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ON POSITIVE SOLUTIONS FOR STURM-LIOUVILLE BOUNDARY VALUE PROBLEMS OF THE FRACTIONAL DIFFERENTIAL EQUATIONS ON THE INFINITE INTERVAL

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ABSTRACT. In this article, we establish the existence and uniqueness results for the positive solutions to Sturm-Liouville boundary value problems of the nonlinear fractional differential equation on the infinite interval. Our analysis rely on the well known fixed point theorems. Some known results are generalized.

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1. INTRODUCTION

Recently there has been a large number of papers concerning with the solvability of the boundary value problems for the fractional differential equations, see the text books [11,17,19] and the papers [1,3,7,10,12,13,15,18,21,23].

This paper is motivated by [23]. Zhao and Ge studied the following boundary value problem for the fractional differential equations

$$(1.1)\qquad \begin{cases} D_{0^+}^{\alpha}u(t)+f(t,u(t))=0, \quad 0<t<\infty, 1<\alpha<2,\\ u(0)=0,\\ \lim_{t\to\infty}\ D_{0^+}^{\alpha-1}u'(t)=0\,, \end{cases}$$

by using the properties of the Green's function of the corresponding BVP, together with the Schauder fixed point theorem. It was proved that BVP (1.1) has at least one positive solution.

Recently, Agarwal Benchohra, Hamani and Pinelas [1], Arara, Benchohra, Hamidi, and Nieto [3] studied the existence of solutions of the following boundary value problem for fractional differential equation

$$(1.2)\qquad \begin{cases} D_{0^+}^{\alpha}u(t)=f(t,u(t)), \quad 0<t<\infty, 1<\alpha<2,\\ u(0)=u_0,\\ u \text{ is bounded on } [0,\infty). \end{cases}$$

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Here D_{0+}^{α} is the Caputo fractional derivative of order α in [3] and the Riemann-Liouville fractional derivative in [1].

One notes that the well known Sturm-Liouville BVP of the the ordinary differential equations is as follows:

$$\begin{cases} u''(t) + f(t, u(t)) = 0, \quad t \in (0, \infty), \\ au(t) - bu'(0) = 0, \\ cu'(1) + du(1) = 0, \end{cases} \tag{1.3}$$

where $f(t, u)$ is continuous and nonnegative on $[0, 1] \times [0, \infty)$, $a \geq 0$, $b \geq 0$, $c \geq 0$ and $d \geq 0$ with $ab + cd + ab > 0$. Such a problem was studied in [2,4-6,8,20,22]. This problem has been generalized to the case of BVP on the half line see [14,15,16].

Motivated by above papers, in this paper, we discuss the existence and uniqueness of the positive solutions to the Sturm-Liouville boundary value problems of the nonlinear fractional differential equation of the form

$$\begin{cases} D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad t \in (0, \infty), 1 < \alpha < 2, \\ a \lim_{t \to 0} t^{2-\alpha} u(t) - b \ \lim_{t \to 0} \ D_{0+}^{\alpha-1} u(t) = 0, \\ c \lim_{t \to \infty} \ D_{0}^{\alpha-1} u(t) + d \lim_{t \to \infty} \frac{1}{1+t^{\alpha-1}} u(t) = 0, \end{cases} \tag{1.4}$$

where $a, b, c, d \in [0, \infty)$, D_{0+}^{α} (D^{α} for short) is the Riemann-Liouville fractional derivative of order α, and $f\left(t, \frac{1}{\delta(t)} x\right)$ defined on $[0, \infty) \times [0, \infty)$ is nonnegative and continuous and satisfies that for each $r > 0$ there exists $\phi_r \in L^1[0, \infty)$ such that $f\left(t, \frac{1}{\delta(t)} x\right) \leq \phi_r(t)$ for every $x \in [0, r]$, where $\delta(t) = \min\left\{t^{2-\alpha}, \frac{1}{1+t^{\alpha-1}}\right\}$. We obtain the results on the existence and uniqueness of the positive solutions about this boundary-value problem by using the fixed point theorems.

2. PRELIMINARY RESULTS

For the convenience of the readers, we present here the necessary definitions from the fractional calculus theory. These definitions and results can be found in the literatures [11,17,19].

Definition 2.1. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, \infty) \to R$ is given by

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} f(s) ds,$$

provided that the right-hand side exists.

Definition 2.2. The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f : (0, \infty) \to R$ is given by

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n - \alpha)} \frac{d^{n+1}}{dt^{n+1}} \int_0^t \frac{f(s)}{(t - s)^{\alpha - n + 1}} ds,$$

where $n-1<\alpha\leq n$, provided that the right-hand side is point-wise defined on $(0,\infty)$.

Definition 2.3. The Riemann-Liouville fractional integral of order $\alpha>0$ of a function $f:(0,\infty)\to R$ is given by

$$I_{0+}^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds,$$

provided that the right-hand side exists.

