Dynamic Systems and Applications 21 (2012) 467-490

BOUNDARY CONTROL AND HIDDEN TRACE REGULARITY OF A SEMIGROUP ASSOCIATED WITH A BEAM EQUATION AND NON-DISSIPATIVE BOUNDARY CONDITIONS

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ABSTRACT. The aim of this work is to develop analyses suitable for studying beam and plate equations equipped with non-monotone feedback boundary conditions. While the analysis of monotone structures is well known by now and based on applications of a suitable version of monotone semigroup theory, in the non-monotone case detailed analysis (microlocal) on the boundary seems necessary. In fact, it is shown that boundary traces display a rather peculiar type of "hidden regularity" which is instrumental in showing that (i) the resulting semigroup is of Gevrey's class, and (ii) the associated control system is "well-posed" within a standard finite energy space and with controls that are **not necessarily collocated**. The result is valid for finite and infinite horizon control problems. This is the first control result of this type in hyperbolic-like dynamics and a non-collocated framework. The unexpected beneficial role of breaking monotonicity is proved to have critical influence on the well-posedness of the control system with control actuators placed at different boundary conditions than the damping.

Numerical simulations reveal spectral properties of the operators complementing theoretical findings.

Key words: Beam and plate equations, non-monotone feedback boundary conditions, hidden boundary regularity, Gevrey's class, spectral analysis, admissible control systems

AMS (MOS) Subject Classification. Primary: 35L05, 35L20 Secondary: 58G16

1. INTRODUCTION

In this paper we study second order in time PDE scalar equations with boundary conditions that are *non-monotone*. This type of model arises in the context

The research of Lasiecka has been supported by the NSF Grant DMS-0606682 and the AFOSR Grant FA9550-09-1-0459. Received November 15, 2011

of modeling long and flexible robot arms -see [7] and references therein. Since the boundary terms involved are not bounded by the topology of the underlying phase spaces, we deal with non-monotone problems with boundary conditions that are not defined on the phase space. As a consequence, this type of problem is not amenable to perturbation or fixed point methods.

It turns out that microlocal analysis estimates allows one not only to prove well-posedness and appropriate energy estimates exhibited by traces of solutions, but also to infer (rather unexpectedly) regularity of solutions that is classified as Gevrey's class [2, 15]. In the present paper we continue the analysis further by discussing asymptotic behavior of the associated semigroups along with control theoretic consequences of the resulting *control* $\rightarrow$ *state* maps. The theoretical findings are illustrated by numerical simulations.

In order to keep this paper focused and simple, we choose to illustrate the method on a simple example of a beam equation. However, the methodology presented is applicable to more general, multidimensional problems [15].

Accordingly, we shall consider the following initial boundary value problem defined for the forced beam equation

$$u_{tt} + u_{xxxx} = f,\ x \in \Omega = (0,1), t > 0 \tag{1.1}$$

with homogeneous-clamped boundary conditions on one end

$$u(0,t) = u_x(0,t) = 0, \tag{1.2}$$

and non-homogeneous and non-monotone boundary conditions on the other end given by:

$$u_{xxx}(1,t) = g_2(t),\ u_{xx}(1,t) = -ku_t(1,t) + g_1(t),\ k \geq 0 \tag{1.3}$$

and standard finite energy initial conditions:

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), x \in \Omega.$$

Function $g_2(t)$ corresponds to force control and function $g_1(t)$ corresponds to moment control.

We are interested in well-posedness and regularity of the corresponding solutions. Well-posedness will be considered within the so called *finite energy space* -i.e. $H \equiv H^2_{cl}(\Omega) \times L_2(\Omega)$, where

$$H^2_{cl}(\Omega) \equiv \{u \in H^2(\Omega), u(0) = u_x(0) = 0\}$$

The energy function associated with the model is standard and given by:

$$E(t) = \frac{1}{2}\int_0^1 \left[|u_{xx}(t,x)|^2 + |u_t(t,x)|^2\right] dx.$$

In order to gain some perspective on the general questions asked, let's take a detour and consider the abstract control system given by

$$y_t = Ay + Bg, y(0) = y_0 \in H \tag{1.4}$$

where H is a Hilbert space, A is a generator of a continuous semigroup e^{At} on H, and B is a control operator acting on $U \to [D(A^*)]'$ where U is another Hilbert space. Motivated by boundary or point control theories [13], we assume that $B \in \mathcal{L}(U, [D(A^*)]')$, which allows for control actions to reside outside the phase space H. When the control action is bounded, i.e., $B \in \mathcal{L}(U, H)$, then the *control to state map* - $L_T : L_p(0, T, U) \to H$, for $1 \le p \le \infty$ is written via the classical variation of parameters formula as:

$$L_T g = \int_0^T e^{A(T-s)} B g(s) ds.$$

However, in the case when the range of B is outside H then the correct representation of the control to state operator is the following *weak-dual* form:

$$(L_T g, \phi)_H = \left(\int_0^T e^{A(T-s)} R(\lambda, A) B g(s) ds, (\lambda - A^*)\phi \right)_H$$

for all $\phi \in D(A^*)$ and some $\lambda \in \rho(A)$. Here $R(\lambda, A) = [\lambda I - A]^{-1}$ for λ in the resolvent set $\rho(A)$.

The above representation implies that the control to state map L_T must be interpreted as an element of a dual space $[D(A^*)]'$, i.e., $L_T \in \mathcal{L}(L_p(U), (D(A^*))')$. Here $L_p(U)$ is short-hand for $L_p(0, T, U)$. One of the fundamental problems in infinite dimensional control theory is to be able to show that such unbounded control action may still generate a bounded control-state map from U into the control space H. This is to say that one would like to know under which conditions

$$L_T \in \mathcal{L}(L_p(U), H) \tag{1.5}$$

In fact, a simple argument in functional analysis [13] reveals that the validity of (1.5) is equivalent to the "admissibility" of the control operator expressed as

$$B^* e^{A^* t} \in \mathcal{L}\left(H, L_{\bar{p}}(U)\right). \tag{1.6}$$

Such a property is typical for dynamics that exhibit some smoothing, which is rare in hyperbolic-like systems.

On the other hand, the above property is critical for perturbation theory, stability theory and other control-theoretic considerations such as they arise in the analysis of optimal control and controllability. Our aim is to investigate this question within the framework of the system (1.1) -(1.3) defined above. Clearly, boundary control operators associated with this model are intrinsically unbounded, thus the meaning of the control-state map is subtle.

In order to put the PDE problem within the abstract framework described above, we shall identify the state $y \equiv (u, u_t)$ and the control $g \equiv (g_1, g_2)$, so that $U = R^2$. Within this framework, questions that we are asking are the following:

- Does the system (1.1), (1.2), (1.3) with *zero* controls $g_1 = g_2 = 0$ generate a semigroup on H?
- If so, what are the properties of the corresponding *control-to-state* map?
- Is this map bounded as in (1.5) and, if so, for which $p \geq 1$?

The answer to these questions is fundamental from the point of view of control theory. Many control theoretic properties depend as a starting point on the above regularity result [12, 13].

The problem considered is a one dimensional linear Euler - Bernoulli equation with feedback boundary conditions. This class of problems has been studied extensively in the literature, in fact in a much more challenging version when $\Omega \subset R^n$ (see [12, 8] and many references therein). Thus, a natural question that arises is the following: *what is special about this particular model?* It turns out that *boundary conditions destroy natural dissipativity* of the underlying generator (biharmonic operator), thus raising a fundamental question of well-posedness of finite energy solutions, and of validity of *some* energy inequality. On the other hand, this kind of boundary condition arises naturally in modeling of rotating beams under boundary force feedback control [7] and references therein. Thus, the model is of both mathematical and physical interest.

