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A NUMERICAL METHOD FOR AN IMPULSIVE OPTIMAL CONTROL PROBLEM WITH SENSITIVITY CONSIDERATION

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ABSTRACT. In this paper, we consider a class of optimal control problem governed by an impulsive systems with constraints in which some of its data are subject to variation. We formulate this optimal control problem as a new optimal control problem, where the sensitivity of the variation of parameters is minimized subject to an additional constraint indicating the allowable reduction in the optimal cost. The gradient formulae of the cost functional are obtained. On this basis, a gradient-based computational method is established, and the optimal control software, MISER 3.3, can be applied. For illustration, two numerical examples are solved by using the proposed method.

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1. INTRODUCTION

In general, a controlled dynamic system is an idealized description of the actual behavior of a physical or engineering system. During the life span of the system, the values of some of its coefficients may change. For an optimal control problem governed by a dynamical system, the optimal cost obtained is under the assumption that the coefficients of the dynamical system are fixed. Since some of these system coefficients are subject to variation, the sensitivity of the variation of these coefficients should be taken into consideration (see [9]). The sensitivity issue has important practical implication. For example, when optimal control techniques are applied to analyze economic policies (see [1]). However, it is rather complicated and there are few papers dedicated to this important issue. Ref. [4] is a relevant paper. For systems involving state jumps, they are occurred in numerous areas, including electrical engineering, mechanical engineering, medicine, and biological sciences (see [3], [8], [7], [13], and [14]).

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In this paper, we consider an optimal control problem governed by a impulsive differential equations with constraints. Data may subject to perturbations that are modeled by a parameter $a \in R^m$. The optimal control problem will be referred to as follows.

Problem P(a).

$$\text{Minimize}_{(u,\xi,z)\in\mathcal{U}\times\Gamma\times Z} J(u,\xi,z;a) := \Phi_0(x(T),z,a) + \int_0^T L_0(t,x(t),u(t),\xi,a)dt \tag{1.1}$$

subject to

$$\begin{cases} \dot{x}(t) = f(t,x(t),u(t),a), & t \in (0,T) \setminus \{t_1,\dots,t_{N-1}\} \\ x(0) = x_0(a) \\ x(t_i+) = \phi_i(x(t_i-),z,a), & i = 1,2,\dots,N-1 \end{cases} \tag{1.2}$$

with the equality constraints

$$g_i(u,\xi,z;a) = \Phi_i(x(T),z,a) + \int_0^T L_i(t,x(t),u(t),\xi,a)dt = 0, \quad i = 1,\dots,M_e, \tag{1.3}$$

and the inequality constraints

$$g_i(u,\xi,z;a) = \Phi_i(x(T),z,a) + \int_0^T L_i(t,x(t),u(t),\xi,a)dt \le 0, \quad i = M_e+1,\dots,M, \tag{1.4}$$

where $t_i, i = 1,2,\dots,N-1$ are switching times, $x : [0,T] \to R^n$ is the system state, $u : [0,T] \to R^r$ is the control, $x_0 : R^m \to R^n$ is a given initial state, and $z \in R^k$ is a vector, determining the strengths of the jumps at $t = t_i$, $i = 1,\dots,N-1$. The controls, the switching times together with state jumps are decision variables. The assumptions on the problem (P) are made precise in Section 2.

We are concerned here with the cost sensitivity of P(a) with respect to perturbations of a near a nominal value a^0. The purpose of this paper is to propose a computational approach to solving this class of optimal control problems, where the cost sensitivity is minimize.

Computational methods for solving this kind of optimal control problem are given in [11] without impulses and [12] for switch controlled system. They appended to the cost function by using the concept of a penalty function, giving rise to a new appended cost function. It is based on the gradient formulae of an appended cost function, which is obtained by appending the cost sensitivity to the original cost function, forming a new appended cost function. Then, the optimal control software, MISER3 [11], is used after substantial modifications.

In this paper, a new optimal control problem (SP) is introduced which minimize the cost sensitivity of the variation of parameter a around a^0 subject to an additional constraint indicating the allowable reduction in the optimal cost. By solving this new optimal control problem, we obtain new optimal controls and trajectories. Then, a cost is given by substituting the optimal solution into the original cost function formula. We called it a suboptimal cost, which is useful in practice.

The rest of this paper is organized as follows. In Section 2, assumptions of the optimal control problem are introduced. Then, by taking into account the cost sensitivity with respect to the variations of parameters, a new optimal control problem is formulated. In Section 3, we describe how the solution of dynamical system changed as the parameter a is varied. In Section 4, we show that an optimal control problem governed by impulsive dynamical system is equivalent to a standard optimal control problem involving ordinary dynamical systems but with mixed boundary conditions by the control parametrization enhancing transform method(see [10]). Then, the optimal control problem can be solved as a mathematical programming problem by any gradient-based method. We can use the optimal control software, MISER 3.3 (see [6]), without any modification. In Section 5, a numerical method for the new optimal control problem is introduced in Section 5. For illustration, two numerical examples are solved using the proposed method in Section 6.

2. PROBLEM STATEMENT

The following notations and assumptions are used throughout this paper. Let $[0, T]$ be fixed. Define

$$U_1 = \{v = (v_1, \ldots, v_r)^T \in R^r : (E^i)^T v \leq b_i, i = 1, \ldots, q\},$$

where $E^i \in R^r, i = 1, \ldots, q$ and $b_i, i = 1, \ldots, q$ are real numbers; and

$$U_2 = \{v = (v_1, \ldots, v_r)^T \in R^r : u_i^l \leq v_i \leq u_i^h, i = 1, 2, \ldots, r\}$$

where u_i^l and $u_i^h, i = 1, \ldots, r$ are given real numbers. Let

$$U = U_1 \bigcap U_2.$$

Clearly, U is a compact and convex subset of R^r. Set

$$\mathcal{U} = \{u = (u_1, \ldots, u_r)^T : [0, T] \to R^r \text{ is measurable and } , u(t) \in U, a.e., t \in [0, T]\}.$$

$\mathcal{U}$ is said to be an admissible control set. We also define

$$\Gamma = \{\xi = (t_1, \ldots, t_{N-1})^T \in R^{N-1} : t_i - t_{i-1} \geq 0, i = 1, \ldots, N \text{ with } t_0 = 0 \text{ and } t_N = T\}$$

and

$$Z = \{z = (z_1, \ldots, z_k)^T \in R^k : z_i^l \leq z_i \leq z_i^u, i = 1, \ldots, k\}$$

where z_i^l and z_i^u are given constants which are respectively the upper and lower bounds for the corresponding element z_i of the vector z.