Definition 2.4. The Riemann-Liouville fractional derivative of order $\alpha>0$ of a continuous function $f:(0,\infty)\to R$ is given by

$$D_{0+}^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\frac{d^{n+1}}{dt^{n+1}}\int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}}ds,$$

where $n-1<\alpha\leq n$, provided that the right-hand side is point-wise defined on $(0,\infty)$.

Lemma 2.5. *Let $n-1<\alpha\leq n$, $u\in C^0(0,\infty)\bigcap L^1(0,\infty)$. Then*

$$I_{0+}^{\alpha}D_{0+}^{\alpha}u(t)=u(t)+C_1t^{\alpha-1}+C_2t^{\alpha-2}+\cdots+C_nt^{\alpha-n},$$

where $C_i\in R$, $i=1,2,\ldots n$.

Lemma 2.6. *The relations*

$$I_{0+}^{\alpha}I_{0+}^{\beta}\varphi=I_{0+}^{\alpha+\beta}\varphi,\ D_{0+}^{\alpha}I_{0+}^{\alpha}=\varphi$$

are valid in following case

$$Re\beta>0,\ Re(\alpha+\beta)>0,\ \varphi\in L_1(0,\infty).$$

Lemma 2.7. *Let $n-1<\alpha\leq n$, $u\in C^0(0,\infty)\bigcap L^1(0,\infty)$. Then*

$$I_{0+}^{\alpha}D_{0+}^{\alpha}u(t)=u(t)+C_1t^{\alpha-1}+C_2t^{\alpha-2}+\cdots+C_nt^{\alpha-n},$$

where $C_i\in R$, $i=1,2,\ldots n$.

Lemma 2.8. *The relations*

$$I_{0+}^{\alpha}I_{0+}^{\beta}\varphi=I_{0+}^{\alpha+\beta}\varphi,\ D_{0+}^{\alpha}I_{0+}^{\alpha}=\varphi$$

are valid in following case

$$Re\beta>0,\ Re(\alpha+\beta)>0,\ \varphi\in L_1(0,\infty).$$

Lemma 2.9. *Suppose that $a\neq 0, c+d\neq 0$. Given $h\in C[0,1]$, the unique solution of*

$$\begin{cases} D_{0+}^{\alpha}u(t)+h(t)=0, 0<t<\infty,\\ a\lim_{t\to 0}t^{2-\alpha}u(t)-b\ \lim_{t\to 0}\ D^{\alpha-1}u(t)=0,\\ c\lim_{t\to\infty}\ D_0^{\alpha-1}u(t)+d\lim_{t\to\infty}\frac{1}{1+t^{\alpha-1}}u(t)=0,\end{cases}\tag{2.1}$$

is

$$u(t)=\int_0^1 G(t,s)h(s)ds, \tag{2.2}$$

where

$$G(t,s)=\begin{cases} \frac{-a(t-s)^{\alpha-1}+at^{\alpha-1}+b\Gamma(\alpha)t^{\alpha-2}}{a\Gamma(\alpha)}, s\le t,\\ \frac{at^{\alpha-1}+b\Gamma(\alpha)t^{\alpha-2}}{a\Gamma(\alpha)}, t\le s.\end{cases} \tag{2.3}$$

Proof. We may apply Lemma 2.7 to reduce BVP (2.1) to an equivalent integral equation

$$u(t)=-\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds+c_1t^{\alpha-1}+c_2t^{\alpha-2}, t\in(0,1)$$

for some $c_i\in R$, $i=1,2$. Note that $\Gamma(0)=\infty$. We get

$$\frac{1}{1+t^{\alpha-1}}u(t)=-\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)(1+t^{\alpha-1})}h(s)ds+c_1\frac{t^{\alpha-1}}{1+t^{\alpha-1}}+c_2\frac{t^{\alpha-2}}{1+t^{\alpha-1}}$$

$$t^{2-\alpha}u(t)=-t^{2-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds+c_1t+c_2$$

and

$$D^{\alpha-1}u(t)=-\int_0^t h(s)ds+c_1\Gamma(\alpha).$$

Since

$$\left|\int_0^t \frac{(t-s)^{\alpha-1}}{1+t^{\alpha-1}}h(s)ds\right|\le\int_0^\infty h(s)ds<\infty,$$

we get

$$\lim_{t\to\infty}\int_0^t \frac{(t-s)^{\alpha-1}}{1+t^{\alpha-1}}h(s)ds=\int_0^\infty h(s)ds.$$

From the boundary conditions in (2.1), since $\lim_{s\to0}\Gamma(s)=\infty$, we get

$$ac_2-bc_1\Gamma(\alpha)=0,$$
$$c\left(-\int_0^\infty h(s)ds+c_1\Gamma(\alpha)\right)+d\left(-\int_0^\infty \frac{1}{\Gamma(\alpha)}h(s)ds+c_1\right)=0.$$

It follows that

$$c_1=\frac{1}{\Gamma(\alpha)}\int_0^\infty h(s)ds,$$

and

$$c_2=\frac{b}{a}\int_0^\infty h(s)ds.$$

Therefore, the unique solution of BVP (1.3) is

$$u(t)=-\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds+\frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^\infty h(s)ds+\frac{bt^{\alpha-2}}{a}\int_0^\infty h(s)ds$$
$$=\int_0^\infty G(t,s)h(s)ds.$$