In order to gain an insight into the problem and the challenges it presents let us recall that the *standard monotone* boundary conditions associated with (1.1) and (1.2) [9] are the following:

$$u_{xxx}(1,t) = 0, u_{xx}(1,t) = -k u_{xt}(1,t). \tag{1.7}$$

Actually, the model (1.1) with $f = 0$, clamped end at $x = 0$ and absorbing moments as in (1.7) is a classical model of a contraction semigroup that is exponentially stable. (This property is well known not only for beams but also plates, where the analysis proves substantially more technical [5, 11].) The energy identity for model (1.1), (1.2), (1.7) takes a very simple form

$$E(t) + k \int_0^t u_{xt}^2(1,s)ds = E(0) + \int_0^t \int_0^1 u_t(x,s) f(x,s) dx ds. \tag{1.8}$$

Thus, when $f = 0$ the dissipation rate is proportional to the square of $u_{xt}(x = 1, t)$. Instead, in the case of non-monotone boundary conditions (1.3), the situation is very different as no apparent dissipation rate emerges from the energetic calculations. Indeed, standard energy arguments applied to an unforced beam (with $f = 0$, $b = 0$)

gives:

$$E(t) + k\int_0^t u_t(1,s)u_{xt}(1,s)ds = E(0). \tag{1.9}$$

Thus, in contrast to (1.8), the energy relation in (1.9) *does not yield*, (even with $f = 0$), any *a-priori* bound for the energy. The boundary term does not seem to provide any information about an additional boundary regularity of solutions (which is the case in all problems with monotone boundary dissipation). Even more, boundary terms display a troublesome unboundedness on the boundary that is not controlled by the energy. In short, the *non-monotone boundary conditions considered above do not seem to yield any dissipative law.* Thus, the issue is not only of the "admissibility" of the control to state map, but the even more fundamental issue of the generation of a semigroup.

Based on the discussion above, one easily concludes that the problem is not within the realm of the theory of dissipative semigroups. This, of course, does not mean that there is no semigroup structure behind the model. However, should such exist it is definitely not obvious and of rather hidden structure. In fact, this issue has attracted the attention of several researchers [4, 18] who studied the problem by Riesz basis techniques. On the other hand, it is well known that Riesz basis techniques, besides being computationally intensive, are limited in their applicability due to the famous "gap condition" that a-priori restricts the analysis to - essentially - one-dimensional models. This has motivated our interest in studying the problem from a more intrinsic and general PDE point of view without any reliance on Riesz basis generation [2]. In fact, the analysis based on microlocal estimates in [2] explains the mechanism of a rather peculiar smoothing induced by the boundary conditions which eventually leads to Gevrey's semigroup. In addition, the microlocal estimates lead to energy inequalities that reveal additional smoothing of the boundary traces which *gain* $1/2$ anisotropic derivative, which, via duality will provide a key for resolving the issue of the admissibility of the control to state map L_T.

Surprisingly, the methods employed in [2] are not as elementary as the simplicity of the model might suggest. The main idea is to represent the original semi-flow as a suitable "perturbation" of a "good" semi-flow generated by dissipative boundary conditions similar to these in (1.7). We say "suitable" since the perturbation is defined only at the microlocal level. The main tool for achieving this is a technique, recently developed in [20], that allows for microlocal decomposition of the traces corresponding to hyperbolic-like equations. By using the microlocal analysis tools we will be able to exhibit some dissipative law, but valid only on a *finite time* horizon. This explains the fact that the semigroup is neither contractive nor dissipative. However, the finite time dissipative law exhibits an additional regularizing effect caused by the boundary conditions.

The goal of the present paper is to consider *control* systems driven by feedback dynamics resulting from non-dissipative laws and associated Gevrey's generators. We shall show that the associated control systems are well-posed on both finite and infinite horizon in the language of system theory. This allows us to apply a variety of tools in control theory (Riccati equations, mini-max) where a perquisite is well-posedness or "admissibility" of control actions [1, 13]. The results obtained are illustrated by numerical analysis and computational simulations.

2. MAIN RESULTS

We begin with a definition of Gevrey's class of semigroups.

Definition 2.1. A strongly continuous semigroup e^{At} is of Gevrey's class δ for $t > t_0$ if e^{At} is infinitely differentiable for $t > t_0$ and for every compact $K \subset (t_0, \infty)$ and each $\theta > 0$, there exists a constant $C = C(K, \theta)$ such that

$$\left\| \left(e^{At}\right)^{(n)} \right\|_{\mathcal{L}(H)} \leq C\theta^n (n!)^\delta, \forall t \in K, n = 0, 1, 2 \ldots$$

Let's consider the operator $A : D(A) \subset H \to H$ given by

$$A(u, v) = (v, -u_{xxxx})$$

$$D(A) \equiv \{u \in H^2_{cl}(\Omega) \cap H^4(\Omega), v \in H^2_{cl}(\Omega),\ u_{xxx}(1) = 0, u_{xx}(1) = -kv(1)\}.$$

The following result was obtained in [2]:

Theorem 2.2. *Let $0 < k \neq 1$. The semigroup e^{At}, introduced above with a generator A on H, is of Gevrey's class $\delta > 2$ with $t_0 = 0$.*

Remark 2.3. Gevrey's regularity is described in terms of the bounds on all derivatives of the semigroup. These bounds are weaker than the ones corresponding to the characterization of analyticity, but they are stronger than the ones corresponding to differentiability (see [25, 3, 17]).

The importance of Gevrey's class is that its membership determines spectral properties of the underlying semigroup. In particular, Gevrey's semigroups satisfy spectrum determined conditions. This amounts to saying that an exponential bound of the semigroup is determined by the location of the least stable eigenvalue (with the largest real part).

We recall the definition of a well-posed control system generated by (A, B) where A is a generator of a strongly continuous semigroup on a Hilbert space H and $B : U \to [D(A^*)]'$ is an unbounded (on H) control operator.

Definition 2.4. We say that the control system (A, B) is (p, T), $p \geq 1$ well-posed iff the control-to-state map

$$(Lg)(t) \equiv A \int_0^t e^{A(t-s)} A^{-1} B g(s) ds$$

is bounded on

$$L_p(0, T; U) \to C([0, T]; H).$$

Remark 2.5. When the control operator B is bounded, i.e; $B \in \mathcal{L}(U, H)$ and A is a generator of a strongly continuous semigroup, then any (A, B) system is $(1, T)$ well-posed for any finite T. The above conclusion easily follows from the variation of parameters formula.

We note that $T - \mathit{wellposedness}$, by virtue of the semigroup property, implies T_1 well-posedness for any $T < T_1 < \infty$. Trivially, (p, T) well-posedness implies (q, T) well-posedness for any $q \geq p, T < \infty$. The limiting case $T_1 = \infty$ is allowed when the semigroup e^{At} is exponentially stable.

Remark 2.6. Well-posed control systems provide for an important class of control systems [1, 13]. In fact many properties such as feedback stabilization, solvability of Riccati equations, and optimal control do depend on well-posedness of control systems. In the case when $p < 2$ the well-posedness offers attractive features for nonlinear analysis and the possibility of considering nonlinear perturbations.

Our main results are formulated below.

Theorem 2.7. 1. *With reference to uncontrolled model* (1.1)–(1.3), $k > 0$, *we have the following energy inequality satisfied for the forced equation: There exists a constant* $c > 0$ *such that for all* $t > 0$

$$E(t) + k|u_t(1,t)|^2_{H^{\frac{1}{4}}(0,t)} + k|u_{tx}(1,t)|^2_{H^{-\frac{1}{4}}(0,t)} \leq c\left(E(0) + \int_0^t |f(s)|^2_{L_2(0,1)} ds\right). \tag{2.1}$$

2. *For any* $k \geq 0$ *the control system defined by the state equation* (1.1) *with* (1.2) *and shear control actions* g_2 *(and* $g_1 = 0$*) is well-posed, with respect to the state space* $H \equiv H^2_{cl}(\Omega) \times L_2(\Omega)$, *with* $p = 4/3$ *on both finite and infinite horizon,* $T \leq \infty$. *More specifically, taking* $g_1 = 0$ *in* (1.3), *the following estimate holds with any* $0 < t \leq T$, $T < \infty$*:*

$$E(t) + k|u_t(1,t)|^2_{L_4(0,T)} \leq c\left(E(0) + \int_0^t |f(s)|^2_{L_2(0,1)} ds + k^{-1}|g_2|^2_{L_{\frac{4}{3}}(0,T)}\right) \tag{2.2}$$

3. *If additionally* $k \neq 1$ *then the estimates are valid also for* $T = \infty$.

Here $H^s(0,t)$ denotes Sobolev spaces of order s.

Remark 2.8. Note that the inequality in (2.1) implies for $k \geq 0$ a strong regularizing effect on the boundary. There is a gain of $\frac{1}{4}$ time derivative for the velocity component of the boundary trace.