Assumptions.

For the functions f, $L_i, i = 0, 1, \ldots, M$, $\phi_i, i = 1, 2, \ldots, N-1$, and $\Phi_i, i = 0, 1, \ldots, M$, we assume that the following conditions are satisfied.

(A1) The functions $f : [0,T] \times R^n \times R^r \times R^m \to R^n$ and $L_i : [0,T] \times R^n \times R^r \times R^{N-1} \times R^m \to R, \quad i = 0, 1, 2, \ldots, M$ are given. f and L_i together with their first and second orders partial derivatives with respect to each of the components of x, u and a are piecewise continuous on $[0,T]$ and continuous on $R^n \times R^r \times R^m$ for each t.

(A2) For any fixed $a \in R^m$ and any compact subset $V \subset R^r$, there exist positive constants C_1 and C_2 such that

$$|f(t,x,u,a)| \leq C_1(1+|x|), \text{ for all } (t,x,u) \in [0,T] \times R^n \times V;$$

(A3) The given functions $\phi_i : R^n \times R^k \times R^m \to R^n, i = 1, 2, \ldots, N-1$ are twice continuously differentiable with respect to x, a and z.

(A4) The given functions $\Phi_i : R^n \times R^k \times R^m \to R, i = 0, 1, 2, \ldots, M$ are twice continuously differentiable with respect to x, a and z.

Once Problem P(a) is solved, an optimal control solution (u^*, ξ^*, z^*) is obtained for $a = a^0$ which is the nominal value of a. But any perturbation of parameters a will result in the optimal cost $J(u^*, \xi^*, z^*; a)$ being violated. Sometimes the vary of $J(u^*, \xi^*, z^*; a)$ is much when a is changed small. It occurred in many practical problems (see [1], [9]). In this case, we called that Problem P(a) is sensitive with the parameter a. Suppose that the optimal cost is allowed reduction in the interval $[J(u^*, \xi^*, z^*; a^0), (1+\alpha)J(u^*, \xi^*, z^*; a^0)]$ for some constant α. We wish the sensitivity of J for the variation of a in minimized. One know that the sensitivity of J in the neighborhood of a^0 is given by the gradient of J with respect to a in $a = a^0$, we denote $D_a J(u, \xi, z; a^0)$. Therefore, in order to ensure that the cost obtained is not sensitive by the variation of a from a^0, one approach is to minimize the cost sensitivity function defined by

$$G(u, \xi, z; a^0) = \left[D_a J(u, \xi, z; a^0)\right] \left[D_a J(u, \xi, z; a^0)\right]^T. \tag{2.1}$$

The new optimal control problem with sensitivity consideration is formulated as follows.

Problem (SP). Given the dynamical system (1.2), find a feasible ternary $(\hat{u}, \hat{\xi}, \hat{z}) \in \mathcal{U} \times \Gamma \times Z$ such that

$$G(\hat{u}, \hat{\xi}, \hat{z}; a^0) \leq G(u, \xi, z; a^0), \quad \forall (u, \xi, z) \in \mathcal{U} \times \Gamma \times Z. \tag{2.2}$$

and subject to the equality constraints (1.3), the inequality constraints (1.4) and

$$J(u,\xi,z;a^0) \le (1+\alpha)J(u^*,\xi^*,z^*;a^0), \tag{2.3}$$

where $\alpha \ge 0$ is a given constant that specifies the allowable amount of increase of the optimal cost functional.

Once $(\hat{u},\hat{\xi},\hat{z})$ is obtained, the $J(\hat{u},\hat{\xi},\hat{z};a^0)$ is regarded as a suboptimal cost.

3. DIFFERENTIABILITY WITH RESPECT TO PARAMETER

In order to study the gradient of J with respect to a, we have to know how the solution of the following controlled system changes as the parameter $a \in \mathbb{R}^m$ is varied when $u \in \mathcal{U}$, $\xi \in \Gamma$, and $z \in \mathcal{Z}$ are fixed.

Consider the impulsive equation:

$$\begin{cases} \dot{x}(t) = f(t,x(t),u(t),a), \quad t \in (0,T)\setminus\{t_1,\dots,t_{N-1}\}, \\ x(0) = x_0(a), \\ x(t_i+) = \phi_i(x(t_i-,z,a), \quad i = 1,2,\dots,N-1. \end{cases} \tag{3.1}$$

Throughout the following, we denotes by $x(t,a)$ the solution of (3.1), while $D_xf(t,x,u,a)$ is the $n\times n$ Jacobian matrix of the first order partial derivatives $\frac{\partial f_i}{\partial x_j}$ $(i = 1,2,\dots,n; j = 1,2,\dots,n)$, $D_af(t,x,u,a)$ is the $n \times m$ matrix of partial derivatives $\frac{\partial f_i}{\partial a_j}$ $(i = 1,2,\dots,n; j = 1,2,\dots,m)$ at point (t,x,u,a).

In the following, we discuss the differentiability of $x(t,a)$ at $a = a^0$, where $a^0 \in \mathbb{R}^m$ is a given vector.