Here G is defined by (2.3). Reciprocally, let u satisfy (2.2). Then

$$a\lim_{t\to0}t^{2-\alpha}u(t)-b\lim_{t\to0}D^{\alpha-1}u(t)=0, c\lim_{t\to\infty}D_0^{\alpha-1}u(t)+d\lim_{t\to\infty}\frac{1}{1+t^{\alpha-1}}u(t)=0,$$

furthermore, we have $D_0^\alpha u(t) = -h(t)$. The proof is complete. $\square$

Lemma 2.10. *Suppose that $a > 0, c + d \neq 0$. Then*

$$G(t,s) \leq \frac{at^{\alpha-1} + b\Gamma(\alpha)t^{\alpha-2}}{a\Gamma(\alpha)}, \text{ for all } s, t \in (0,\infty),$$

and

$$G(t,s) \geq 0 \text{ for all } s, t \in (0,\infty).$$

Proof. One sees from (2.3) that

$$G(t,s) \leq \frac{at^{\alpha-1} + b\Gamma(\alpha)t^{\alpha-2}}{a\Gamma(\alpha)}.$$

On the other hand, we have from (2.3) that

$$\begin{aligned} G(t,s) &\geq \begin{cases} \frac{-at^{\alpha-1}+at^{\alpha-1}+b\Gamma(\alpha)t^{\alpha-2}}{a\Gamma(\alpha)}, s \leq t, \\ \frac{at^{\alpha-1}+b\Gamma(\alpha)t^{\alpha-2}}{a\Gamma(\alpha)}, t \leq s. \end{cases} \\ &= \begin{cases} \frac{b\Gamma(\alpha)t^{\alpha-2}}{a\Gamma(\alpha)}, s \leq t, \\ \frac{at^{\alpha-1}+b\Gamma(\alpha)t^{\alpha-2}}{a\Gamma(\alpha)}, t \leq s. \end{cases} \\ &\geq 0. \end{aligned}$$

The proof is completed. $\square$

For our construction, we let

$$X = \left\{ x \in C(0,\infty) : \text{ there exist the limits } \lim_{t\to 0} t^{2-\alpha}u(t) \text{ and } \lim_{t\to\infty} \frac{1}{1+t^{\alpha-1}}x(t) \right\}.$$

Let $\delta(t) = \min\left\{t^{2-\alpha}, \frac{1}{1+t^{\alpha-1}}\right\}$. One sees that δ is continuous on $[0,\infty)$ and $\lim_{t\to 0}\delta(t) = 0$, $\lim_{t\to\infty}\delta(t) = 0$. For $x \in X$, let

$$\|x\| = \sup_{0<t<\infty} \delta(t)|u(t)|.$$

Then X is a Banach space. We seek solutions of BVP (1.3) that lie in the cone

$$P = \{u \in X : u(t) \geq 0, \ 0 < t < \infty\}.$$

Suppose that $\delta = bd\Gamma(\alpha) + ad + ac\Gamma(\alpha) > 0$. Define the operator $T : P \to P$, by

$$Tu(t) = \int_0^\infty G(t,s)f(s,u(s))ds.$$

Lemma 2.11. *Let $V = \{x \in X : ||x|| < l\}$, $l > 0$, and $V_1 = \{\delta(t)x(t) : x \in V\}$. If V_1 is equicontinuous on any compact interval of $(0,\infty)$ and equicontinuous at infinity and zero point, then V is relative compact on X [16].*

Remark 2.12. V_1 is equicontinuous at infinity if and only if for all $\epsilon > 0$, there exists $v = v(\epsilon) > 0$ such that for all $x \in V$, $t_1, t_2 \geq v$, it holds

$$\left| \frac{x(t_1)}{1+t_1^{\alpha-1}} - \frac{x(t_2)}{1+t_2^{\alpha-1}} \right| < \epsilon.$$

V_1 is equicontinuous at zero if and only if for all $\epsilon > 0$, there exists $v = v(\epsilon) > 0$ such that for all $x \in V$, $0 \leq t_1, t_2 \leq v$, it holds

$$\left| t_1^{2-\alpha} x(t_1) - t_2^{2-\alpha} x(t_2) \right| < \epsilon.$$

Lemma 2.13. *Suppose that* $a > 0, c+d \neq 0$, $f\left(t, \frac{1}{\delta(t)} u\right)$ *is continuous. Then* T *is completely continuous.*

Proof. We divide the proof into three steps.

Step 1. We prove that T is continuous.