Remark 2.9. The energy inequality in (2.1) allows us to study semilinear problems with both interior and boundary nonlinear terms. Because of space limitations, this topic is not pursued here.

Remark 2.10. As a matter of comparison with collocated damping, one notices that the *dissipative* boundary conditions

$$u_{xxx}(1,t) = u_t(1,t) + g_2$$

would produce control -to state map well-posedness with $p = 2$ [13]. In other words the additional collocated "hidden regularity" yields

$$u_t(1,t) \in L_2(0,T)$$

rather than non-dissipative and non-collocated damping that gives $u_t(1,t) \in H^{1/4}(0,T) \subset L_4(0,T)$. Thus, the non-monotone boundary conditions, contrary to the appearance, provide an additional 1/2 of anisotropic regularity which translates into 1/4 of a time derivative.

Remark 2.11. Related "hidden" regularity results for two-dimensional plates with moment and shear boundary forcing are established in [15]. The method of proof employed in [15] relies on a natural extension of the microlocal argument used in [2].

The control-theoretic version of the results given above reads as follows:

Corollary 2.12. *The control to state operator L_T associated with* (1.1)–(1.3) *and g_2 is $L_4(U) \to H$ bounded and the control system (A,B) is $(4,\infty)$ well-posed.*

3. TRACE INEQUALITY

The proof of theorem 2.7 is based on microlocal estimates. The main idea goes back to the so-called microlocal decomposition of traces corresponding to boundary value problems [25, 20]. Indeed, the goal is to express one boundary condition in terms of the remaining three, modulo a perturbation that is "smooth". Since we already know that a "good" dissipative (monotone) feedback has the form $u_{xx} = -ku_{tx}$, $x = 1$, the aim is to rewrite (microlocally) the imposed nondissipative boundary conditions with the term $-ku_t$. This is done by algebraic-microlocal decomposition where $u_x(1)$ can be written as a linear combination of the other three traces $u(1), u_{xx}(1), u_{xxx}(1)$ with appropriate PDE coefficients representing time regularity and additional interior terms (resulting from commutators) that can be shown to be of lower order.

Of course, the price for doing this is the introduction of lower order terms that destroy contractivity of the semigroup. Thus, at the end of the process we obtain a good energy estimate but polluted by lower order terms.

In what follows we shall use by now classical anisotropic notation $H^s_a(\Sigma)$ and $H^s_a(Q)$, denoting anisotropic Sobolev spaces that are of anisotropic order s (see [6, 20, 11]). By $H^s_a(Q)$ we mean that s derivatives in Ω and $\frac{s}{2}$ derivatives in time are square integrable. (This is in line with the canonical scaling of the principal part of the operator corresponding to Euler - Bernoulli, Schrödinger and heat operators).

Motivated by the considerations elaborated in the introduction, it is clear that the crux of the matter and the difficulty of the problem lies in the boundary behavior of the underlying PDE. Thus, our main goal is to analyze this behavior and to derive appropriate estimates for the corresponding traces. In order to formulate the result, let's consider a formal adjoint problem which consists of equation (1.1) with $f = 0$, clamped boundary conditions on one end, $x = 0$, and the following feedback boundary conditions on the other end, $x = 1$:

$$
\begin{aligned}
v_{tt} + v_{xxxx} &= 0, x \in \Omega, \quad t > 0 \\
v(0,t) = v_x(0,t) &= 0, \quad t > 0 \\
v_{xxx}(1,t) = kv_{tx}(1,t), \ v_{xx}(1,t) &= 0 \\
v(x,0) = v_0, v_t(x,0) &= v_1, \quad \text{in } \Omega.
\end{aligned} \tag{3.1}
$$

By Theorem 2.2 , the adjoint semigroup $e^{A^*t}(v_0, v_1) = (v(t), v_t(t))$ is strongly continuous and of Gevrey's class.

The technical result needed for the proof of Theorem 2.7 is contained in the Lemma below -proved in [2]- which provides the following trace estimates for solutions (u, u_t) to (1.1) -(1.3) with $f = 0, g_i = 0, i = 1, 2$ and also solutions to the adjoint problem in (3.1).

Lemma 3.1. *Let $k > 0$. For any solutions u to (1.1)–(1.3) with $g_i = 0, i = 1, 2$ and also v satisfying (3.1) the following a priori trace regularity is valid: $\forall t > 0$, $\exists C_{tk} > 0$ such that:*

$$
\begin{aligned}
|u_t|^2_{H^{1/4}(\Sigma_t)} + |u_{tx}|^2_{H^{-1/4}(\Sigma_t)} &\le C_{t,k}\left[E_u(0) + |f|^2_{L_2((0,t)\times(0,1))}\right] \\
|v_t|^2_{H^{1/4}(\Sigma_t)} + |v_{tx}|^2_{H^{-1/4}(\Sigma_t)} &\le C_{t,k}\left[E_v(0) + |f|^2_{L_2((0,t)\times(0,1))}\right]
\end{aligned} \tag{3.2}
$$

where E_u (resp. E_v) denotes the energy corresponding to u (resp. v) and $\Sigma_t \equiv \{x = 1\} \times (0, t)$.

The proof of the lemma is given in [2]. The above trace estimates will allow us to use control theoretic arguments in order to prove the estimates in Theorem 2.7. The needed control theoretic results are developed in the subsequent section.

4. CONTROL FRAMEWORK

4.1. **Uncontrolled dynamics.** We shall represent the system (1.1) - (1.3) as a control system.

We introduce the following spaces and operators:

- $H \equiv H^2_{cl}(\Omega) \times L_2(\Omega)$
- $\mathcal{A}u \equiv u_{xxxx}$ in $\mathcal{D}(\mathcal{A}) \equiv \{u \in H^2_{cl}(\Omega) \cap H^4(\Omega),\, u_{xx}(1) = u_{xxx}(1) = 0\}$
- Green's map $G_1 : L_2(\Gamma) \to H^2_{cl}(\Omega)$ given by $v \equiv G_1 g$ iff

$$v_{xxxx} = 0, \text{ in } \Omega, v_{xxx}(1) = 0, v_{xx}(1) = g$$

- Green's map $G_2 : L_2(\Gamma) \to H^2_{cl}(\Omega)$ given by $v = G_2 g$ iff

$$v_{xxxx} = 0, \text{ in } \Omega, v_{xxx}(1) = g, v_{xx}(1) = 0$$

With the above notation we define

$$A \equiv \begin{pmatrix} 0 & I \\ -\mathcal{A} & 0 \end{pmatrix}$$

and $B_i : L_2(\Gamma) \to [D(A)]'$ given by $B_i \equiv \begin{pmatrix} 0 \\ -\mathcal{B}_i \end{pmatrix}$ where $\mathcal{B}_i = \mathcal{A}G_i$. Then it is known [13] that with adjoints considered as pivots in the L_2 topology

$$\mathcal{B}_1^*(v) = -v_x(1)$$

and

$$B_1^*(\vec{v}) = [0, \mathcal{B}_1^* v_2].$$

Similarly,

$$\mathcal{B}_2^*(v) = v(1)$$

and

$$B_2^*(\vec{v}) = [0, \mathcal{B}_2^* v_2].$$

Note that with $U \equiv L_2(\Gamma)$, $\mathcal{B}_i : U \to [D(\mathcal{A}^{1/2})]'$ and hence $\mathcal{D}(\mathcal{A}^{1/2}) = H^2_{cl}(\Omega)$ we have $\mathcal{B}_i^* : \mathcal{D}(\mathcal{A}^{1/2}) \to U$ is bounded.

In order to understand better the role of "collocated" feedback, let us start with a "good" model which corresponds to a dissipative feedback.