Theorem 3.1. *Assume that* (A1) *and* (A3) *are hold. Let $\hat{x}(t)$ be the solution of* (3.1) *with $a = a^0$ defined for $t \in [0,T]$. For $\rho^0 \in \mathbb{R}^m$, call $v(\cdot)$ the solution of the linear impulsive system*

$$\begin{cases} \dot{v}(t) = D_xf(t,\hat{x}(t),u(t),a^0)v(t) + D_af(t,\hat{x}(t),u(t),a^0)\rho^0, \\ \qquad\qquad t \in (0,T)\setminus\{t_1,t_2,\dots,t_{N-1}\}, \\ v(0) = D_ax_0(a^0)\rho^0, \\ v(t_i+) = [D_x\phi_i(\hat{x}(t_i-),z,a^0) - E]v(t_i-) + D_a\phi_i(\hat{x}(t_i-),z,a^0)\rho^0, \end{cases} \tag{3.2}$$

where E is $n\times n$ identity matrix. Then

$$\lim_{\varepsilon\to 0+}\Big|\frac{x(t,a^0+\varepsilon\rho^0)-\hat{x}(t)}{\varepsilon} - v(t)\Big| = 0. \tag{3.3}$$

Proof. For $\varepsilon > 0$ sufficiently small define

$$x_\varepsilon(t) = x(t,a^0+\varepsilon\rho^0), y_\varepsilon(t) = \hat{x}(t) + \varepsilon v(t).$$

To prove the theorem, we need to show that

$$\lim_{\varepsilon\to 0}\Big|\frac{x_\varepsilon(t)-y_\varepsilon(t)}{\varepsilon}\Big| = 0. \tag{3.4}$$

Observe that $x_\varepsilon(\cdot)$ is the fixed point of the map $w \mapsto \Psi(a^0+\varepsilon\rho^0, w)$ with

$$\begin{aligned}\Psi(a^0+\varepsilon\rho^0, w) &= x_0(a^0+\varepsilon\rho^0)+\int_0^t f(s,w(s),u(s),a^0+\varepsilon\rho^0)ds \\ &\quad + \sum_{0<t_i\le t}\phi_i(w(t_i),z,a^0+\varepsilon\rho^0),\end{aligned}$$

where $\Psi:\mathbb{R}^m\times X\to X$ satisfies the assumptions of Contraction Mapping Theorem (see in [2], A.2, pp265) on the spaces $\mathbb{R}^m$ and $X=PC([0,T];\mathbb{R}^n)$ with norm $\|w\|_1 = \max\limits_{t\in(0,T)\setminus D} e^{-2Lt}|w(t)|$.

Thinking of $y_\varepsilon(\cdot)$ as an approximate fixed point, using the estimate (see in [2], A.6, pp 265) with $k=1/2$, we obtain

$$\frac{1}{\varepsilon}\|x_\varepsilon - y_\varepsilon\|_1 \le \frac{2}{\varepsilon}\|\Phi(a^0+\varepsilon\rho^0, y_\varepsilon)-y_\varepsilon\|_1.$$

To prove (3.4), it therefore sufficient to show that

$$\begin{aligned}&\lim_{\varepsilon\to 0}\sup_{t\in(0,T)\setminus D}\frac{1}{\varepsilon}\big|x_0(a^0+\varepsilon\rho^0)+\int_0^t f(s,\hat{x}(s)+\varepsilon v(s),u(s),a^0+\varepsilon\rho^0)ds \\ &+\sum_{0<t_i<t}\phi_i(\hat{x}(t_i-)+\varepsilon v(t_i-),z,a^0+\varepsilon\rho^0)-\hat{x}(t)-\varepsilon v(t)\big| = 0.\end{aligned}$$

Since

$$\begin{aligned}&\frac{1}{\varepsilon}\big|\{x_0(a^0+\varepsilon\rho^0)+\int_0^t f(s,\hat{x}(s)+\varepsilon v(s),u(s),a^0+\varepsilon\rho^0)ds\} \\ &\quad+\sum_{0<t_i<t}\phi_i(\hat{x}(t_i-)+\varepsilon v(t_i-),z,a^0+\varepsilon\rho^0)-\hat{x}(t)-\varepsilon v(t)\big| \\ &=\frac{1}{\varepsilon}\big|x_0(a^0)+D_a x_0(a^0)\cdot\varepsilon\rho^0+\int_0^t f(s,\hat{x}(s),u(s),a^0)ds \\ &\quad+\int_0^t D_x f(s,\hat{x}(s),u(s),a^0)\varepsilon v(s)ds \\ &\quad+\int_0^t D_a f(s,\hat{x}(s),u(s),a^0)\varepsilon\rho^0 ds-\hat{x}(t)-\varepsilon v(t) \\ &\quad+\sum_{0<t_i\le t}[\phi_i(\hat{x}(t_i-),z,a^0)+D_x\phi_i(\hat{x}(t_i-),z,a^0)\varepsilon v(t_i-)+D_a\phi_i(\hat{x}(t_i-),z,a^0)\varepsilon\rho^0] \\ &\quad+\varepsilon G(t,\varepsilon)\big| \\ &=\big|D_a x_0(a^0)\rho^0+\int_0^t D_x f(s,\hat{x}(s),u(s),a^0)v(s)ds+\int_0^t D_a f(s,\hat{x}(s),u(s),a^0)\rho^0 ds \\ &\quad+\sum_{0<t_i\le t}[D_x\phi_i(\hat{x}(t_i-),z,a^0)v(t_i-)+D_a\phi_i(\hat{x}(t_i-),z,a^0)\rho^0]-v(t)+G(t,\varepsilon)\big| \\ &=|G(t,\varepsilon)|.\end{aligned}$$