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in X. Let

$$r = \max\left\{ \sup_{n \in N} ||y_n||, ||y|| \right\}.$$

One sees that

$$\int_0^\infty \left| \left(t, \frac{1}{\delta(t)} y_n(s)\right) - \left(t, \frac{1}{\delta(t)} y(s)\right) \right| ds \leq 2 \int_0^\infty \phi_r(s) ds.$$

Then for $t \in [0, \infty)$, we have

$$\begin{aligned}
& \delta(t)|(Ty_n)(t) - (Ty)(t)| \\
= & \left| \int_0^\infty \delta(t) G(t,s) f(s, y_n(s)) ds - \int_0^\infty \delta(t) G(t,s) f(s, y(s)) ds \right| \\
\leq & \int_0^\infty \delta(t) G(t,s) |f(s, y_n(s)) - f(s, y(s))| ds \\
\leq & \int_0^\infty \delta(t) \frac{a t^{\alpha-1} + b\Gamma(\alpha) t^{\alpha-2}}{a\Gamma(\alpha)} \left| \left(t, \frac{1}{\delta(t)} y_n(s)\right) - \left(t, \frac{1}{\delta(t)} y(s)\right) \right| ds \\
\leq & \sup_{t \in (0,\infty)} \delta(t) \frac{a t^{\alpha-1} + b\Gamma(\alpha) t^{\alpha-2}}{a\Gamma(\alpha)} \int_0^\infty \left| \left(t, \frac{1}{\delta(t)} y_n(s)\right) - \left(t, \frac{1}{\delta(t)} y(s)\right) \right| ds.
\end{aligned}$$

By the definitions of f, we have $||Ty_n - Ty|| \to 0$ as $n \to \infty$.

Step 2. We prove that T maps bounded sets into bounded sets in X.

It suffices to show that for each $l > 0$, there exists a positive number $L > 0$ such that for each $x \in M = \{y \in X : ||y|| \leq l\}$, we have $||Ty|| \leq L$. By the definition of T, we get

$$\begin{aligned}
\delta(t)|(Ty)(t)| &= \int_0^\infty \delta(t) G(t,s) f(s, y(s)) ds \\
&\leq \int_0^\infty \delta(t) G(t,s) f\left(s, \frac{1}{\delta(s)} \delta(s) y(s)\right) ds
\end{aligned}$$

$$\le \int_0^\infty \delta(t)\frac{at^{\alpha-1}+b\Gamma(\alpha)t^{\alpha-2}}{a\Gamma(\alpha)}f\left(s,\frac{1}{\delta(s)}\delta(s)y(s)\right)ds$$
$$\le \sup_{t\in(0,\infty)}\delta(t)\frac{at^{\alpha-1}+b\Gamma(\alpha)t^{\alpha-2}}{a\Gamma(\alpha)}\int_0^\infty \phi_l(s)ds.$$

It follows that

$$||Ty||\le \sup_{t\in(0,\infty)}\delta(t)\frac{at^{\alpha-1}+b\Gamma(\alpha)t^{\alpha-2}}{a\Gamma(\alpha)}\int_0^\infty \phi_l(s)ds$$

for each $y\in\{y\in X:||y||\le l\}$. Then T maps bounded sets into bounded sets in X.

Step 3. We prove that T maps bounded sets into equicontinuous sets in X.

Firstly, we prove that T is equicontinuous on each compact interval of $(0,\infty)$. Let $t_1,t_2\in[m,n]\subset(0,\infty)$ with $t_1<t_2$ and $y\in M=\{y\in X:||y||\le l\}$. We have

$$\begin{aligned}
&|\delta(t_1)(Ty)(t_1)-\delta(t_2)(Ty)(t_2)|\\
=&\left|\int_0^\infty \delta(t_1)G(t_1,s)f(s,y(s))ds-\int_0^\infty \delta(t_2)G(t_2,s)f(s,y(s))ds\right|\\
\le&\int_0^\infty |\delta(t_1)G(t_1,s)-\delta(t_2)G(t_2,s)|\,f\left(s,\frac{1}{\delta(s)}\delta(s)y(s)\right)ds\\
\le&\int_0^{t_1} |\delta(t_1)G(t_1,s)-\delta(t_2)G(t_2,s)|\,\phi_l(s)ds\\
&+\int_{t_1}^{t_2} |\delta(t_1)G(t_1,s)-\delta(t_2)G(t_2,s)|\,\phi_l(s)ds\\
&+\int_{t_2}^{\infty} |\delta(t_1)G(t_1,s)-\delta(t_2)G(t_2,s)|\,\phi_l(s)ds\\
=&\int_0^{t_1}\left|\frac{-a[\delta(t_1)(t_1-s)^{\alpha-1}-\delta(t_2)(t_2-s)^{\alpha-1}]}{a\Gamma(\alpha)}\right|\phi_l(s)ds\\
&+\int_0^{t_1}\left|\frac{a[\delta(t_1)t_1^{\alpha-1}-\delta(t_2)t_2^{\alpha-1}]}{a\Gamma(\alpha)}\right|\phi_l(s)ds\\
&+\int_0^{t_1}\left|\frac{b\Gamma(\alpha)[\delta(t_1)t_1^{\alpha-2}-\delta(t_2)t_2^{\alpha-2}]}{a\Gamma(\alpha)}\right|\phi_l(s)ds\\
&+\int_{t_1}^{t_2} |\delta(t_1)G(t_1,s)-\delta(t_2)G(t_2,s)|\,\phi_l(s)ds\\
&+\int_{t_2}^{\infty}\left|\frac{a[\delta(t_1)t_1^{\alpha-1}-\delta(t_2)t_2^{\alpha-1}]+b\Gamma(\alpha)[\delta(t_1)t_1^{\alpha-2}-\delta(t_2)t_2^{\alpha-2}]}{a\Gamma(\alpha)}\right|\phi_l(s)ds\\
\le&\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\left|\delta(t_1)(t_1-s)^{\alpha-1}-\delta(t_2)(t_2-s)^{\alpha-1}\right|\phi_l(s)ds\\
&+\left|\frac{a[\delta(t_1)t_1^{\alpha-1}-\delta(t_2)t_2^{\alpha-1}]}{a\Gamma(\alpha)}\right|\int_0^\infty \phi_l(s)ds\\
&+\left|\frac{b\Gamma(\alpha)[\delta(t_1)t_1^{\alpha-2}-\delta(t_2)t_2^{\alpha-2}]}{a\Gamma(\alpha)}\right|\int_0^\infty \phi_l(s)ds
\end{aligned}$$

