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ON THE REGULARITY OF THE SOLUTION MAP FOR DIFFERENTIAL INCLUSIONS

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ABSTRACT. In this paper we give sufficient conditions for the solution map of the differential inclusion $y' \in F(y)$, with $F : K \rightsquigarrow X$ a multifunction and K a nonempty subset of a finite dimensional space X, to be lower semicontinuous. We present an application on the propagation of the continuity of the state constrained minimum time function associated with the given differential inclusion and the target zero.

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1. INTRODUCTION

Consider the differential inclusion

$$y'(t) \in F(y(t)), \tag{1.1}$$

where $F : K \rightsquigarrow X$ is a multifunction with nonempty values and K is a nonempty subset of a finite dimensional space X. For every $x \in K$, denote by $\mathcal{S}(x)$ the set of all solutions of (1.1) starting from x. The definition of solution is given below. The aim of this paper is to establish sufficient conditions for the solution map $\mathcal{S}$ of (1.1) to be lower semicontinuous.

The properties of the solution map of differential inclusions have been extensively studied in the literature. The first contributions in this area are the papers of Filippov [8, 9] and Plis [16]. The central assumption of Filippov theorem is the Lipschitz condition of the right-hand side with respect to the state variable. In Plis' result it is required a special uniform continuity condition instead of Lipschitz continuity. There are many extensions of these results under various frames, with various assumptions on F: Lipschitz type, one sided Lipschitz, one sided Kamke, continuous-like (see, e.g., [10], [17], [21], [7], [13], [12]).

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In [4], it is proved that the solution map of a semilinear differential inclusion is lower semicontinuous, the assumption on the multifunction F being of continuity type with a prescribed modulus, as in [16]. In this paper we require a weaker condition on F then the one in [4], condition introduced in [6], and we prove, under appropriate assumptions, the lower semicontinuity of the solution map $\mathcal{S}$ of (1.1).

Finally, we give an application to the propagation of continuity of the state constrained minimum time function associated with (1.1) and the target zero. For other results on the regularity of the minimum time function see [18], [20], [15], [14]. See also [2], [1], [3] for some regularity results of the minimum time function associated with semilinear control systems in Banach spaces.

We present now some preliminary results. First, let us recall that by a solution of (1.1) on $[0,T]$ we mean an absolutely continuous function $y:[0,T]\to K$ which satisfies (1.1) for almost all $t\in[0,T]$. A solution of (1.1) on the semi-open interval $[0,T)$ is defined similarly. A solution $y:[0,T)\to K$ of (1.1) is called noncontinuable if there is no other solution $\widetilde{y}:[0,\widetilde{T})\to K$ of (1.1), with $T<\widetilde{T}$ and satisfying $y(t)=\widetilde{y}(t)$ for all $t\in[0,T)$.

Let $K\subseteq X$ and $\xi\in K$. We say that $\eta\in X$ is tangent to the set K at the point ξ if

$$\liminf_{h\downarrow 0}\frac{1}{h}\operatorname{dist}(\xi+h\eta;K)=0.$$

It is easy to see that $\eta\in X$ is tangent to K at $\xi\in K$ if and only if there exist two sequences $(h_n)_n$ in $\mathbb{R}_+$ and $(q_n)_n$ in X with $h_n\downarrow 0$, $\lim_{n\to\infty}q_n=\eta$ and such that $\xi+h_nq_n\in K$ for each $n\in\mathbb{N}$. The set of all tangent vectors to K at $\xi\in K$ is denoted by $T_K(\xi)$ and is called the Bouligand tangent cone to K at ξ.

Definition 1.1. The set K is viable with respect to F if for each $\xi\in K$ there exists $T>0$ such that (1.1) has at least one solution $y:[0,T]\to K$ with $y(0)=\xi$.

The next viability theorem can be found, for instance, in [5].

Theorem 1.2. *Let X be finite dimensional, let $K\subseteq X$ be nonempty and locally closed and let $F:K\rightsquigarrow X$ be an upper semicontinuous multifunction with nonempty, compact and convex values. A necessary and sufficient condition in order that K be exact viable with respect to F is the tangency condition*

$$F(\xi)\cap T_K(\xi)\neq\emptyset$$

for each $\xi\in K$.

We recall that a set K is locally closed if for each $x\in K$ there exists $B(x,r)$ such that $K\cap B(x,r)$ is closed, where by $B(x,r)$ we denoted the closed ball of center x and radius r in X.