where

$$
\begin{aligned}
&G(t,\varepsilon)\\
&=\int_0^t\int_0^1[D_xf(s,\hat{x}(s)+\sigma\varepsilon v(s),u(s),d^0)-D_xf(s,\hat{x}(s),u(s),a^0)]\cdot v(s)d\sigma ds\\
&\quad+\int_0^t\int_0^1[D_af(s,\hat{x}(s)+\sigma\varepsilon v(s),u(s),d^0)-D_af(s,\hat{x}(s),u(s),a^0)]\cdot \rho^0 d\sigma ds\\
&\quad+\sum_{0<t_i\le t}\int_0^1[D_x\phi_i(\hat{x}(t_i-)+\sigma\varepsilon v(t_i-),z,d^0)-D_x\phi_i(\hat{x}(t_i-),z,a^0)]d\sigma\cdot v(t_i-)\\
&\quad+\sum_{0<t_i\le t}\int_0^1[D_a\phi_i(\hat{x}(t_i-)+\sigma\varepsilon v(t_i-),z,d^0)-D_a\phi_i(\hat{x}(t_i-),z,a^0)]d\sigma\cdot \rho^0
\end{aligned}
$$

and $d^0=a^0+\sigma\varepsilon\rho^0$. Hence

$$
\begin{aligned}
&|G(t,\varepsilon)|\\
&\le\int_0^T\int_0^1\parallel D_xf(s,\hat{x}(s)+\sigma\varepsilon v(s),u(s),d^0)-D_xf(s,\hat{x}(s),u(s),a^0)\parallel\cdot|v(s)|d\sigma ds\\
&\quad+\int_0^T\int_0^1\parallel D_af(s,\hat{x}(s)+\sigma\varepsilon v(s),u(s),d^0)-D_af(s,\hat{x}(s),u(s),a^0)\parallel\cdot|\rho^0|d\sigma ds\\
&\quad+\sum_{i=1}^{N-1}\int_0^1\parallel D_x\phi_i(\hat{x}(t_i-)+\sigma\varepsilon v(t_i-),z,d^0)-D_x\phi_i(\hat{x}(t_i-),z,a^0)\parallel d\sigma\cdot|v(t_i-)|\\
&\quad+\sum_{i=1}^{N-1}\int_0^1\parallel D_a\phi_i(\hat{x}(t_i-)+\sigma\varepsilon v(t_i-),z,d^0)-D_a\phi_i(\hat{x}(t_i-),z,a^0)\parallel d\sigma\cdot|\rho^0|.
\end{aligned}
$$

By the Lebesgue Dominated Convergence Theorem, the right hand side of inequality above convergence to zero as $\varepsilon\to 0$. In turn, this proves (3.4), hence (3.3). $\square$

Theorem 3.1 states the existence of all directional derivatives for map $a\mapsto x(t,a)$. Therefore, we conclude that the map $a\mapsto x(t,a)$ is differentiable. Its Jacobian matrix at a given a^0 is $D_ax(t,a)|_{a=a^0}=[\frac{\partial x_i(t,a^0)}{\partial a_j}]_{n\times m}$. If we denote the jth column of $D_ax(t,a)|_{a=a^0}$ by

$$
w_j(t)=\left(\frac{\partial x_1(t,a^0)}{\partial a_j},\frac{\partial x_2(t,a^0)}{\partial a_j},\dots,\frac{\partial x_n(t,a^0)}{\partial a_j}\right)^T\in\mathbb{R}^n,\quad j=1,2,\dots,m
$$

and

$$
w(t)=(w_1(t),w_2(t),\dots,w_m(t)),
$$

then $w_j(\cdot)$ are solutions of the following problem

$$
\tag{3.5}
\begin{cases}
\dot{w}_j(t)=D_xf(t,\hat{x}(t),u(t),a^0)w_j(t)+D_af(t,\hat{x}(t),u(t),a^0)e_j,\\
\qquad\qquad t\in(0,T)\setminus\{t_1,\dots,t_{N-1}\},\\
w_j(0)=\frac{\partial x_0(a^0)}{\partial a_j},\\
w_j(t_i+)=[D_x\phi_i(\hat{x}(t_i-),z,a^0)-E]w_j(t_i-)+D_a\phi_i(\hat{x}(t_i-),z,a^0)e_j
\end{cases}
$$

where $e_j = (0, 0, \ldots, 1, \ldots, 0)$ is the standard basis in $\mathbb{R}^n$.

4. NUMERICAL METHOD FOR IMPULSIVE OPTIMAL CONTROL PROBLEMS

For any $a \in R^m$ fixed, Problem P(a) is a standard optimal impulsive control problem and it can be solved by many existing optimal control computational techniques. One of difficulties to solve the optimal impulsive control problem P(a) is that we have no insight of how the switching times t_i, i.e., ξ are distributed. The control parametrization enhancing transform (CEPT) method introduced in [10] can be used to solving this problem. We state in brief as follows.

At first, it is shown that the optimal impulsive control problem P(a) is equivalent to a standard optimal control problem with mixed boundary conditions.

Consider a new time scale $s \in [0, N]$. Define a transformation from $t \in [0, T]$ to $s \in [0, N]$ as the following differential equation.

$$\frac{dt(s)}{ds} = v(s) \tag{4.1}$$

with the initial condition

$$t(0) = 0. \tag{4.2}$$

Then a mapping $t(s)$ from $t \in [0, T]$ to $s \in [0, N]$ maps the variable jump points

$$0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$$

into the fixed points $0, 1, \ldots, N$.

A scalar function $v(s)$ for all $s \in [0, N]$ is called a time scale control if it is a piecewise constant function with possible discontinuous at the pre-fixed knots $s = 1, 2, \ldots, N-1$, i.e.,

$$v(s) = \sum_{i=1}^{N} \delta_i \chi_{[i-1,i]}(s), \tag{4.3}$$

where $\chi_I(\cdot)$ is the indicator function of the interval I as defined by

$$\chi_I(t) = \begin{cases} 1 & \text{if } t \in I, \\ 0 & \text{otherwise,} \end{cases}$$

$\delta_i, i = 1, 2, \ldots, N$ are parameters which satisfy

$$\delta_i \geq 0, \quad i = 1, 2, \ldots, N \text{ and } \sum_{i=1}^{N} \delta_i = T. \tag{4.4}$$

Let $\delta = (\delta_1, \ldots, \delta_{N-1})$ and

$$\Theta = \left\{ \delta = (\delta_1, \ldots, \delta_{N-1}) : \delta_i \geq 0, i = 1, \ldots, N-1, \sum_{i=1}^{N-1} \delta_i \leq T \right\}.$$

If δ is given, $\delta_N = T - \sum_{i=1}^{N-1} \delta_i$ and jump points $t_i, i = 1, 2, \ldots, N-1$ are determined uniquely by δ.