$$+2\sup_{t\in(0,\infty)}\delta(t)G(t,s)\int_0^\infty \phi_l(s)ds|t_1-t_2|$$
$$+\frac{a+b\Gamma(\alpha)}{a\Gamma(\alpha)}\left|\delta(t_1)t_1^{\alpha-1}-\delta(t_2)t_2^{\alpha-1}\right|\int_0^\infty \phi_l(s)ds.$$

Since $\delta(t)G(t,s)$ is continuous on $(0,\infty)\times[0,\infty)$ and there exist the limits

$$\lim_{t\to 0}\delta(t)G(t,s) \text{ and } \lim_{t\to\infty}\delta(t)G(t,s),$$

we get that the right-hand side of the above inequality tends to zero as $t_1 \to t_2$.

Secondly, we prove that T is equicontinuous at infinity. For $y \in M = \{y \in X : ||y|| \leq l\}$, since

$$\delta(t)(Ty)(t) = \int_0^\infty \delta(t)G(t,s)f(s,y(s))ds \leq \sup_{t\in(0,\infty)}\delta(t)\frac{at^{\alpha-1}+b\Gamma(\alpha)t^{\alpha-2}}{a\Gamma(\alpha)}\int_0^\infty \phi_l(s)ds,$$

we get

$$\lim_{t\to\infty}\delta(t)(Ty)(t) = \lim_{t\to\infty}\int_0^\infty \frac{1}{1+t^{\alpha-1}}G(t,s)f(s,y(s))ds = 0$$

uniformly. Then for each $\epsilon > 0$ there is $H > 0$ such that

$$\left|\frac{1}{1+t_1^{\alpha-1}}(Ty)(t_1) - \frac{1}{1+t_2^{\alpha-1}}(Ty)(t_2)\right| < \epsilon, t_1, t_2 > H, y \in M.$$

Hence T is equicontinuous at infinity.

Lastly, we prove that T is equicontinuous at zero point. For $y \in M = \{y \in X : ||y|| \leq l\}$, since

$$\delta(t)(Ty)(t) = \int_0^\infty \delta(t)G(t,s)f(s,y(s))ds \leq \sup_{t\in(0,\infty)}\delta(t)\frac{at^{\alpha-1}+b\Gamma(\alpha)t^{\alpha-2}}{a\Gamma(\alpha)}\int_0^\infty \phi_l(s)ds,$$

we get

$$\lim_{t\to 0}\delta(t)(Ty)(t) = \lim_{t\to\infty}\int_0^\infty t^{2-\alpha}G(t,s)f(s,y(s))ds = \frac{b}{a}\int_0^\infty f(s,y(s))ds$$

uniformly. Then for each $\epsilon > 0$ there is $H > 0$ such that

$$|t_1^{2-\alpha}(Ty)(t_1) - t_2^{2-\alpha}(Ty)(t_2)| < \epsilon, t_1, t_2 > H, y \in M.$$

Hence T is equicontinuous at zero point.

From above discussion, we see from Lemma 2.7 that T is completely continuous. □

3. MAIN RESULTS

In this section, we prove the main results. It is supposed that $f\left(t, \frac{1}{\delta(t)}x\right)$ defined on $[0,\infty)\times[0,\infty)$ is nonnegative and continuous and satisfies that for each $r > 0$ there exists $\phi_r \in L^1[0,\infty)$ such that $f\left(t, \frac{1}{\delta(t)}x\right) \leq \phi_r(t)$ for all $t \in [0,\infty)$ and $x \in [0,r]$.

