\}_{k\in\mathbb{N}}$ is equiabsolutely integrable. By the Dunford-Pettis theorem [9, 10.3.i] there is a function $\eta \in L^1([0,T], \mathbb{R}^{n_1+n_2})$ such that $\ddot{x}^k \rightharpoonup \eta$ (weakly in $L^1([0,T], \mathbb{R}^{n_1+n_2})$). For arbitrary but fixed point $t_0 \in [0,T]$ we have

$$\dot{x}_i^k(t_0) = \int_0^{t_0} \ddot{x}_i^k(t)\,dt + \dot{x}_i^k(0) = \int_0^T \chi_{[0,t_0]}(t)\,\ddot{x}_i^k(t)\,dt + \dot{x}_i^k(0),$$

where $\dot{x}_i^k$ denotes the $i-$th coordinate function and $\chi_{[0,t_0]}$ denotes the characteristic function of the interval $[0,t_0]$. Next, passing with $k\to\infty$, we get by (5.8) that

$$\dot{x}^0(t_0) = \int_0^{t_0} \eta(t)\,dt + \dot{x}^0(0).$$

In consequence, $\ddot{x}^0(t) = \eta(t)$ for a.e. $t \in [0,T]$, $\ddot{x}^k \rightharpoonup \ddot{x}^0$ (weakly in $L^1([0,T], \mathbb{R}^{n_1+n_2})$) and $f(t, x^0(t), u^0(t)) = \eta(t)$ for a.e. $t \in [0,T]$ and moreover $(x^0, u^0) \in A \subset \mathcal{D}$. By the lower semicontinuity of I, we obtain

$$\begin{aligned}\inf_{(x,u)\in\mathcal{D}} I(x,u) &= \inf_{(x,u)\in A} I(x,u) \le I\left(x^0,u^0\right) \overset{lsc}{\le} \liminf_{k\to+\infty} I\left(x^k,u^k\right)\\ &= \lim_{k\to+\infty} I\left(x^k,u^k\right) = \inf_{(x,u)\in A} I(x,u).\end{aligned}$$

Thus, $\inf_{(x,u)\in\mathcal{D}} I(x,u) = \inf_{(x,u)\in A} I(x,u) = I(x^0,u^0)$ and the proof is complete. □

Example 5.4. Let us introduce the set of coordinates in $\mathbb{R}^4$ as (x^1, y^1, x^2, y^2). Consider the following control system

$$\begin{cases} \dot{x}^1(t) = y^1(t) \\ \dot{y}^1(t) = -x^1(t) + 4t^2\left(x^1(t)\right)^3 + x^2(t)\,u(t) - 2t \\ \dot{x}^2(t) = y^2(t) \\ \dot{y}^2(t) = -x^2(t) + 2\left(x^2(t)\right)^5 - x^1(t)\,u(t) + t^2 \end{cases} \tag{5.9}$$

with the boundary conditions $x^1(0) = x^2(0) = 0$, $x^1(T) = x^2(T) = 1$ and the cost functional

$$I(x,u) = |\dot{x}(0)|^2 - 2\dot{x}^1(0) + |\dot{x}(T)|^2 + \dot{x}^2(T) \tag{5.10}$$

$$+\int_0^T \left(|x(t)|^2 + x^1(t)(u(t))^3 + x^2(t)(u(t))^2\right) dt$$

where $x = (x^1, x^2)$. For simplicity, we assume that $T = \frac{\sqrt{2}}{2}\pi$ and $M = [-1, 1]$. It is easy to notice that system (5.9) does not satisfy the condition (1.4). Therefore, the following properties cannot be guaranteed:

(i) for admissible controls exist trajectories defined on the whole interval $[0, T]$,

(ii) the set of all trajectories are commonly bounded,

and we cannot apply classical Filippov's theorem on the existence of optimal processes (cf. [9, Theorem 9.3.i]). It is easy to notice that the system (5.9) can be reformulated as

$$(5.11)\quad \begin{cases} \ddot{x}^1(t) = -x^1(t) + 4t^2(x^1(t))^3 + x^2(t)u(t) - 2t, \ x^1(0) = 0, \ x^1(T) = 1, \\ \ddot{x}^2(t) = -x^2(t) + 2(x^2(t))^5 - x^1(t)u(t) + t^2, \ x^2(0) = 0, \ x^2(T) = 1. \end{cases}$$

The vector field given by the vector function on the right hand side of equations in (5.11) is a gradient of the potential Φ defined as

$$\begin{aligned} \Phi\left(t, x^1, x^2, u\right) &= -\frac{1}{2}\left(x^1\right)^2 + t^2\left(x^1\right)^4 + x^1x^2u - 2tx^1 \\ &\quad - \left(-\frac{1}{2}\left(x^2\right)^2 + \frac{1}{3}\left(x^2\right)^6 + t^2x^2\right). \end{aligned}$$

Moreover, the functions

$$\begin{aligned} \varphi^1\left(t, x^1, x^2, u\right) &= \Phi_{x^1}\left(t, x^1, x^2, u\right), \\ \varphi^2\left(t, x^1, x^2, u\right) &= \Phi_{x^2}\left(t, x^1, x^2, u\right), \\ \psi_1\left(x^1\right) &= \Phi\left(t, x^1, x^2(t), u\right) + \alpha_1\left|x^1\right|^2, \\ \psi_2\left(x^2\right) &= \Phi\left(t, x^1(t), x^2, u\right) - \alpha_2\left|x^2\right|^2 \end{aligned}$$

satisfy assumptions (A1)–(A4) with $\alpha_1 = \alpha_2 = \frac{1}{2}$. From Theorem 3.1 it follows that for any admissible control u there is the unique trajectory x_u of the system (5.10) defined on the whole interval $[0, T]$ and the set of all trajectories are commonly bounded. Furthermore, it is easy to check that the following functions

$$\begin{aligned} g^0(\dot{x}(0), \dot{x}(T)) &= |\dot{x}(0)|^2 - 2\dot{x}^1(0) + |\dot{x}(T)|^2 + \dot{x}^2(T), \\ f^0(t, x, \dot{x}, u) &= |x(t)|^2 + x^1(t)(u(t))^3 + x^2(t)(u(t))^2 \end{aligned}$$

satisfy assumptions (A5), (A6). Therefore, by applying Theorem 5.3, we get that the control problem consisting of system (5.9) or (5.11) and the cost functional (5.10) possesses an optimal solution.

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