We present a result concerning the differential inclusion (1.1), which belongs to Wazewski [19] (see also [6]).

Proposition 1.3. *Let the multifunction F be upper semicontinuous with compact and convex values. Then, for every $x \in K$ and for every solution $y : [0, \sigma) \to K$ of* (1.1) *with $y(0) = x$, there exist $p \in F(x)$ and a sequence $(s_n)_n$ in $(0, \sigma)$ converging to* 0 *such that the sequence $\left(\frac{y(s_n)-x}{s_n}\right)_n$ converges to p.*

The next result, proved in [6], will be used later in the paper.

Proposition 1.4. *Let the multifunction F be lower semicontinuous with closed and convex values and let the set K be locally closed. Then $F(x) \subseteq T_K(x)$, for every $x \in K$, if and only if for every $x \in K$ and for every $p \in F(x)$, there exists a solution $y : [0, \sigma) \to K$ of the differential inclusion* (1.1)*, with $y(0) = x$, such that for every sequence $(s_n)_n$ in $(0, \sigma)$ converging to* 0*, the sequence $\left(\frac{y(s_n)-x}{s_n}\right)_n$ converges to p.*

We end this section by presenting a result on the upper semicontinuity of the solution map for a differential equation in $\mathbb{R}$, that will play a key role in the proof of the main result. We recall that a continuous function $G : [0, \infty) \to \mathbb{R}$ with $G(0) = 0$ such that the differential equation $z'(t) = G(z(t))$ has the null function as the unique solution with $z(0) = 0$ is called a Perron function. The following result is a consequence of [11, Lemma 3.1].

Proposition 1.5. *Let G be a Perron function. Then, for any $\varepsilon > 0$ and any $\theta > 0$, there exists $\delta = \delta(\varepsilon, \theta) > 0$ such that for any $\xi \geq 0$, $\xi < \delta$ and any $z : [0, \sigma_z) \to \mathbb{R}$, noncontinuable solution of $z' = G(z)$ with $z(0) = \xi$, we have $\theta < \sigma_z$ and $|z(t)| < \varepsilon$ for all $t \in [0, \theta]$.*

2. MAIN RESULTS

Before presenting the following result, we mention that we denote by $[x, y]_+$ the right directional derivative of the norm calculated at x in the direction y, i.e.

$$[x, y]_+ = \lim_{h \downarrow 0} \frac{\|x + hy\| - \|x\|}{h}.$$

Theorem 2.1. *Let X be a finite dimensional space, K a locally closed subset of X, $F : K \rightsquigarrow X$ a continuous multifunction, with convex and compact values. Assume that there exists $G : [0, \infty) \to [0, \infty)$ a continuous function such that*

$$\sup_{p \in F(x)} \inf_{q \in F(y)} [x - y, p - q]_+ \leq G(\|x - y\|) \tag{2.1}$$

for any $x, y \in K$. Moreover, assume that, for any $x \in K$,

$$F(x) \subseteq T_K(x). \tag{2.2}$$

Let $x_0 \in K$ *and let* $y_0 : [0,T] \to K$ *be a solution of* (1.1) *with* $y_0(0) = x_0$. *Then, for any* $x_1 \in K$, *any* $z_0 \geq 0$ *with* $\|x_0 - x_1\| \leq z_0$, *there exist a noncontinuable solution* $y_1 : [0,\sigma_y) \to K$ *of* (1.1) *with* $y_1(0) = x_1$ *and* $z_1 : [0,\sigma_z) \to \mathbb{R}$ *a noncontinuable solution of* $z'(s) = G(z(s))$ *with* $z_1(0) = z_0$ *such that*

$$\|y_1(s) - y_0(s)\| \leq z_1(s),$$

for all $s \in [0,T] \cap [0,\sigma_y) \cap [0,\sigma_z)$.