Theorem 3.1. *Suppose that* $a > 0, c + d \neq 0,\ f$ *satisfies*

$$\left|f\left(t, \frac{1}{\delta(t)}u\right) - f\left(t, \frac{1}{\delta(t)}v\right)\right| \leq \alpha(t)|u - v|, t \in [0, \infty), u, v \in [0, \infty). \tag{3.1}$$

Then BVP (1.3) *has an unique positive solution if*

$$\int_0^\infty \alpha(s)\delta(s)\frac{as^{\alpha-1} + b\Gamma(\alpha)s^{\alpha-2}}{a\Gamma(\alpha)}ds < 1. \tag{3.2}$$

Proof. We shall prove that under the assumptions (3.1) and (3.2), T^n is a contraction operator for n sufficiently large. Indeed, by the definition of $G(t, s)$ for $u, v \in P$, from Lemma 2.10 and (3.1), we have the estimate

$$\begin{aligned}
&\delta(t)|(Tu)(t) - (Tv)(t)| \\
= &\int_0^\infty \delta(t)G(t, s)\left|f\left(s, \frac{1}{\delta(s)}\delta(s)u(s)\right) - f\left(s, \frac{1}{\delta(s)}\delta(s)v(s)\right)\right| ds \\
\leq &\ \delta(t)\frac{at^{\alpha-1} + b\Gamma(\alpha)t^{\alpha-2}}{a\Gamma(\alpha)}\int_0^\infty \alpha(s)\delta(s)|u(s) - v(s)|ds \\
\leq &\ \delta(t)\frac{at^{\alpha-1} + b\Gamma(\alpha)t^{\alpha-2}}{a\Gamma(\alpha)}\int_0^\infty \alpha(s)ds||u - v||.
\end{aligned}$$

Hence

$$\begin{aligned}
&\delta(t)|(T^2u)(t) - (T^2v)(t)| \\
= &\int_0^\infty \delta(t)G(t, s)|f(s, (Tu)(s)) - f(s, (Tv)(s))|ds \\
\leq &\ \delta(t)\frac{at^{\alpha-1} + b\Gamma(\alpha)t^{\alpha-2}}{a\Gamma(\alpha)}\int_0^\infty \alpha(s)\delta(s)|(Tu)(s) - (Tv)(s)|ds \\
\leq &\ \delta(t)\frac{at^{\alpha-1} + b\Gamma(\alpha)t^{\alpha-2}}{a\Gamma(\alpha)}\int_0^\infty \alpha(s)\delta(s)\frac{as^{\alpha-1} + b\Gamma(\alpha)s^{\alpha-2}}{a\Gamma(\alpha)}\int_0^\infty \alpha(s)ds||u - v||ds \\
= &\ \delta(t)\frac{at^{\alpha-1} + b\Gamma(\alpha)t^{\alpha-2}}{a\Gamma(\alpha)}\int_0^\infty \alpha(s)\delta(s)\frac{as^{\alpha-1} + b\Gamma(\alpha)s^{\alpha-2}}{a\Gamma(\alpha)}ds\int_0^\infty \alpha(s)ds||u - v||.
\end{aligned}$$

Similarly, one gets

$$\begin{aligned}
&\delta(t)|(T^3u)(t) - (T^3v)(t)| \\
= &\int_0^\infty \delta(t)G(t, s)|f(s, (T^2u)(s)) - f(s, (T^2v)(s))|ds \\
\leq &\ \delta(t)\frac{at^{\alpha-1} + b\Gamma(\alpha)t^{\alpha-2}}{a\Gamma(\alpha)} \times \\
&\left(\int_0^\infty \alpha(s)\delta(s)\frac{as^{\alpha-1} + b\Gamma(\alpha)s^{\alpha-2}}{a\Gamma(\alpha)}ds\right)^2\int_0^\infty \alpha(s)ds||u - v||.
\end{aligned}$$

By induction methods, we get

$$\begin{aligned}
&\delta(t)|(T^nu)(t) - (T^nv)(t)| \\
\leq &\ \delta(t)\frac{at^{\alpha-1} + b\Gamma(\alpha)t^{\alpha-2}}{a\Gamma(\alpha)} \times
\end{aligned}$$

$$\left(\int_0^\infty \alpha(s)\delta(s)\frac{as^{\alpha-1}+b\Gamma(\alpha)s^{\alpha-2}}{a\Gamma(\alpha)}ds\right)^{n-1}\int_0^\infty \alpha(s)ds||u-v||$$
$$\le \sup_{t\in(0,\infty)}\delta(t)\frac{at^{\alpha-1}+b\Gamma(\alpha)t^{\alpha-2}}{a\Gamma(\alpha)}\times$$
$$\left(\int_0^\infty \alpha(s)\delta(s)\frac{as^{\alpha-1}+b\Gamma(\alpha)s^{\alpha-2}}{a\Gamma(\alpha)}ds\right)^{n-1}\int_0^\infty \alpha(s)ds||u-v||.$$