With the above notation, the "good -dissipative- model" with $\mathcal{B}_1$ control action corresponds to a dissipative feedback $\mathcal{B}_1^* u_t$, and the resulting second order equation becomes

$$u_{tt} + \mathcal{A}u + k\mathcal{B}_1\mathcal{B}_1^* u_t = 0$$

or written in the first order form as $\vec{y} = (u, u_t)$

$$\vec{y}_t - A\vec{y} + kB_1B_1^*\vec{y} = 0. \tag{4.1}$$

The above model corresponds to dissipative boundary conditions given by

$$u_{xx}(1,t) = -ku_{xt}(1,t).$$

Standard by now semigroup methods (Lumer Phillips Theorem, [17]) allow us to show that (4.1) generates a strongly continuous contraction semigroup, which additionally is exponentially stable for any $k > 0$ [9, 13]. This latter conclusion follows from the intrinsic dissipativity property combined with the Lyapunov-multipliers method for proving decay rates of linear dissipative PDE's. Dissipative feedback provides an additional "regularity" property on the boundary so that

$$\int_0^\infty |u_{xt}(1,t)|^2 dt \le Ck^{-1}E_u(0). \tag{4.2}$$

We note that (4.2) is equivalent to saying

$$B_1^* e^{(A-kB_1B_1^*)t} \in \mathcal{L}(H, L_2(U)), \forall T > 0. \tag{4.3}$$

In a similar vein one can consider a dissipative model corresponding to $\mathcal{B}_2$ control action with dissipative shear forces feedback given by $\mathcal{B}_2^* u_t$, which then leads to monotone dynamics:

$$u_{tt} + \mathcal{A}u + k\mathcal{B}_2\mathcal{B}_2^* u_t = 0$$

or written in the first order form as $\vec{y} = (u, u_t)$

$$\vec{y}_t - A\vec{y} + kB_2B_2^*\vec{y} = 0. \tag{4.4}$$

The above model corresponds to dissipative boundary conditions given by

$$u_{xxx}(1,t) = ku_t(1,t).$$

As before, standard energy methods give

$$\int_0^\infty |u_t(1,t)|^2 dt \le Ck^{-1}E_u(0) \tag{4.5}$$

and semigroup methods yield both well-posedness and exponential decay of the resulting semigroup. As before, (4.5) is equivalent to saying

$$B_2^* e^{(A-kB_2B_2^*)t} \in \mathcal{L}(H, L_2(U)). \tag{4.6}$$

Remark 4.1. Note that (4.3) (resp. (4.6)) represent an "admissibility" property with $p = 2$ (1.6) valid for the control systems $(A - kB_1B_1^*, B_1)$ (resp. $(A - kB_2B_2^*, B_2)$). In fact, these properties are obtained almost for "free" -as they result from structural symmetry and collocation of the control action and the feedback.

We now analyze the situation where control and feedbacks are non-collocated -as in our non-dissipative model with control action $\mathcal{B}_2$. We notice first:

$$G_2^* \mathcal{A} v = v(1).$$

Thus the boundary conditions $u_{xx}(1,t) = -k u_t(1,t)$ correspond to

$$u_{xx}(1,t) = k G_2^* \mathcal{A} u_t(1,t).$$

This leads to the following abstract model written with control acting via moments $\mathcal{B}_1 \equiv \mathcal{A} G_1$ and the feedback $\mathcal{B}_2^* u_t$:

$$u_{tt} + \mathcal{A} u + k \mathcal{B}_1 \mathcal{B}_2^* u_t = 0$$

or written in the first order form as $\vec{y} = (u, u_t)$

$$\vec{y}_t - A\vec{y} + k B_1 B_2^* \vec{y} = 0. \tag{4.7}$$

The structure in (4.7) reveals an intrinsic lack of dissipativity in the equation. Thus, we deal with the **non-dissipative** and also *not relatively bounded perturbation* $B_1 B_2^*$. However, Theorem 2.2 shows that the operator

$$A_{B_1} \equiv A - k B_1 B_2^*$$

generates a strongly continuous semigroup which is of Gevrey's class. Moreover, the said semigroup $e^{A_{B_1} t}$ is *exponentially stable.* This latter conclusion follows from Gevrey's property combined with spectral analysis in section 5.3 which demonstrates strict negativity of real parts of eigenvalues associated with A_{B_1}. Since Gevrey's semigroups satisfy the *spectrum determined growth condition* [24], exponential stability is equivalent to the fact that the spectrum of the generator is located in the left complex plane -see Thm 1.1 [3]. The generator of the semigroup has compact resolvent. This can be seen from standard elliptic theory.

$$(u,v) \in \mathcal{D}(A) \to v \in H^2(\Omega), u_{xxxx} \in L_2(\Omega)$$

with the boundary conditions:

$$u(0) = u_x(0) = 0, u_{xxx}(1) = 0, u_{xx}(1) = v_x(1) \in H^{1/2}(\Gamma).$$

Elliptic theory ensures $u \in H^3(\Omega) \subset H^2(\Omega)$ with compact injection. Supplying the above result with eigenvalue analysis of section 5 leads to the final conclusion of exponential stability of the semigroup. The same result holds for the adjoint semigroup $e^{A_{B_2}^* t}$ where

$$A_{B_2} \equiv A - k B_2 B_1^*$$

Now the estimates in the first part of the Theorem 2.7 follow from Lemma 3.1 after taking into consideration exponential decay of the Gevrey's semigroup. The proof of the first part of Theorem 2.7 is thus complete.

4.2. **Controlled dynamics.** The main goal of this paper is to study the effects of non-collocated *controlled dynamics* -part 2 in Theorem 2.7. We thus consider controlled dynamics given by

$$\vec{y}_t + A_{B_1}(\vec{y}) = B_2 g_2 \tag{4.8}$$

where now we know that $A_{B_1} \equiv A - kB_1B_2^*$ generates a strongly continuous (Gevrey's) semigroup. The above equation in (4.8) can be written as

$$\vec{y}_t - [A - kB_1B_2^*]\vec{y} - B_2 g_2 = 0$$

or as a second order equation

$$u_{tt} + \mathcal{A}[u + kG_1G_2^*u_t - G_2g_2] = 0.$$

The existence of regular solutions requires that

$$u + kG_1G_2^*u_t - G_2g_2 \in \mathcal{D}(\mathcal{A})$$

This in particular implies compatibility conditions on the boundary which require

$$u_{xx}(1,t) + kG_2^*u_t(1,t) = 0,\ u_{xxx}(1,t) = g_2(t)$$

which is equivalent to

$$u_{xx}(1,t) + ku_t(1,t) = 0,\ u_{xxx}(1,t) = g_2(t)$$

We recall that the corresponding dissipative and collocated model takes the form:

$$u_{xx}(1,t) + ku_{xt}(1,t) = g_1,\ u_{xxx}(1,t) = 0.$$

If one carries out a similar analysis for the control actuating moments,

$$\vec{y}_t + A_{B_2}(\vec{y}) = B_1 g_1 \tag{4.9}$$

then the corresponding boundary conditions become:

$$u_{xx}(1,t) = g_1(t),\ u_{xxx}(1,t) = ku_{tx}(1,t).$$

We also recall that the corresponding "dissipative" and collocated model will take the form

$$u_{xx}(1,t) = 0,\ u_{xxx}(1,t) = u_t(1,t) + g_2(t).$$

Going back to system (4.8) -this is a classical example of an unbounded input control system. PDE interpretation is as follows:

$$u_{tt} + u_{xxxx} = 0,\ u_{xx}(1,t) = -ku_t(1,t),\ u_{xxx}(1,t) = g_2(t)$$

which is the control system actuated by shear forces.

In order to further develop the theory, one would like to classify this control action as an "admissible control" because this class enjoys many properties relevant to control theory.

We recall admissibility conditions and focus on (p,T) admissibility of shear controls, i.e., with $B = B_2$. These amount to verification of the following conditions.

Assumption 4.2. 1. $A_B^{-1}B_2 \in \mathcal{L}(U,H)$,

2. $\int_0^T \left|B_2^* e^{A_B^* t}\vec{y}\right|_U^q dt \leq C_T \, |\vec{y}|_H^q$, $p^{-1}+q^{-1}=1$

with respect to controlled dynamics

$$\vec{y}_t + A_B(\vec{y}) = B_2 g_2 \tag{4.10}$$

where $A_B \equiv A + kB_iB_2^*$. The following result has been shown in [13].

Theorem 4.3. *Assume that system* (4.10) *is p-admissible. This is to say Assumption* 4.2 *is satisfied with* $q \geq 1$. *Then the control system* (4.10) *is well posed with* $p = \bar{q}$, $p^{-1}+q^{-1}=1$. *In particular the control-state map*

$$\vec{g} \to \vec{y}$$

is bounded from

$$L_p(0,T;U) \to C([0,T];H).$$

It is well known that the collocated system $B_i = B_2$ is *admissible* with $q = 2$. This follows from (4.6)-see also more general arguments in [13].