Proof. Let us take $x_0 \in K$ and let $y_0 : [0,T] \to K$ be a solution of (1.1) with $y_0(0) = x_0$. Since F is continuous and has compact and convex values, the solution $y_0(\cdot)$ can be continued up to a noncontinuable one, denoted $\widetilde{y}_0 : [0,\widetilde{\sigma}) \to K$, $\widetilde{\sigma} > T$. Consider the space $\mathcal{X} = \mathbb{R} \times X \times \mathbb{R}$, the set

$$\mathcal{K} = \{(\tau,x,\lambda) \in \mathcal{X};\ \tau \in [0,\widetilde{\sigma}),\ x \in K,\ \lambda \in \mathbb{R},\ \|\widetilde{y}_0(\tau) - x\| \leq \lambda\}$$

and the multifunction $\mathcal{F} : \mathcal{K} \rightsquigarrow \mathcal{X}$ defined by

$$\mathcal{F}(\tau,x,\lambda) = \{1\} \times F(x) \times \{G(\lambda)\}.$$

First, we shall prove that the tangency condition

$$T_{\mathcal{K}}(\tau,x,\lambda) \cap \mathcal{F}(\tau,x,\lambda) \neq \emptyset \tag{2.3}$$

holds for any $(\tau,x,\lambda) \in \mathcal{K}$. Assume first that $(\tau,x,\lambda) \in \mathcal{K}$ with $\|\widetilde{y}_0(\tau) - x\| = \lambda$. By Proposition 1.3, there exist $p \in F(\widetilde{y}_0(\tau))$ and a sequence $(s_n)_n \subset [0,\widetilde{\sigma})$, $s_n \downarrow 0$ such that the sequence $\dfrac{\widetilde{y}_0(\tau + s_n) - \widetilde{y}_0(\tau)}{s_n}$ converges to p. By (2.1) we have that

$$\inf_{q \in F(x)} [\widetilde{y}_0(\tau) - x, p - q]_+ \leq G(\|\widetilde{y}_0(\tau) - x\|) = G(\lambda).$$

Since $F(x)$ is compact, there exists $q \in F(x)$ such that $[\widetilde{y}_0(\tau) - x, p - q]_+ \leq G(\lambda)$.

By (2.2) and Proposition 1.4, there exists $\widetilde{y}(\cdot)$ solution of (1.1) with $\widetilde{y}(0) = x$ such that $q_n := (1/s_n)(\widetilde{y}(s_n) - x)$ converges to q and $x + s_n q_n \in K$, for every $n \in \mathbb{N}$. We have that

$$\begin{aligned} \|\widetilde{y}_0(\tau + s_n) - (x + s_n q_n)\| &\leq \|\widetilde{y}_0(\tau) - x\| + s_n [\widetilde{y}_0(\tau) - x, p - q]_+ + s_n r_n \\ &\leq \lambda + s_n G(\lambda) + s_n r_n, \end{aligned}$$

where $r_n = \left\| \frac{\widetilde{y}_0(\tau+s_n) - \widetilde{y}_0(\tau)}{s_n} - p \right\| + \frac{\|\widetilde{y}_0(\tau) - x + s_n(p-q)\| - \|\widetilde{y}_0(\tau) - x\|}{s_n} - [\widetilde{y}_0(\tau) - x, p - q]_+ + \|q - q_n\|$ converges to 0.

So, we obtained that

$$(\tau + s_n, x + s_n q_n, \lambda + s_n G(\lambda) + s_n r_n) \in \mathcal{K}$$

for every $n \in \mathbb{N}$, hence the tangency condition (2.3) holds.

If $(\tau,x,\lambda) \in \mathcal{K}$ and $\|\widetilde{y}_0(\tau) - x\| < \lambda$, then $T_{\mathcal{K}}(\tau,x,\lambda) = T_{[0,\widetilde{\sigma})}(\tau) \times T_K(x) \times \mathbb{R}$. Since $1 \in T_{[0,\widetilde{\sigma})}(\tau)$ and $F(x) \subseteq T_K(x)$, (2.3) holds too.

Then, by Theorem 1.2, the set $\mathcal{K}$ is viable with respect to $\mathcal{F}$. Since $(0, x_1, z_0) \in \mathcal{K}$, there exist $\theta > 0$ and a solution $w = (t, y, z)$ of the problem $w' \in \mathcal{F}(w)$, on $[0, \theta]$, with $w(0) = (0, x_1, z_0)$, such that $(t(s), y(s), z(s)) \in \mathcal{K}$ for all $s \in [0, \theta]$. Hence

$$\|\widetilde{y}_0(s) - y(s)\| \leq z(s)$$

for all $s \in [0, \theta]$. By a continuation argument, there exists a pair $(\overline{y}, \overline{z}) : [0, c) \to K \times \mathbb{R}$, $\overline{y}(\cdot)$ solution of (1.1) with $\overline{y}(0) = x_1$ and $\overline{z}(\cdot)$ solution of $z' \in G(z)$ with $\overline{z}(0) = z_0$ satisfying