After the transform, the dynamical system (1.2) take form

$$\frac{dx(s)}{ds} = \widetilde{f}(s, x(s), u(s), \delta, a), \quad s \in (0, N], \tag{4.5}$$

with initial condition

$$x(0) = x_0(a), \tag{4.6}$$

and jump conditions

$$x(i+) = \widetilde{\phi}_i(x(i-), z, a), \quad i = 1, \ldots, N-1. \tag{4.7}$$

where

$$\widetilde{f}(s, x(s), u(s), \delta, a) = v(s) f(t(s), x(t(s)), u(t(s)), a).$$

and

$$\widetilde{\phi}_i(x(i), z, a) = \phi_i(x(t_i), z, a).$$

Constrains (1.3) and (1.4) are changed into the following form.

$$\begin{aligned} \widetilde{g}_i(u, \delta, z; a) = {} & \widetilde{\Phi}_i(x(N), z, a) \\ & + \int_0^N \widetilde{L}_i(s, x(s), u(s), \delta, a) ds = 0, \quad i = 1, \ldots, M_e, \end{aligned} \tag{4.8}$$

and the inequality constraints

$$\begin{aligned} \widetilde{g}_i(u, \delta, z; a) = {} & \widetilde{\Phi}_i(x(N), z, a) \\ & + \int_0^N \widetilde{L}_i(s, x(s), u(s), \delta, a) ds \leq 0, \quad i = M_e + 1, \ldots, M, \end{aligned} \tag{4.9}$$

where

$$\begin{aligned} \widetilde{\Phi}_i(x(N), z, a) &= \Phi_i(x(T), z, a), \\ \widetilde{L}_i(s, x(s), u(s), \delta, a) &= v(s) L_i(t(s), x(t(s)), u(t(s)), \xi(\delta), a), \\ & \quad i = 1, \ldots, M. \end{aligned}$$

The equivalent optimal control problem of P(a) may now be stated as follows.

Problem P1(a). Subject to the impulsive dynamics system (4.5)and (4.1) with initial conditions (4.6) and (4.2), jump conditions (4.7), and constrains (4.8) and (4.9), find a control and parameters $(u, \delta, z) \in \mathcal{U} \times \Theta \times Z$ such that the cost function

$$\widetilde{J}(u, \delta, z; a) = \widetilde{\Phi}_0(x(N), z, a) + \int_0^N \widetilde{L}_0(s, x(s), u(s), \delta, a) ds, \tag{4.10}$$

is minimized. Where

$$\begin{aligned} \widetilde{\Phi}_0(x(N), z, a) &= \Phi_0(x(T), z, a), \\ \widetilde{L}_0(s, x(s), u(s), \delta, a) &= v(s) L_0(t(s), x(t(s)), u(t(s)), \xi(\delta), a). \end{aligned}$$

Until now, jump points have been fixed. But the problem could not be solved directly because of jump points. Define

$$y_i(s) = x(s+i-1), \quad i=1,\dots,N, \quad s\in[0,1], \tag{4.11}$$

$$\widetilde{u}_i(s) = u(s+i-1), \quad i=1,\dots,N, \quad s\in[0,1], \tag{4.12}$$

$$\tau_i(s) = t(s+i-1), \quad i=1,\dots,N, \quad s\in[0,1]. \tag{4.13}$$

and let $\tau=(\tau_1,\dots,\tau_N)$ be the switching parameter vector, $y=(y_1,\dots,y_N)$ be the state vector and $\widetilde{u}=(\widetilde{u}_1,\dots,\widetilde{u}_N)$ be the control vector. Then for each $i=1,2,\dots,N$, y_i and τ_i satisfy the following differential equation:

$$\begin{cases} \dot{y}_i(s) = \widehat{f}_i(s,y_i(s),\widetilde{u}_i(s),\delta,a), \quad i=1,\dots,N, s\in[0,1], \\ \dot{\tau}_i(s)=\delta_i, \quad i=1,\dots,N, s\in[0,1], \\ y_1(0)=x_0(a), \\ y_i(0)=\phi^i(y_{i-1}(1),z,a), \quad i=2,\dots,N, \\ \tau_1=0, \\ \tau_i(0)=\tau_{i-1}(1), \quad i=2,\dots,N. \end{cases} \tag{4.14}$$

where

$$\widehat{f}_i(s,y_i(s),\widetilde{u}_i(s),\delta,a) = \widetilde{f}(s,x(s+i-1),u(s+i-1),\delta,a), s\in[0,1]. \tag{4.15}$$

Let

$$\widehat{L}_0^i(s,y_i(s),\widetilde{u}_i(s),\delta,a) = \widetilde{L}_0(s,x(s+i-1),u(s+i-1),\delta,a), s\in[0,1]. \tag{4.16}$$

Then the equivalent optimal control problem is stated in the following.

Problem P2(a). Given dynamical system (4.14), find a feasible control and parameters $(\widetilde{u},\delta,z)\in\widetilde{\mathcal{U}}\times\Theta\times Z$ such that the cost function

$$\widehat{J}(\widetilde{u},\delta,z;a) = \widehat{\Phi}_0(y_N(1),z,a) + \sum_{i=1}^{N}\int_0^1 \widehat{L}_0^i(s,y_i(s),\widetilde{u}_i(s),\delta,a)ds \tag{4.17}$$

is minimized subject to constrains (4.8) and (4.9), where $\widetilde{u}=[\widetilde{u}_1,\dots,\widetilde{u}_N]^T\in R^{Nr}$, $\widetilde{u}_i=[\widetilde{u}_{i,1},\dots,\widetilde{u}_{i,r}]\in R^r$, $\widetilde{\mathcal{U}}$ is an admissible control set on corresponding to set of $\mathcal{U}$.