Our task, however, is to verify the admissibility property for the non-dissipative model.

Lemma 4.4. *Control system* (4.10) *is "admissible" for* $i = 1$ *with* $q = 4$ *hence* $p = \frac{4}{3}$.

Proof. Step 1 Condition 1. Solving:

$$A_B^{-1}B_2 g = [y_1, y_2]$$

gives

$$y_2 = 0, \mathcal{B}_2 g = \mathcal{A}y_1 + k\mathcal{B}_2\mathcal{B}_1^* y_2.$$

Hence

$$y_2 = 0, y_1 = \mathcal{A}^{-1}\mathcal{B}_2 g$$

which gives

$$A_B^{-1}B_2 g = [\mathcal{A}^{-1}\mathcal{B}_2 g, 0] \in \mathcal{D}(\mathcal{A}^{1/2}) \times \{0\} \in H.$$

This proves the first condition of admissibility.

Step 2: The second condition is more subtle. It can be written with $\vec{\psi} = e^{A_B^* t}\vec{y_0}$ as

$$\mathcal{B}_1^* \psi_t \in L_4(0,T;U)$$

which is equivalent to

$$\int_0^T |\psi_t(1,t)|^4 dt \leq C|\vec{y}_0|_H^4.$$

But the above inequality follows from the trace inequality in Lemma 3.1 after applying Sobolev's embedding $H^{1/4}(0,T) \subset L_4(0,T)$. So, the system with force control is p-admissible with $p = 4/3$ which is the conjugate exponent to $\bar{p} = 4$. □

Applying Theorem 4.3 we obtain that the maps

$$\int_0^t e^{A_B(t-s)} B_2 g_2(s) ds : L_p(0,T;U_i) \to C([0,T];H) \tag{4.11}$$

are bounded. The above implies the estimate

$$|u(t)|_{H^2(\Omega)} + |u_t(t)|_{L_2(\Omega)} \leq c_t |g_2|_{L_p(U)}.$$

This gives the desired estimate in the second part of the main theorem. In order to establish the third part, we exploit exponential stability of the underlying semigroup. The exponential stability of the semigroups makes it possible to extend the estimates to $T = \infty$. This follows from a duality argument. Our goal is to show that

$$\int_0^\infty \left| B_2^* e^{A_B^* t} y \right|_U^{\bar{p}} dt \leq C |y|_H^{\bar{p}}. \tag{4.12}$$

Validity of (4.12), by duality, implies the statement in the third part of Theorem 2.7. For (4.12) we let $\bar{p} = q$ and calculate with any fixed $T > 0$ so that Lemma 4.3 (and Theorem 4.3) hold.

$$\begin{aligned} \int_0^{nT} |B_2^* e^{A_B^* t} y|_U^q dt &= \sum_i^{n-1} \int_{iT}^{(i+1)T} |B_2^* e^{A_B^* t} y|_U^q dt \\ &\leq C \sum_i^n \int_0^T e^{-\omega i T} |B_2^* e^{A_B^* t} y|_U^q dt \\ &\leq C |y|_H^q \sum_i^n e^{-\omega i T} \leq C |y|_H^q \end{aligned} \tag{4.13}$$

where we exploited the exponential decay of the semigroup. Letting $n \to \infty$ gives the final conclusion of Theorem 2.7.

5. NUMERICAL AND ASYMPTOTIC ANALYSIS

Numerical analysis for these problems can all be reduced to root-finding by observing that

$$\begin{aligned} u(x,t) = e^{\lambda t} \Bigg[& c_1 \sin \sqrt{\frac{\lambda}{2}} x \sinh \sqrt{\frac{\lambda}{2}} x + \\ & c_2 \left(\cos \sqrt{\frac{\lambda}{2}} x \sinh \sqrt{\frac{\lambda}{2}} x - \sin \sqrt{\frac{\lambda}{2}} x \cosh \sqrt{\frac{\lambda}{2}} x \right) \Bigg] \in H, \end{aligned} \tag{5.1}$$

solves $u_{tt} + u_{xxxx} = 0$, where $\lambda \in \mathbb{C}$ and c_1 and c_2 are complex constants chosen so that u is real-valued. Various boundary conditions at $x = 1$ can be satisfied by relating c_1 and c_2 in (5.1) and then by solving the resulting transcendental equation.

5.1. **Dissipative Cases.**

5.1.1. *Case 1.* The boundary condition $u_{xxx}(1,t)=0$ requires

$$c_2 = \frac{c_1}{2}\left(\tanh\sqrt{\frac{\lambda}{2}} - \tan\sqrt{\frac{\lambda}{2}}\right)$$

and $u_{xx}(1,t) + ku_{xt}(1,t) = 0$, along with several trigonometric identities, gives the transcendental equation

$$k\sqrt{2\lambda}\left(\sin\sqrt{2\lambda} + \sinh\sqrt{2\lambda}\right) + \cos\sqrt{2\lambda} + \cosh\sqrt{2\lambda} + 2 = 0 \tag{5.2}$$

for the eigenvalues. If $k=0$, then there is no dissipation and the beam has a free right end. In that case, $\lambda = \pm i\beta$, $\beta \geq 0$, is purely imaginary and (5.2) further reduces to

$$1 + \cos\sqrt{\beta}\cosh\sqrt{\beta} = 0.$$

Since $\cos\sqrt{\beta} = -\text{sech}\sqrt{\beta}$ is exponentially small for large β, $\cos\sqrt{\beta}\approx 0$ and $\lambda \approx \pm i(2n-1)^2\pi^2/4$ for large integers n. At the other extreme as $k\to\infty$, there is again no dissipation and (5.2) reduces to

$$\cos\sqrt{\beta}\sinh\sqrt{\beta} + \sin\sqrt{\beta}\cosh\sqrt{\beta} = 0.$$

Since $\tan\sqrt{\beta} = -\tanh\sqrt{\beta}$ is exponentially close to -1 for large β, $\lambda \approx \pm i(4n-1)^2\pi^2/16$ for large integers n. Seeking a formal asymptotic series of the form

$$\lambda = \pm i\frac{(4n-1)^2\pi^2}{16} + a_1\left(\frac{1}{k}\right) + \mathcal{O}\left(\frac{1}{k}\right)^2,$$

substituting into (5.2), expanding in powers of $1/k$, and ignoring exponentially small terms gives

$$a_1 = -\frac{\pi(4n-1)\left(\sqrt{2}(-1)^n + 2\,\text{sech}\frac{(4n-1)\pi}{4}\right)}{\sqrt{2}(-1)^n((4n-1)\pi - 2) + 4\sin\frac{(4n+1)\pi}{4}\tanh\frac{(4n-1)\pi}{4}}.$$

The calculations were done in *Mathematica* and we choose to spare the reader the details. For large arguments, sech is exponentially close to zero and tanh is exponentially close to one. Therefore, for large n, $a_1 \approx -1$ and

$$\lambda \approx -\frac{1}{k} \pm i\frac{(4n-1)^2\pi^2}{16}. \tag{5.3}$$

Figure 1 shows the spectrum for various values of k. Note how well the approximation in (5.3) holds even for small values of k. Figure 2 shows the paths of the first few eigenvalues as a function of k. Note that the eigenvalues all begin ($k=0$) and end ($k=\infty$) on the imaginary axis with the $k=0$ eigenvalues closer to the origin. Also, the "first" eigenvalue pair is the closest to the imaginary axis for all $k>0$. This implies the existence of an optimal gain parameter k, which was determined numerically to be approximately 0.400072.