$$\|\widetilde{y}_0(s) - \overline{y}(s)\| \leq \overline{z}(s) \tag{2.4}$$

for all $s \in [0, c)$, noncontinuable with this property. In conclusion, there exist a noncontinuable solution $y_1 : [0, \sigma_y) \to K$ of (1.1) with $y_1(0) = x_1$ and $z_1 : [0, \sigma_z) \to \mathbb{R}$ a noncontinuable solution of $z' \in G(z)$ with $z_1(0) = z_0$ such that

$$\|\widetilde{y}_0(s) - y_1(s)\| \leq z_1(s) \tag{2.5}$$

for all $s \in [0, \min\{\widetilde{\sigma}, \sigma_y, \sigma_z\})$. □

Remark 2.2. If, in Theorem 2.1, the set K is closed, then we get the following conclusion: for any $x_1 \in K$, any $z_0 \geq 0$ with $\|x_0 - x_1\| \leq z_0$, there exist a noncontinuable solution $z_1 : [0, \sigma_z) \to \mathbb{R}$ of $z'(s) = G(z(s))$ with $z_1(0) = z_0$ and a noncontinuable solution $y_1 : [0, \sigma_y) \to K$ of (1.1) with $y_1(0) = x_1$, such that $\sigma_y \geq \min\{\widetilde{\sigma}, \sigma_z\}$ and (2.5) holds for all $s \in [0, \min\{\widetilde{\sigma}, \sigma_z\})$. Indeed, if the solution $\overline{z} : [0, c) \to \mathbb{R}$ of $z' \in G(z)$ with $\overline{z}(0) = z_0$, obtained in the proof of Theorem 2.1, is noncontinuable, then $\sigma_z = c \leq \min\{\widetilde{\sigma}, \sigma_y\}$ and (2.5) holds for $s \in [0, \sigma_z)$. If $\overline{z} : [0, c) \to \mathbb{R}$ is continuable, then we prove that $c = \widetilde{\sigma}$. To this end, assume by contradiction that $c < \widetilde{\sigma}$. Then $\overline{z}(\cdot)$ is bounded on $[0, c)$ and there exists $\lim_{s \uparrow c} z(s) \in \mathbb{R}$. By (2.4) we have that $\overline{y}(\cdot)$ is bounded on $[0, c)$ and, since F is compact valued, we have that there exists $y^* := \lim_{s \uparrow c} \overline{y}(s)$. As K is a closed set, $y^* \in K$. Moreover, by (2.4) we get that $\|\widetilde{y}_0(c) - y^*\| \leq \overline{z}(c)$. Applying now Theorem 1.2 for $(c, y^*, \overline{z}(c)) \in \mathcal{K}$ we deduce that $(\overline{y}, \overline{z})$ can be continued to the right of c with property (2.4), which contradicts the maximality of $(\overline{y}, \overline{z})$. Hence $c = \widetilde{\sigma}$. In conclusion, we have that there exist $z_1 : [0, \sigma_z) \to \mathbb{R}$, $\sigma_z > \widetilde{\sigma}$, a noncontinuable solution of $z'(s) = G(z(s))$ with $z_1(0) = z_0$ and $y_1 : [0, \sigma_y) \to K$, $\sigma_y > \widetilde{\sigma}$, a noncontinuable solution of (1.1) with $y_1(0) = x_1$, such that (2.5) holds for all $s \in [0, \widetilde{\sigma})$.

Now we give the main result of the paper on the lower-semicontinuity of the solution map $\mathcal{S}$.

Theorem 2.3. *Let X be a finite dimensional space, K a locally closed subset of X, $F : K \rightsquigarrow X$ a continuous multifunction, with convex and compact values. Assume that there exists $G : [0, \infty) \to [0, \infty)$ a Perron function such that* (2.1) *holds for any $x, y \in K$. Moreover, assume that the tangency condition* (2.2) *is verified for any*

$x \in K$. Then, for every $x_0 \in K$, for every $y_0 : [0,T] \to K$ solution of (1.1) *with $y_0(0) = x_0$ and for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every $x_1 \in K$ with $\|x_0 - x_1\| < \delta$, there exists a solution $y_1 : [0,T] \to K$ of* (1.1) *with $y_1(0) = x_1$ such that*

$$\|y_1(s) - y_0(s)\| < \varepsilon,$$

for all $s \in [0,T]$.