Theorem 4.1. *Problem* P(a) *is equivalent to Problem* P2(a) *in the sense that* (u^*,ξ^*,z^*) *is a solution of Problem* P(a) *if and only if* $(\widetilde{u^*},\delta^*,z^*)$ *is a solution of Problem* P2(a), *where* (u^*,ξ^*,z^*) *and* $(\widetilde{u^*},\delta^*,z^*)$ *are related by*

$$\widetilde{u^*}_i(s) = u^*(s+i-1), \quad i=1,\dots,N, s\in[0,1],$$

and

$$\delta_i^* = t_i^* - t_{i-1}^*, \quad i=1,\dots,N.$$

Moreover, for a fixed $a=a^0$, *we have*

$$J(u^*,\xi^*,z^*;a^0) = \widehat{J}(\widetilde{u^*},\delta^*,z^*;a^0). \tag{4.18}$$

and

$$\begin{aligned} u^*(t) &= \widetilde{u}_1^*(t^*(s)), \quad && \text{if} \quad 0 \le t \le t_1, \\ u^*(t) &= \widetilde{u}_2^*(t^*(s)), \quad && \text{if} \quad t_1 < t \le t_2, \\ &\cdots && \cdots \\ u^*(t) &= \widetilde{u}_N^*(t^*(s)), \quad && \text{if} \quad t_{N-1} < t \le T, \end{aligned}$$

where $t^*(s)$ *is determined by* (4.1) *and* (4.14), *and*

$$t_i^* = \sum_{j=1}^{i} \delta_j^*.$$

The Proof is similar to Theorem 3.1 of [8] or [13]. We omit it.

Now Problem P2(a) is a standard optimal control problem for any fixed a. It can be solve numerically by many numerical methods of optimal control problem. Thus, the optimal control software package, MISER 3.3, can be applicable to solve it. For the detail, one can refer Chapter 8 of [10].

5. A NUMERICAL METHOD WITH MINIMUM SENSITIVITY ON PARAMETERS VARIATION

Problem (SP) is different with Problem P(a). CEPT which is an efficient one for solving Problem P(a) cannot be used directly for solving Problem (SP) involving the cost functional of the form (2.1). The reason is that the control parametrization technique is a gradient-based method and the gradient of G is rather cumbersome even without impulse (see [11]). Now we construct a numerical method to compute the gradient of G with respect to a by the method introduced in paper [12].

The idea is to introduce new variables and formulate (2.1) to a standard optimal control cost.

Theorem 5.1. *Suppose that* $D_a x(t, a^0)$ *is Jacobian matrix at* $a = a^0$ *and* $x(t) = x(t, a)$ *is a solution of* (1.2) *at* $a^0 \in R^m$ *and* $(u, \xi, z) \in \mathcal{U} \times \Gamma \times Z$. *If* $y(t)$ *is a solution of the following system of impulsive differential equations*

(5.1)
$$\begin{cases} \frac{dy(t)}{dt} = D_x L_0(t, x(t), u(t), \xi, a^0) \cdot D_x(t, a^0) + D_a L_0(t, x(t), u(t), \xi, a^0), \\ \qquad\qquad t \in (0, T) \setminus \{t_1, t_2, \ldots, t_{N-1}\}, \\ y(0) = 0, \\ y(t_i+) = D_x \Phi_i(x(t_i-), z, a^0) \cdot D_a x(t_i-, a^0) + D_a \Phi_i(x(t_i-), z, a)^T + y(t_i-), \\ \qquad\qquad i = 1, 2, \ldots, N-1. \end{cases}$$

Then, the gradient of the cost functional J *is*

$$D_a J(u, \xi, z; a^0) = y^T(T) + \widetilde{\Phi}_0^T(x(T), z, a^0). \tag{5.2}$$

where

$$\widetilde{\Phi}_0^T(x(T), z, a^0) = D_x \Phi_0(x(T), z, a^0) D_a x(T, a^0) + D_a \Phi_0(x(T), z, a^0). \tag{5.3}$$

Now one can easy to compute the cost function of the optimal control problem (SP), that is,

$$\begin{aligned} G(u,\xi,z;a^0) &= [D_a J(u,\xi,z;a^0)][D_a J(u,\xi,z;a^0)]^T \\ &= \|y(T)\|^2 + y^T(T)\widetilde{\Phi}_0(x(T),z,a^0) \\ &\quad + \widetilde{\Phi}_0^T(x(T),z,a^0)y(T) + \|\widetilde{\Phi}_0(x(T),z,a^0)\|^2. \end{aligned}$$

If we set $\widetilde{x}(t) = [x(t), w(t), y(t)]^T$, where $x(t)$ is a solution of Eq.(1.2), $w(t)$ is a solution of Eq.(3.5), and $y(t)$ is a solution of Eq.(5.1), then the optimal control problem (SP) can be reformulate as (SP1).