It follows from (3.2) that for sufficiently large n we have

$$\left(\int_0^\infty \alpha(s)\delta(s)\frac{as^{\alpha-1}+b\Gamma(\alpha)s^{\alpha-2}}{a\Gamma(\alpha)}ds\right)^{n-1}$$
$$\le \frac{1}{2\sup_{t\in(0,\infty)}\delta(t)\frac{at^{\alpha-1}+b\Gamma(\alpha)t^{\alpha-2}}{a\Gamma(\alpha)}\int_0^\infty \alpha(s)ds}.$$

Therefor, we get for such n that

$$||T^nu-T^nv||\le\frac{1}{2}||u-v||.$$

Hence the contraction map principle implies that BVP (1.3) has an unique positive solution. The proof is completed. □

Theorem 3.2. *Suppose that* $a>0, c+d>0$*,* $f(t,u)$ *satisfies* $f(t,0)\not\equiv 0$ *for* $t\in(0,\infty)$ *and there exists* $B\in L^1[0,\infty)$ *such that*

$$0\le\overline{\lim_{x\to\infty}}\max_{t\in[0,\infty)}\frac{f\left(t,\frac{1}{\delta(t)}x\right)}{xB(t)}<\frac{1}{\sup_{t\in(0,\infty)}\delta(t)\frac{at^{\alpha-1}+b\Gamma(\alpha)t^{\alpha-2}}{a\Gamma(\alpha)}\int_0^\infty B(s)ds}. \tag{3.3}$$

Then, BVP (1.3) *has at least one positive solution.*

Proof. It follows from (3.3) that there is $M>0$ and $H>0$ such that

$$0\le\frac{f\left(t,\frac{1}{\delta(t)}x\right)}{xB(t)}\le M<\frac{1}{\sup_{t\in(0,\infty)}\delta(t)\frac{at^{\alpha-1}+b\Gamma(\alpha)t^{\alpha-2}}{a\Gamma(\alpha)}\int_0^\infty B(s)ds}, t\in[0,\infty), x>H.$$

Let

$$C(t)=\max_{x\in[0,H]}f\left(t,\frac{1}{\delta(t)}x\right).$$

It follows that $B,C\in L^1[0,\infty)$ and

$$0\le f\left(t,\frac{1}{\delta(t)}x\right)\le MxB(t)+C(t), t\in[0,\infty), x\in[0,\infty).$$

Choose $R>0$ sufficiently large such that

$$R = \sup_{t\in(0,\infty)}\delta(t)\frac{at^{\alpha-1}+b\Gamma(\alpha)t^{\alpha-2}}{a\Gamma(\alpha)}\times$$
$$\max\left\{\int_0^\infty C(s)ds,\ \frac{M\int_0^\infty B(s)ds\int_0^\infty C(s)ds\sup_{t\in(0,\infty)}\delta(t)\frac{at^{\alpha-1}+b\Gamma(\alpha)t^{\alpha-2}}{a\Gamma(\alpha)}}{1-M\sup_{t\in(0,\infty)}\delta(t)\frac{at^{\alpha-1}+b\Gamma(\alpha)t^{\alpha-2}}{a\Gamma(\alpha)}\int_0^\infty B(s)ds}\right\}.$$

Let

$$B_R = \left\{ x \in P : \left|\left| x - \int_0^\infty G(t,s)C(s)ds \right|\right| \le R \right\}.$$

It is easy to see that B_R is a convex, bounded, and closed subset of the Banach space X. For $x \in B_R$, we have

$$\begin{aligned} ||x|| &\le \left|\left| x - \int_0^\infty G(t,s)C(s)ds \right|\right| + \left|\left| \int_0^\infty G(t,s)C(s)ds \right|\right| \\ &\le R + \left|\left| \int_0^\infty G(t,s)C(s)ds \right|\right| \\ &\le R + \sup_{t\in(0,\infty)} \delta(t) \frac{at^{\alpha-1} + b\Gamma(\alpha)t^{\alpha-2}}{a\Gamma(\alpha)} \int_0^\infty C(s)ds. \end{aligned}$$

One sees that

$$-C(t) \le f\left(t, \frac{1}{\delta(t)}x\right) - C(t) \le MxB(t).$$

It follows that

$$\left| f\left(t, \frac{1}{\delta(t)}x\right) - C(t) \right| \le \max\{C(t), M|x|B(t)\}.$$