Proof. Let $x_0 \in K$, $y_0 : [0,T] \to K$ solution of (1.1) with $y_0(0) = x_0$ and let $\varepsilon > 0$. The solution $y_0(\cdot)$ can be continued up to a noncontinuable one, denoted $\widetilde{y}_0 : [0,\widetilde{\sigma}) \to K$, $\widetilde{\sigma} > T$.

As K is locally closed and y_0 is continuous on $[0,T]$, there exist $\eta < \varepsilon$ and a compact set $M \subseteq K$ such that $B(y_0(s), \eta) \cap K \subseteq M$ for all $s \in [0,T]$. Let $\delta := \delta(\eta, T) > 0$ given by Proposition 1.5. Let $x_1 \in K$ with $\|x_0 - x_1\| < \delta$. By Theorem 2.1, there exist a noncontinuable solution $y_1 : [0,\sigma_y) \to K$ of (1.1) with $y_1(0) = x_1$ and $z : [0,\sigma_z) \to \mathbb{R}$, a noncontinuable solution of $z'(s) = G(z(s))$, with $z(0) = \|x_0 - x_1\|$, such that

$$\|y_1(s) - \widetilde{y}_0(s)\| \leq z(s) \tag{2.6}$$

for all $s \in [0, \min\{\widetilde{\sigma}, \sigma_y, \sigma_z\})$. By Proposition 1.5 we have that $\sigma_z > T$ and

$$|z(s)| < \eta \tag{2.7}$$

for all $s \in [0,T]$. We only have to show that $\sigma_y > T$. Indeed, let us assume by contradiction that $\sigma_y \leq T$. By (2.6) we have that $y_1(\cdot)$ is bounded on $[0,\sigma_y)$ and, since F is compact valued, we have that there exists $\lim_{s\uparrow\sigma_y} y_1(s)$. Denote this limit by y_1^*. On the other hand, by (2.6) and (2.7) we have that $y_1(s) \in B(y_0(s), \eta) \cap K \subseteq M$ for all $s \in [0,\sigma_y)$, and, since M is a compact subset of K, passing to the limit for $s \uparrow \sigma_y$ we obtain that $y_1^* \in K$. Using this observation, since K is viable (by Theorem 1.2), we conclude that y_1 can be continued to the right of σ_y, which contradicts the fact that y_1 is noncontinuable. The proof is complete. $\square$

When K is a closed subset of X we obtain a stronger version of Theorem 2.3.

Theorem 2.4. *Let X be a finite dimensional space, K a closed subset of X, $F : K \rightsquigarrow X$ a continuous multifunction, with convex and compact values. Assume that there exists $G : [0,\infty) \to [0,\infty)$ a Perron function such that* (2.1) *holds for any $x, y \in K$. Moreover, assume that the tangency condition* (2.2) *is verified for any $x \in K$. Then, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every $x_0, x_1 \in K$, with $\|x_0 - x_1\| < \delta$ and for every $y_0 : [0,T] \to K$ solution of* (1.1) *with $y_0(0) = x_0$, there exists a solution $y_1 : [0,T] \to K$ of* (1.1) *with $y_1(0) = x_1$ such that*

$$\|y_1(s) - y_0(s)\| < \varepsilon,$$

for all $s \in [0,T]$.

Proof. Fix $T > 0$ and $\varepsilon > 0$ and $\delta(\varepsilon,T) > 0$ be given by Proposition 1.5. Let $x_0, x_1 \in K$ with $\|x_0 - x_1\| < \delta$ and $y_0 : [0,T] \to K$ solution of (1.1) with $y_0(0) = x_0$. By Remark 2.2, there exist $y_1 : [0,\sigma_y) \to K$ a noncontinuable solution of (1.1) with $y_1(0) = x_1$ and $z_1 : [0,\sigma_z) \to \mathbb{R}$ a noncontinuable solution of $z'(s) = G(z(s))$ with $z_1(0) = \|x_0 - x_1\|$, satisfying $\sigma_y \geq \min\{T,\sigma_z\}$ and $\|y_0(s) - y_1(s)\| \leq z_1(s)$ for all $s \in [0, \min\{T,\sigma_z\})$. By Proposition 1.5, we have that $T < \sigma_z$, hence $T \leq \sigma_y$ too, and $|z_1(s)| < \varepsilon$ for all $s \in [0,T]$. This achieves the proof. □

3. APPLICATION

In this section we prove a result on the propagation of continuity of the state constrained minimum time function.