Problem(SP1).

$$\min_{(u,\xi,z)\in\mathcal{U}\times\Gamma\times Z} G(u,\xi,z;a^0) = \Phi(\widetilde{x}(T),z,a^0)$$

subject to

$$\begin{cases} \frac{d\widetilde{x}(t)}{dt} = \widetilde{f}(t,\widetilde{x}(t),u(t),a^0), \quad t\in(0,T)\setminus\{t_1,\dots,t_{N-1}\}, \\ \widetilde{x}(0) = \widetilde{x}_0(a^0), \\ \widetilde{x}(t_i+) = \widetilde{\phi}_i(\widetilde{x}(t_i-),z,a^0), \quad i=1,2,\dots,N-1, \end{cases}$$

where

$$\begin{aligned} &\Phi(\widetilde{x}(T),z,a) = \\ &\quad \|y(T)\|^2 + y^T(T)\widetilde{\Phi}_0(x(T),z,a) + \widetilde{\Phi}_0^T(x(T),z,a)y(T) + \|\widetilde{\Phi}_0(x(T),z,a)\|^2, \\ &\widetilde{\Phi}_0(x(T),z,a) = D_x\Phi_0(x(T),z,a)\cdot[w(T)]^T + D_a\Phi_0(x(T),z,a), \\ &\widetilde{f} = \left[f^T, \quad (D_x f\cdot D_a x + D_a f)^T, \quad (D_x L_0\cdot D_a x + D_a L_0)^T\right]^T, \\ &\widetilde{x_0}(a) = \left[(x_0(a))^T, \quad (D_a x_0(a))^T, \quad 0^T\right]^T, \\ &\widetilde{\phi}_i(\widetilde{x}(t_i+),z,a) = \\ &\quad \left[(\phi_i)^T, [(D_x\phi_i - E)\cdot D_a x(t_i-)]^T, [D_x\Phi_i\cdot D_a x(t_i-) + D_a\Phi_i + y(t_i-)]^T\right]^T \end{aligned}$$

with the equality constrains

$$\Phi_i(x(T),z,a^0) + \int_0^T L_i(t,x(t),u(t),\xi,a^0)dt = 0, \quad i=1,2,\dots,M_e$$

and the inequality constrains

$$\Phi_i(x(T),z,a^0) + \int_0^T L_i(t,x(t),u(t),\xi,a^0)dt \le 0, \quad i=M_e+1,\dots,M,$$

$$\Phi_0(x(T),z,a^0) + \int_0^T L_0(t,x(t),u(t),\xi,a^0)dt - c \le 0,$$

where

$$c = (1+\alpha)J(u^*,\xi^*,z^*;a^0)$$

which is an known constant.

Problem (SP1) is a standard impulsive optimal control problem and can be solved as a mathematical programming problem by the method introduced in Section 4.

Once (SP1) is solved and an optimal solution $(\hat{u}, \hat{\xi}, \hat{s})$ is obtained, $J(\hat{u},\ \hat{\xi},\ \hat{s}; a^0)$ is considered a suboptimal cost with minimize sensitive parameters around a^0.

Summary the numerical method as follows.

ALGORITHM.

STEP 1. Solve Problem P(a) as a mathematical programming by using gradient-based method, which we state in Section 4. We obtain an optimal solution (u^*, ξ^*, z^*) and $J(u^*, \xi^*, z^*; a^0)$.

STEP 2. Compute $c = J(u^*, \xi^*, z^*; a^0)$ and take α. Solve the optimal control problem (SP1) and obtain a solution $(\hat{u}, \hat{\xi}, \hat{z})$ by using the same method in STEP 1.

STEP 3. Compute $J(\hat{u}, \hat{\xi}, \hat{z}; a^0)$ by substituting $(\hat{u}, \hat{\xi}, \hat{z}, a^0)$ into the formula

$$J(\hat{u}, \hat{\xi}, \hat{z}; a^0) = \Phi_0(\hat{x}(T), \hat{z}, a^0) + \int_0^T L_0(t, \hat{x}(t), \hat{u}(t), \hat{\xi}, a^0)dt,$$

where $\hat{x}(\cdot)$ is a solution of the following equation

$$\begin{cases} \dot{x}(t) = f(t, x(t), \hat{u}(t), a^0), \quad t \in (0,T) \setminus \{t_1, \dots, t_{N-1}\}, \\ x(0) = x_0(a^0), \\ x(t_i+) = \phi_i(x(t_i-), \hat{z}, a), \quad i = 1, 2, \dots, N. \end{cases} \tag{5.4}$$

and $\hat{\xi} = (t_1, ..., t_{N-1})$ with $t_0 = 0$ and $t_N = T$.

6. **EXAMPLE**

In this section, two numerical examples are solved to show the efficiency of the method proposed. The first example is optimal control problem without impulse and the second example is an optimal control problem governed by impulsive dynamic system. We use the algorithm Section 5 and Software Miser 3.3 (Ref. [6]) to obtain numerical optimal cost functions and suboptimal cost functions with minimum sensitivity for a fixed. Then, the moving of these cost functions during parameter a varying are observed.

Example 6.1. Consider the following controlled system defined on $[0, 10]$:

$$\begin{cases} \dot{x_1}(t) = x_2(t), \quad t \in (0, 10) \\ \dot{x_2}(t) = -x_1(t) + (1.4 - 0.14x_2^2(t))x_2(t) + 4u(t), \quad t \in (0, 10) \\ x_1(0) = -5.0 + a, \\ x_2(0) = -5.0, \end{cases} \tag{6.1}$$

where $U = \mathbb{R}$ is admissible control set. The cost function is given by

$$J(u; a) = \frac{1}{2}\int_0^{10} (x_1^2(t) + u^2(t))dt. \tag{6.2}$$

Problem (P). Find $u^* \in \mathbb{R}$ subject to dynamics system (6.1) such that

$$J(u^*; a) \leq J(u; a), \quad \forall u \in \mathbb{R}.$$

Taking $a = 0$, we compute u^* and $J(u^*, 0)$ by Miser 3.3 and obtain $J(u^*, 0) = 15.3988$.

For the problem (P), we have one coefficient a which occurs only in the initial conditions. To reduce the sensitivity of the optimal solution with respect to the variation of the parameter a, we introduce an optimal control problem (SP) which minimize the sensitivity.

Problem(SP). Find $\hat{u} \in U$ subject to (6.1) such that

$$G(\hat{u}; 0) := [\frac{\partial J(\hat{u}; 0)}{\partial a}][\frac{\partial J(\hat{u}; 0)}{\partial a}]^T = \min$$

with an extra constraint

$$J(\hat{u}, 0) \leq (1 + \alpha) J(u^*; 0).$$

where α is a constant.