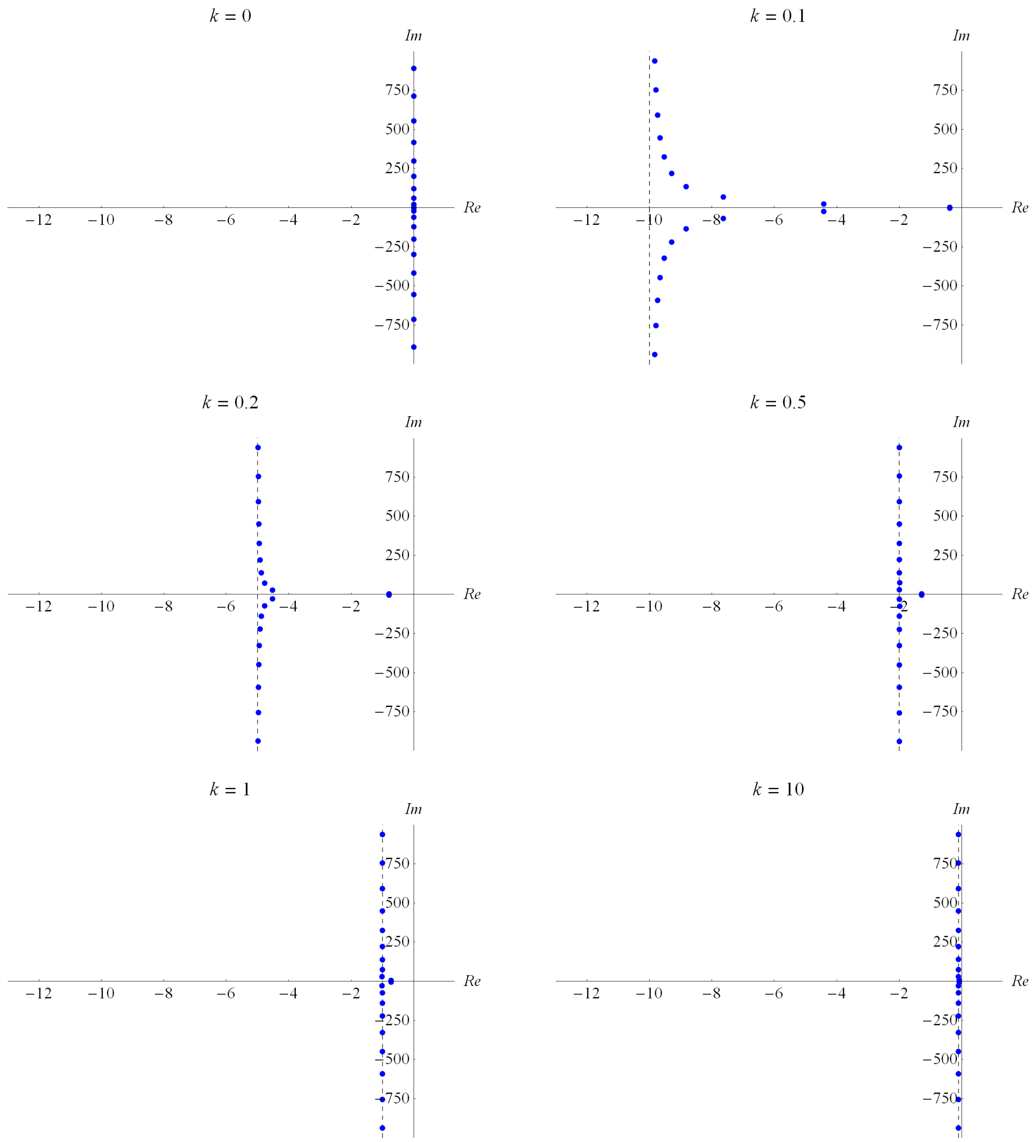

FIGURE 1. Spectrum for different values of k for the dissipative boundary conditions $u_{xxx}(1,t) = 0$, $u_{xx}(1,t) + k u_{xt}(1,t) = 0$. The asymptotes occur at $-1/k$.

5.1.2. *Case 2.* The boundary condition $u_{xx}(1,t) = 0$ requires

$$c_1 = c_2 \left(\tanh \sqrt{\frac{\lambda}{2}} + \tan \sqrt{\frac{\lambda}{2}} \right),$$

and, in turn, $u_{xxx}(1,t) = k u_t(1,t)$ gives the transcendental equation

$$2k \left(\sinh \sqrt{2\lambda} - \sin \sqrt{2\lambda} \right) + 2\sqrt{2\lambda} + \sqrt{2\lambda} \cos \sqrt{2\lambda} + \sqrt{2\lambda} \cosh \sqrt{2\lambda} = 0 \tag{5.4}$$

for the eigenvalues. For $k = 0$, as in Case 1, the beam has a free right end and $\lambda \approx \pm i(2n-1)^2\pi^2/4$, for large integers n. For $k > 0$ but small, an argument similar

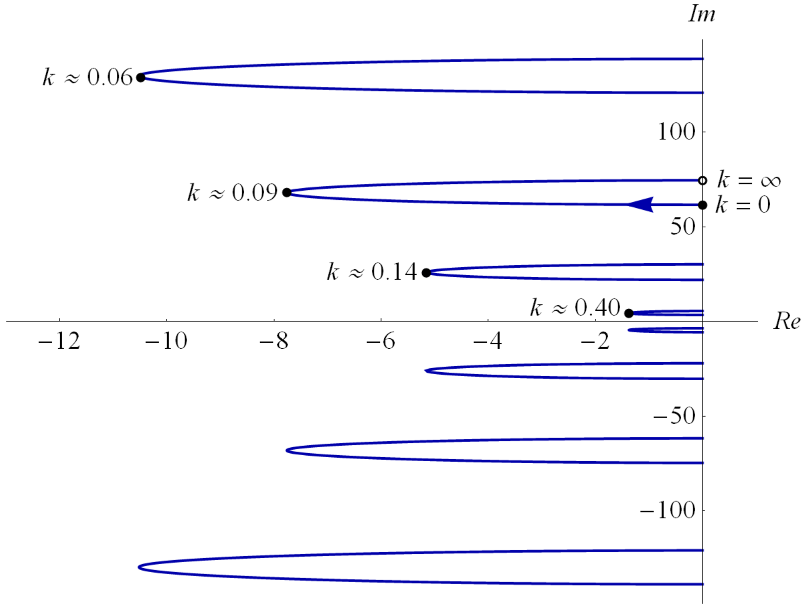

FIGURE 2. Spectral paths for the first four eigenvalue conjugate pairs as a function of k for the dissipative boundary conditions $u_{xxx}(1,t) = 0$, $u_{xx}(1,t) + ku_{xt}(1,t) = 0$. All eigenvalues lie on the imaginary axis for $k = 0$ and as $k \to \infty$, with the $k = 0$ case closer (vertically) to the origin. The optimal value of k is approximately 0.400072.

to the one for Case 1 gives

$$\lambda \approx -2k \pm i\frac{(2n-1)^2\pi^2}{4} \tag{5.5}$$

to within exponentially small terms for large integers n. As $k \to \infty$, the beam approaches a simply supported right end with no dissipation. With $\lambda = \pm i\beta$, $\beta \geq 0$, (5.4) reduces to

$$\cos\sqrt{\beta}\sinh\sqrt{\beta} - \sin\sqrt{\beta}\cosh\sqrt{\beta} = 0$$

and $\lambda \approx \pm i(4n-3)^2\pi^2/16$ for large integers n.

Figure 3 shows part of the spectrum for various values of k. Figure 4 shows the spectral paths for the first few eigenvalues as a function of k. It is similar in appearance to Figure 2, except that the eigenvalues follow counterclockwise paths with increasing k. The numerically determined optimal value of k in this case is 1.65725.