Hence the definition of R implies that

$$\begin{aligned} & \left|\left| Tx - \int_0^1 G(t,s)C(s)ds \right|\right| \\ = & \sup_{t\in[0,\infty)} \delta(t) \left| \int_0^\infty G(t,s) \left(f\left(s, \frac{1}{\delta(s)}\delta(s)x(s)\right) - C(t) \right) ds \right| \\ \le & \sup_{t\in[0,\infty)} \int_0^\infty \delta(t)G(t,s) \max\{C(s),\ MB(s)\delta(s)|x(s)|\}\, ds \\ \le & \sup_{t\in(0,\infty)} \delta(t) \frac{at^{\alpha-1} + b\Gamma(\alpha)t^{\alpha-2}}{a\Gamma(\alpha)} \int_0^\infty \max\{C(s),\ MB(s)\delta(s)|x(s)|\}\, ds \\ \le & \sup_{t\in(0,\infty)} \delta(t) \frac{at^{\alpha-1} + b\Gamma(\alpha)t^{\alpha-2}}{a\Gamma(\alpha)} \times \\ & \max\left\{ \int_0^\infty C(s)ds,\ M \int_0^\infty B(s)ds||x|| \right\} \\ \le & \sup_{t\in(0,\infty)} \delta(t) \frac{at^{\alpha-1} + b\Gamma(\alpha)t^{\alpha-2}}{a\Gamma(\alpha)} \max\left\{ \int_0^\infty C(s)ds, \right. \\ & \left. M \int_0^\infty B(s)ds \left(R + \sup_{t\in(0,\infty)} \delta(t) \frac{at^{\alpha-1} + b\Gamma(\alpha)t^{\alpha-2}}{a\Gamma(\alpha)} \int_0^\infty C(s)ds \right) \right\} \\ \le & R. \end{aligned}$$

So, we have $TB_R \subset B_R$. Since T is completely continuous, the Schauder fixed point theorem assures that operator T has at least one fixed point in B_R and then BVP (1.3) has at least one positive solution. The proof is complete. □

Theorem 3.3. *Suppose that $a > 0, c + d > 0$, f satisfies that*

$$\left| f\left(t, \frac{1}{\delta(t)}x\right)\right| \leq \phi(t)w(|x|), t \in [0,\infty), x \in [0,\infty) \tag{3.4}$$

with $\phi \in L^1[0,1]$ and $w \in C([0,\infty),[0,\infty))$ nondecreasing. If there exists a constant $\sigma > 0$ such that

$$\frac{\sigma}{\sup_{t\in(0,\infty)} \delta(t)\frac{at^{\alpha-1}+b\Gamma(\alpha)t^{\alpha-2}}{a\Gamma(\alpha)} \int_0^\infty \phi(s)dsw(\sigma)} > 1, \tag{3.5}$$

then, BVP (1.3) *has at least one positive solution such that*

$$0 \leq \delta(t)x(t) \leq \sigma, t \in [0,\infty). \tag{3.6}$$

Proof. We consider the BVP of the form

$$\begin{cases} D^\alpha u(t) + \lambda f(t,u(t)) = 0, \quad t \in (0,\infty), 1 < \alpha < 2, \\ a \lim_{t\to 0} t^{2-\alpha}u(t) - b\ \lim_{t\to 0}\ D_{0+}^{\alpha-1}u(t) = 0, \\ c \lim_{t\to\infty}\ D_0^{\alpha-1}u(t) + d \lim_{t\to\infty} \frac{1}{1+t^{\alpha-1}}u(t) = 0, \end{cases} \tag{3.7}$$

for $0 < \lambda < 1$. Solving BVP (3.7) is equivalent to solving the fixed point problem $x = \lambda Tx$.

Let

$$U = \{x \in X : ||x|| \leq \sigma\}.$$

We claim that $x \neq \lambda Tx$ for all ∂U and $\lambda \in (0,1)$. In fact, if $x = \lambda Tx$ for some $x \in \partial U$ and $\lambda \in (0,1)$, we have

$$\begin{aligned} ||x|| &= \sup_{t\in[0,\infty)} \lambda\delta(t)(Tx)(t) \\ &\leq \sup_{t\in[0,\infty)} \delta(t)(Tx)(t) \\ &= \sup_{t\in[0,\infty)} \int_0^\infty \delta(t)G(t,s)f(s,x(s))ds \\ &= \sup_{t\in[0,\infty)} \int_0^\infty \delta(t)G(t,s)f\left(s, \frac{1}{\delta(s)}\delta(s)x(s)\right)ds \\ &\leq \sup_{t\in(0,\infty)} \delta(t)\frac{at^{\alpha-1}+b\Gamma(\alpha)t^{\alpha-2}}{a\Gamma(\alpha)} \int_0^\infty \phi(s)w(\delta(s)x(s))ds \\ &\leq \sup_{t\in(0,\infty)} \delta(t)\frac{at^{\alpha-1}+b\Gamma(\alpha)t^{\alpha-2}}{a\Gamma(\alpha)} \int_0^\infty \phi(s)dsw(\sigma). \end{aligned}$$

So

$$\sigma \leq \sup_{t\in(0,\infty)} \delta(t)\frac{at^{\alpha-1}+b\Gamma(\alpha)t^{\alpha-2}}{a\Gamma(\alpha)} \int_0^\infty \phi(s)dsw(\sigma).$$

It follows that

$$\frac{\sigma}{\sup_{t\in(0,\infty)} \delta(t)\frac{at^{\alpha-1}+b\Gamma(\alpha)t^{\alpha-2}}{a\Gamma(\alpha)} \int_0^\infty \phi(s)dsw(\sigma)} \leq 1,$$

which contradicts with (3.5). Since T is completely continuous, by Schauder's fixed point theorem [3], we see that BVP (1.3) has at least one positive solution x such that (3.6) holds. The proof is complete. □

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