Taking $\alpha = 0.05$, we compute $\hat{u}$ and $G(\hat{u}; 0) =$ by the method introduced in Section 5 and software Miser 3.3, then substitute $\hat{u}$ into (6.2) and obtain $J(\hat{u}, a) = 16.3988$. Taking $\alpha = 0.1$, we obtain $J(\bar{u}, 0) = 17.0988$.

Now we calculate $J(u^*; a)$, $J(\hat{u}; a)$ and $J(\bar{u}; a)$ as the system coefficient a is varied away from $a = 0$. The results obtained are plotted in Figure 1. We note that the change in $J(u^*; a)$ is the greatest as a is varied away from $a = 0$.

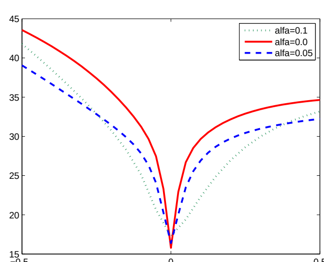

FIGURE 1. Variation in cost value due to the variation of a.

Example 6.2. Consider the following impulsive dynamical system defined on $t \in [0, 10)$:

(6.3)

$$\begin{cases} \dot{x_1}(t) = x_2(t), \quad t \in (0,10) \setminus \{t_1, t_2, t_3\} \\ \dot{x_2}(t) = -x_1(t) + (1.4 - 0.14x_2^2(t))x_2(t) + (4+a)u(t), \ t \in (0,10) \setminus \{t_1, t_2, t_3\} \\ x_1(0) = -5.0, \\ x_2(0) = -5.0, \\ x_1(t_i+) = \frac{x_1(t_i-)+x_2(t_i-)}{x_1^2(t_i-)+x_2^2(t_i-)}, \\ x_2(t_i+) = \frac{4x_2^2(t_i-)}{x_1^2(t_i-)+x_2^2(t_i-)}, \quad i = 1, 2, 3 \end{cases}$$

where $U = \mathbb{R}$ is admissible control set. The cost function is given by

$$J(u, \xi; a) = \frac{1}{2} \int_0^{10} (x_1^2(t) + u^2(t))dt$$

for fixed $a \in \mathbb{R}$, where $\xi = (t_1, t_2, t_3)^T$. Let

$$\Gamma = \{\xi = (t_1, t_2, t_3)^T \in R^3 : t_i - t_{i-1} \geq 0, i = 1, 2, 3, 4 \text{ with } t_0 = 0 \text{ and } t_4 = 10\}.$$

Problem (P). Subject to (6.3), find $u^* \in U$ and $\xi^* \in \Gamma$ such that

$$J(u^*, \xi^*; a) \leq J(u, \xi; a), \quad \forall (u, \xi) \in U \times \Gamma.$$

Let $a = 0$, we compute (u^*, ξ^*) and $J(u^*, \xi^*; a)$ by Miser 3.3 and obtain $J(u^*, \xi^*; a) = 10.03839$.

Problem(SP). Subject to (6.3), find $\hat{u} \in U$ and $\hat{\xi} \in \Gamma$ such that

$$G(\hat{u}, \hat{\xi}; 0) := [\frac{\partial J(\hat{u}, \hat{\xi}; 0)}{\partial a}][\frac{\partial J(\hat{u}, \hat{\xi}; 0)}{\partial a}]^T = \min$$

with constraint

$$J(\hat{u}, \hat{\xi}; 0) \leq (1 + \alpha)J(u^*, \xi^*; 0).$$

Taking $\alpha = 0.1$, we solve Problem (SP) and get a optimal control $\hat{u}$ and $\hat{\xi}$ by Miser 3.3 and $J(\hat{u}, \hat{\xi}; a)$ by Miser 3.3. Solutions and obtain $J(\hat{u}, \hat{\xi}; a) = 10.7767$.

We calculate $J(u^*, \xi^*; a)$ and $J(\hat{u}, \hat{\xi}; a)$ as the parameter a from -0.3 to 0.3. The results obtained are plotted in Figure 2. We note that the changes in $J(u^*, \xi^*; a)$ is much faster than $J(\hat{u}, \hat{\xi}; a)$ as a is varied.

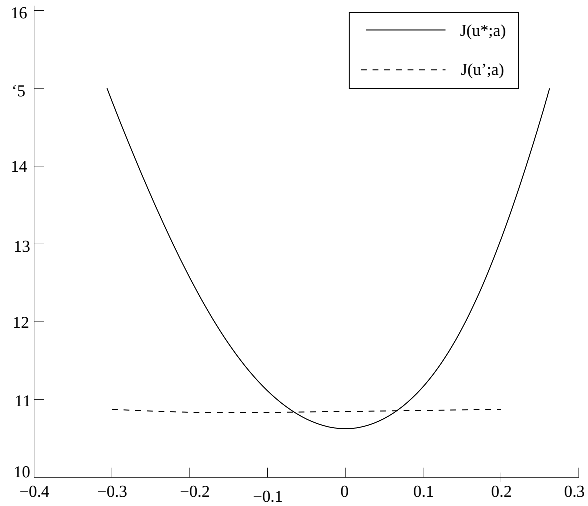

FIGURE 2. Variation in cost value due to the variation of a.

CONCLUSION

It follows from example 6.1 and example 6.2 that changes in $J(u^*, \xi^*; a)$ is much greater than in $J(\hat{u}, \hat{\xi}; a)$ as a is varied away from $a = 0$. Although we increase 0.10 or 0.05 optimal cost $J(u^*, \xi^*; a)$, but the sensitivity of this cost with coefficient variation is much reduced. Therefore, the new formula is useful to solving the cost sensitivity optimal control problem and the numerical method is efficient for optimal control problem governed by impulsive differential equations.

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