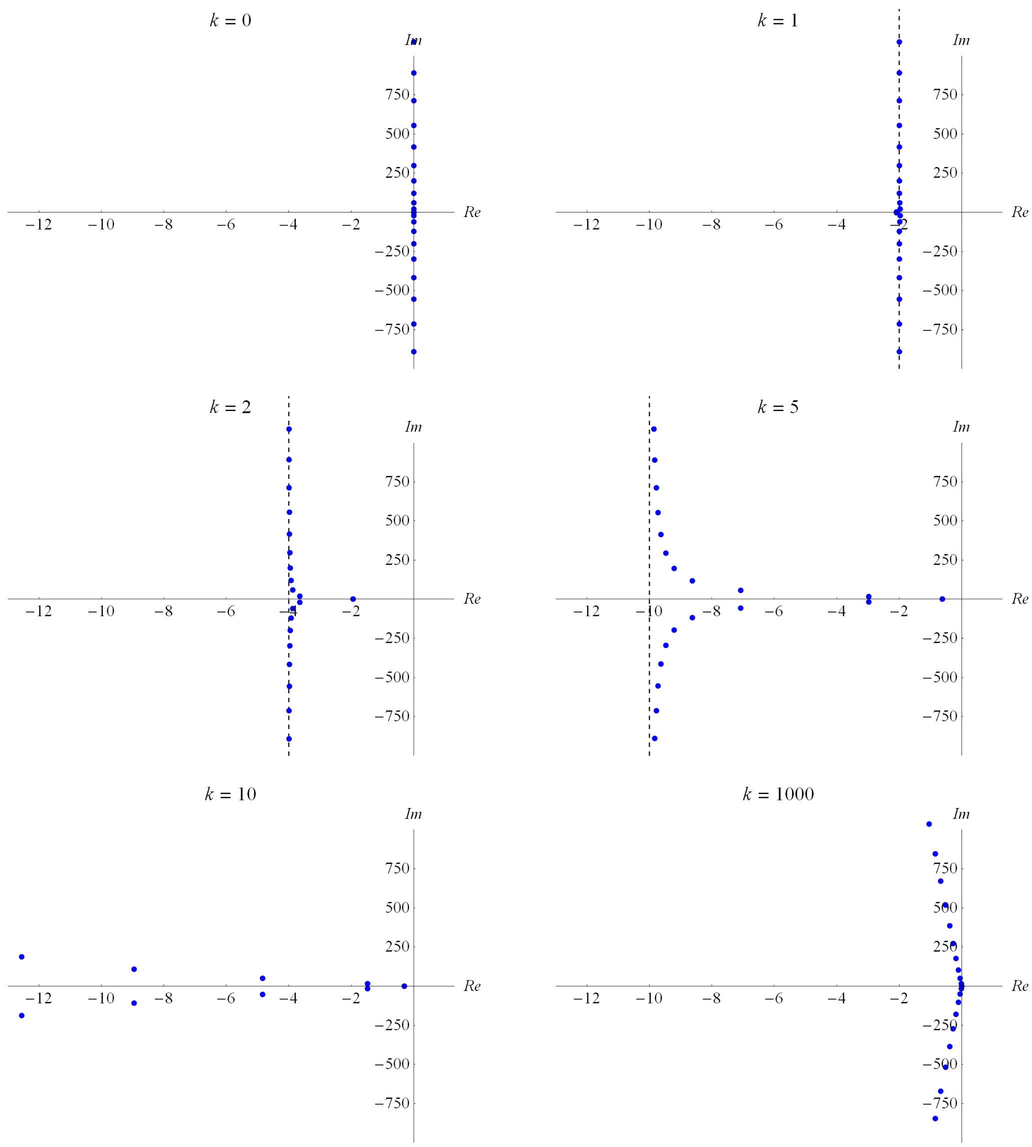

FIGURE 3. Spectrum for different values of k for the dissipative boundary conditions $u_{xx}(1,t) = 0$, $u_{xxx}(1,t) = ku_t(1,t)$. The vertical asymptote occurs near $-2k$.

5.2. **Nondissipative Case.** The boundary condition $u_{xxx}(1,t) = 0$ implies

$$c_2 = \frac{c_1}{2}\left(\tanh\sqrt{\frac{\lambda}{2}} - \tan\sqrt{\frac{\lambda}{2}}\right)$$

and $u_{xx}(1,t) + ku_t(1,t) = 0$ in turn gives the following equation for the eigenvalues:

$$(k+1)\cosh\sqrt{2\lambda} - (k-1)\cos\sqrt{2\lambda} + 2 = 0. \tag{5.6}$$

If $k = 0$, then the right end of the beam is free and $\lambda \approx \pm i(2n-1)^2\pi^2/4$ for large integers n. As $k \to \infty$, (5.6) with $\lambda = \pm i\beta$, $\beta \geq 0$, reduces to

$$\sin\sqrt{\beta}\sinh\sqrt{\beta} = 0,$$

from which we conclude that $\lambda = \pm in^2\pi^2$.

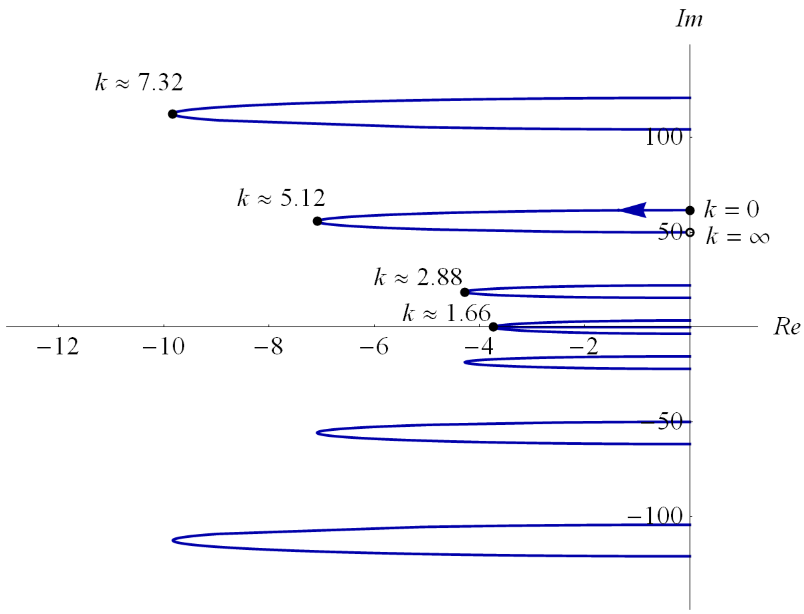

FIGURE 4. Spectral paths for the first four eigenvalue conjugate pairs as a function of k for the dissipative boundary conditions $u_{xx}(1,t) = 0$, $u_x xx(1,t) = ku_t(1,t)$. All eigenvalues lie on the imaginary axis for $k = 0$ and as $k \to \infty$, with the $k = 0$ case further (vertically) from the origin. The optimal value of k is approximately 1.65725.

Of particular interest in this case is $k = 1$ because all of the eigenvalues are – surprisingly – real and are easily computed because (5.6) reduces to

$$\cosh \sqrt{2\lambda} = \cos i\sqrt{2\lambda} = -1,$$

so $\lambda = -(2n-1)^2\pi^2/2$, $n = 1, 2, 3, \ldots$. This is a significant difference from the dissipative cases, where increasing k from zero moves the eigenvalues left and then right again in the complex plane without significant changes in the imaginary parts of the eigenvalues. In the nondissipative case, all of the eigenvalues lie on the imaginary axis for $k = 0$ and then on the real axis for $k = 1$ and back to the imaginary axis as $k \to \infty$. Furthermore, the behavior of the eigenvalues is extremely sensitive near $k = 1$.

It was shown in [2] that for large $|\lambda|$,

$$\lambda \approx \begin{cases} -\frac{(2n-1)\pi}{2} \ln\left(\frac{1+k}{1-k}\right) \pm i\left[\frac{(2n-1)^2\pi^2}{4} - \frac{1}{4}\ln^2\left(\frac{1+k}{1-k}\right)\right] & 0 \le k < 1 \\ -n\pi \ln\left(\frac{k+1}{k-1}\right) \pm i\left[n^2\pi^2 - \frac{1}{4}\ln^2\left(\frac{k+1}{k-1}\right)\right] & k > 1 \end{cases}. \tag{5.7}$$

Note that the approximation in (5.7) includes the results for $k = 0$ and $k \to \infty$ as special cases, but the result for $k = 1$ cannot be recovered by taking limits, which suggests extreme sensitivity of the eigenvalues near $k = 1$. The two approximations in (5.7) can be combined into the quadratic relation

$$\ln^2\left(\frac{k+1}{|k-1|}\right) Im(\lambda) \approx \pm\left[Re(\lambda)^2 - \frac{1}{4}\ln^4\left(\frac{k+1}{|k-1|}\right)\right], \qquad k \neq 1, \tag{5.8}$$

which is included in the plots in Figure 5 for comparison. Figure 5 shows the spectra for different values of k, with an emphasis on values near $k = 1$ to indicate the rapid changes in the eigenvalues that occur there.

Figure 6 shows the paths of the first few eigenvalues as a function of k. Take, for example, the complex conjugate pair of eigenvalues near $\pm 22i$ for $k = 0$. As k increases, they follow the green and orange curves leftward and toward the real axis. At $k = 1$ they meet on the real axis at $-(3\pi)^2/2$, and for $k > 1$, one of them initially decreases and the other increases on the real axis. The decreasing eigenvalue leaves the plot window on the left, but the increasing eigenvalue continues until it meets another decreasing eigenvalue near -11 at $k \approx 1.036$, at which point they split and become a complex conjugate pair and move back toward the imaginary axis at $\pm\pi^2$. Note the very small interval, $k = 1$ to $k \approx 1.036$, for which the eigenvalue is real.