Let $F : X \rightsquigarrow X$ be a multifunction with nonempty values and consider the differential inclusion

$$y'(t) \in F(y(t)). \tag{3.1}$$

Let K be a closed nonempty subset of X with $0 \in K$. The K-constrained minimum time function $T : K \to [0,+\infty]$ is defined by

$$T(x) = \inf\{T \geq 0;\ \exists y(\cdot) \text{ solution of (3.1) satisfying } y(t) \in K\ \forall t \in [0,T],$$
$$y(0) = x,\ y(T) = 0\}.$$

If no solution from x can reach zero then $T(x) = +\infty$. We denote by $\mathcal{R}$ the set of all points $x \in K$ such that $T(x) < +\infty$.

We shall consider the following hypothesis:

(H) For any $\varepsilon > 0$ there exists $\eta(\varepsilon) > 0$ such that any point $x \in X \setminus \{0\}$ with $\|x\| < \eta(\varepsilon)$ can be transferred to 0 by solutions of (1.1) in a time $t \leq \varepsilon$.

We say that the control system (1.1) which satisfies (H) is small time locally controllable. In [14] it is proved that if F is upper semicontinuous with convex and compact values and there exist $r, c, \gamma > 0$ such that, for all $x \in K \cap B(0,r)$,

$$\inf_{u \in F(x) \cap T_K(x)} \langle x, u \rangle \leq c\|x\|^2 - \gamma\|x\|,$$

then $T(\cdot)$ is locally proto-Lipschitz, that is, on a neighborhood of the origin in K, $T(x) \leq M\|x\|$ for some $M > 0$, which clearly implies (H).

Theorem 3.1. *Let X be a finite dimensional space, K a closed subset of X with $0 \in K$, $F : X \rightsquigarrow X$ a continuous multifunction, with convex and compact values. Assume that there exists $G : [0,\infty) \to [0,\infty)$ a Perron function such that* (2.1) *holds for any $x, y \in K$. Moreover, assume that the tangency condition* (2.2) *is verified for any $x \in K$. Assume that* (H) *holds and that $0 \in F(0)$. Then the reachable set $\mathcal{R}$ is* *open in K and the minimum time function $T(\cdot)$ is locally uniformly continuous on $\mathcal{R}$.*

Proof. Let $x \in \mathcal{R}$ and $\varepsilon > 0$. Let z_1, z_2 in K intersected with the closed ball of center x and radius $\delta(\eta(\varepsilon), T(x))$ (given by Proposition 1.5) and such that

$$\|z_1 - z_2\| < \delta(\eta(\varepsilon), T(x) + \varepsilon). \tag{3.2}$$

Let y_x be the optimal solution for x, that is $y_x(0) = x$ and $y_x(T(x)) = 0$. By Theorem 2.4, there exists $y_1 : [0, T(x)] \to K$ a solution of (3.1) with $y_1(0) = z_1$ such that

$$\|y_1(t) - y_x(t)\| < \eta(\varepsilon),$$

for all $t \in [0, T(x)]$. It follows that $\|y_1(T(x))\| < \eta(\varepsilon)$. By (H) we have that $y_1(T(x)) \in \mathcal{R}$ which implies that $z_1 \in \mathcal{R}$ and $T(y_1(T(x))) < \varepsilon$. Applying now Bellman optimality principle, we obtain that

$$T(z_1) \leq T(x) + \varepsilon. \tag{3.3}$$

Now let y_{z_1} be the optimal solution for z_1. As $0 \in F(0)$, we can extend y_{z_1} on the whole interval $[0, \infty)$, putting $y_{z_1}(t) = 0$ for $t > T(z_1)$. Applying again Theorem 2.4, this time for z_1, z_2 and y_{z_1}, and taking into account (3.2), there exists $y_{z_2} : [0, T(x) + \varepsilon] \to K$ solution of (3.1) with $y_{z_2}(0) = z_2$ such that

$$\|y_{z_2}(t) - y_{z_1}(t)\| < \eta(\varepsilon)$$

for every $t \in [0, T(x) + \varepsilon]$. Taking $t = T(z_1)$, by (3.3), we have

$$\|y_{z_2}(T(z_1))\| < \eta(\varepsilon).$$

By (H) and by Bellman optimality principle we obtain that $T(z_2) \leq T(z_1) + \varepsilon$. Switching the roles of z_1 and z_2 we obtain that

$$|T(z_1) - T(z_2)| \leq \varepsilon,$$

which proves the theorem. □

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