6. **CONCLUSIONS**

1. We have shown that the non-monotone feedback provides better properties with respect to regularity of the dynamics than a standard monotone and collocated feedback. Indeed, not only is the solution of the feedback semigroup more regular (Gevrey's class), but the control actions can be taken in less regular spaces, still producing more regularity on the boundary.
2. In fact, the comparison shows that there is a gain of *one full anisotropic derivative* with respect to standard homogeneous boundary conditions and 1/2 anisotropic derivative with respect to monotone and collocated controls. The ability to use non-collocated controls gives much more flexibility in terms of actuating on the system. The above result is the first result, to our best knowledge, which displays hidden regularity in a non-collocated scenario for hyperbolic-like dynamics.
3. The smoothing effect of boundary conditions propagates into the interior –as evidenced by Gevrey's property, and also the distribution of the eigenvalues are confined to "Gevrey's sectors" –rather than having vertical asymptotes as is the case with the standard boundary conditions that are dissipative.
4. As a consequence of the regularity of the boundary traces with non-collocated feedbacks we are able to show that the control system (A, B) with B corresponding to boundary forcing is a well-posed control system with $p = 4/3$.
5. p - well-posedness of the control system with $p < 2$ allows for consideration of semilinear perturbations which are not necessarily bounded linearly at infinity.

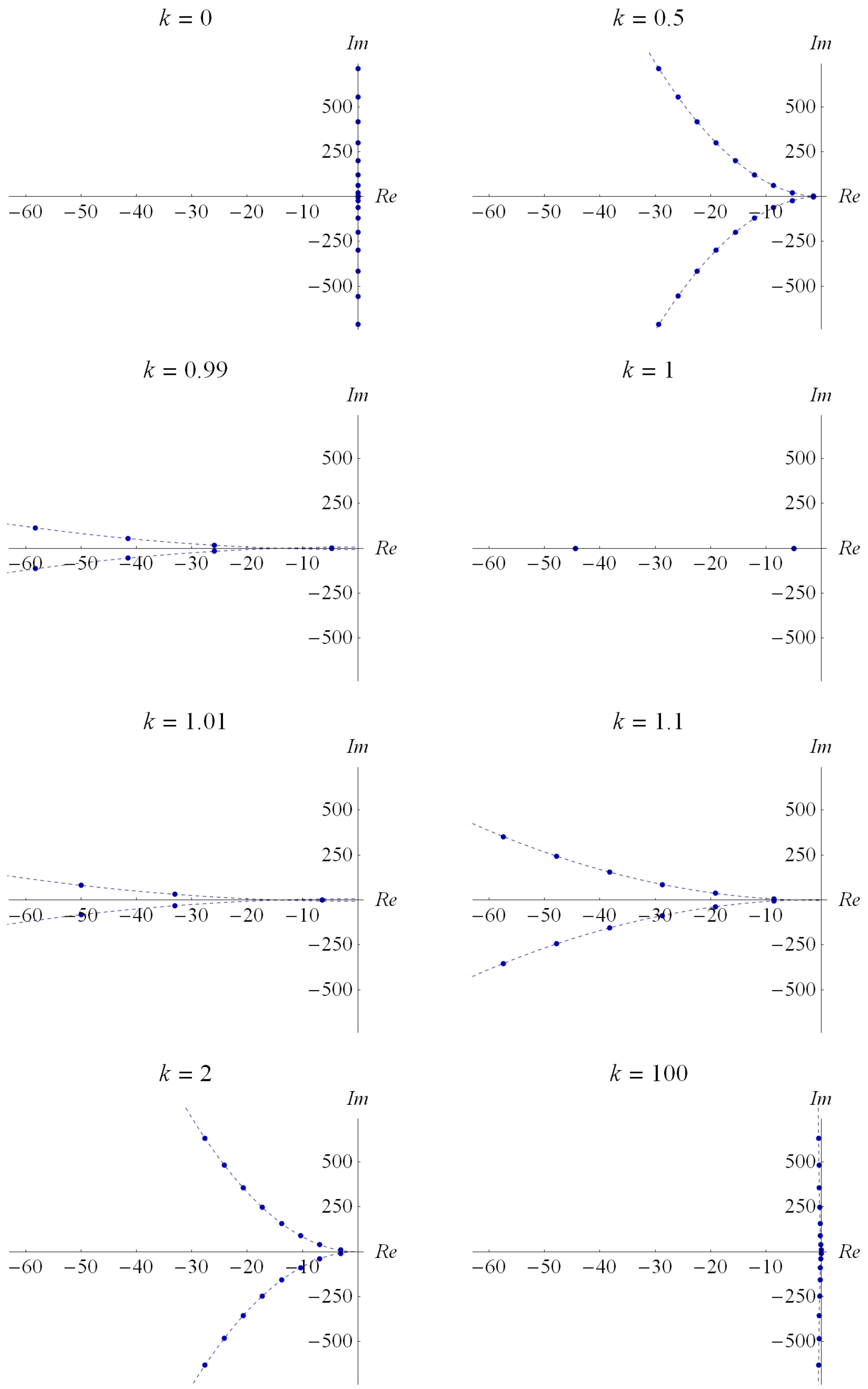

FIGURE 5. Spectrum for different values of k for the nondissipative boundary conditions $u_{xxx} = 0$ and $u_{xx}(1,t) + ku_t(1,t) = 0$. Note how close the eigenvalues are to the parabolas given in (5.8).

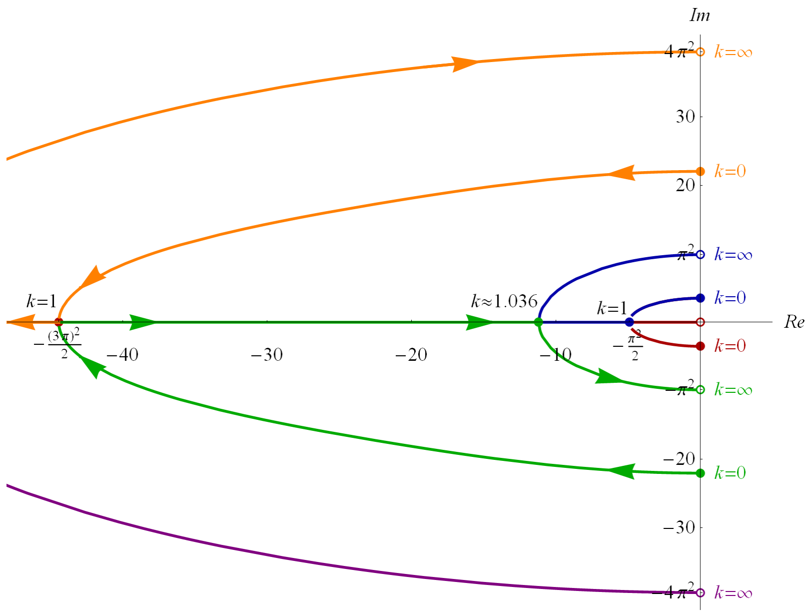

FIGURE 6. Spectral paths for the first few eigenvalues as a function of k for the nondissipative boundary conditions $u_{xxx}(1,t) = 0$ and $u_{xx}(1,t) + ku_t(1,t) = 0$. Note that complex conjugate pairs of eigenvalues coincide on the real axis for $k = 1$.

In addition, $p = 4/3$ allows for consideration of a bilinear control

$$g(t)u_t(t)$$

acting as a shear force on the boundary. The estimate

$$|gu_t(1)|_{L_{4/3}} \leq |g|_{L_2}^{2/3} |u_t(1,t)|_{L_4}^{1/3}$$

can be used (see Theorem 2.7) for the analysis of well-posedness of the bilinear control system.

6. Note that the results of Theorem 2.2 do not require $k \neq 1$. The condition $k \neq 1$ is needed for Theorem 2.7 . Numerical simulations also confirm that this restriction is not essential for the well-posedness of the system. We conjecture that $k \neq 1$ improves robustness of the estimates but it is not essential for stability of the system. However, we are unable to prove this, since the spectral determined condition requires the assumption that $k \neq 1$.
7. The theoretical results obtained above find a full confirmation in the numerical simulation presented in the paper.

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