Dynamic Systems and Applications 21 (2012) 161-168

EXISTENCE AND UNIQUENESS OF POSITIVE SOLUTIONS FOR THIRD ORDER BOUNDARY VALUE PROBLEMS WITH p-LAPLACIAN

XIUGUO LIAN

School of Mathematics Science, Huaiyin Normal University
Huaian, Jiangsu, 223300, China

ABSTRACT. In this paper, we provide an existence theorem for a unique fixed point for a class of increasing operators without continuity and compactness, and apply it to discuss existence and uniqueness of positive solutions of third order boundary value problems with p-Laplacian.

AMS (MOS) Subject Classification. 34B15

1. INTRODUCTION

Third order boundary value problems (BVPs) arise in variety of different areas of applied mathematics and physics. In recent years, many authors have studied them, see, for example, [1,2,4-9] and the references therein. In this paper, we consider the following third order BVPs with p-Laplacian:

$$\begin{cases} (\phi_p(u'))'' + a(t)f(u) = 0, \ 0 < t < 1, \\ u(0) = \xi u(\eta) + \lambda, \ u'(0) = u'(1) = 0, \end{cases} \tag{1.1}$$

where $\phi_p(x) = |x|^{p-2}x, \ p > 1, \ \xi \in [0, 1), \ \eta \in [0, 1], \ \lambda > 0$, and $a : (0, 1) \to [0, +\infty)$ and $f : (0, +\infty) \to [0, +\infty)$ are continuous. We call $u \in C^1[0, 1]$ a positive solution of BVP (1.1) if $u(t) \geq 0$ for $t \in [0, 1]$ but $u(t)$ is not identically vanishing in $[0, 1]$, and $u(t)$ satisfies (1.1). In Eq. (1.1), the function $a(t)$ may be singular at $t = 0$ and $t = 1$, and our nonlinear term $f(u)$ may be singular at $u = 0$.

When the nonlinearity $f(x)$ is regular at $x = 0$, [7] discuss the uniqueness of positive solutions and the dependence of positive solutions on the parameter λ for BVP (1.1). However, to the best of our knowledge, for the nonlocal BVPs with nonhomogeneous BCs, results on the uniqueness of positive solutions are rare in the literature when nonlinearities involved in the associated problems are singular in the phase variable.

In this paper, we first present an existence theorem for a unique fixed point for a class of increasing operators without continuity and compactness, and apply it to

Received June 28, 2011

investigate the existence and uniqueness of positive solutions for the nonlocal singular BVP (1.1).

Here we briefly recall various basic definitions and facts.

Let E be a real Banach space and P a nonempty convex closed subset of E. We say P is a cone in E if P satisfies the following two conditions:

(i) $\lambda x \in P$ for any $x \in P$ and $\lambda \geq 0$;

(ii) $x, -x \in P$ implies $x = \theta$, where θ is the zero element of E.

The Banach space E can be partially ordered by a cone P, that is, $x \leq y$ if and only if $y - x \in P$. Recall that a cone P in E is normal if there exists a constant $N > 0$ such that $\|x\| \leq N\|y\|$ when $\theta \leq x \leq y$. The smallest N satisfying the condition is said to the normality constant of P.

Given $h > \theta$, that is, $h \geq \theta$ and $h \neq \theta$, we let

$$P_h = \{x | x \in E \text{ and there exist } \lambda(x), \mu(x) > 0 \text{ such that } \lambda(x)h \leq x \leq \mu(x)h\}.$$

It is easy to see that $P_h \subset P$. It is clear that for any $x, y \in P_h$ and $k > 0$, we have $x + y \in P_h,\ kx \in P_h$ and there exist $\lambda, \mu > 0$ such that $\lambda x \leq y \leq \mu x$.

Consider the linear space $C[a, b]$ of all real-valued continuous functions $x(t)$ defined on $[a, b]$. $C[a, b]$ is a Banach space when given the norm

$$\|x\| = \max_{t\in[a,b]} |x(t)|.$$

Set $P = \{x | x \in C[a, b],\ x(t) \geq 0,\ t \in [a, b]\}$. Then P is a normal cone in $C[a, b]$ whose normality constant is 1.

Let E be a real Banach space, P a cone in E, D a subset of E. We say an operator $A : D \longrightarrow E$ is increasing on D if $x_1 \leq x_2$ implies that $Ax_1 \leq Ax_2$ for any $x_1, x_2 \in D$.

See [3] for a detailed exposition.

2. FIXED POINT THEOREMS FOR INCREASING OPERATORS

In this section, we give a fixed point theorem for increasing operators.

Theorem 2.1. *Let E be a real Banach space, P a normal cone in E, $h > \theta$, $w \in P_h$, where θ is the zero element of E. Assume that $A : P \longrightarrow P$ is an increasing operator satisfying $A(tx) \geq t^\alpha Ax$ for any $x \in P$ and $t \in (0, 1)$ ($\alpha > 1$). The operator C is given by $Cu = Au + w,\ u \in P$. If there exists $v_0 \in P_h$ such that*

(i) $Cv_0 \leq v_0$;

(ii) $Av_0 \leq \beta w$, where $\beta \in (0, \frac{1}{\alpha-1})$;

then

(a) C has a unique fixed point x^* in $[\theta, v_0]$, and $x^* \in P_h$; and there exists $v_0' \in P_h$ with $v_0' > v_0$ such that C has no fixed points in $[\theta, v_0'] \backslash [\theta, v_0]$;

(b) for any $x_0 \in [\theta, v_0]$, writing $x_{n+1} = Cx_n$, $n = 0, 1, 2, \ldots$, we have $\lim_{n\to\infty} x_n = x^*$. Moreover, there exist $\bar{l}, \gamma \in (0,1)$ such that $\|x_n - x^*\| \leq 2N(1 - \bar{l}^{\gamma^n})\|v_0\|$, $n = 1, 2, \ldots$, where N is the normality constant of P.

Proof. We have

$$Cu \in P_h \text{ for any } u \in [\theta, v_0].$$

For, by $w \in P_h$, there exist $\lambda, \mu > 0$ such that $\lambda h \leq w \leq \mu h$. Hence for any $u \in [\theta, v_0]$, since $Au \in P$, we obtain, relating to the increasing property of A, that

$$\lambda h \leq w \leq Au + w = Cu \leq Av_0 + w \leq \beta w + w = (\beta + 1)w \leq (\beta + 1)\mu h.$$

We also have there exists $\gamma \in (0,1)$ such that

$$C(lu) \geq l^\gamma Cu \text{ for any } l \in (0,1) \text{ and } u \in [\theta, v_0].$$

In fact, since the function $\varphi(s) = \frac{s}{\alpha - s}$ is continuous on $[0,1]$ and $\beta \in (0, \frac{1}{\alpha - 1})$, there exists $\gamma \in (0,1)$ such that $\frac{\gamma}{\alpha - \gamma} > \beta$. It is easily verified that

$$\frac{l^\gamma - 1}{l^\alpha - l^\gamma} > \frac{\gamma}{\alpha - \gamma} \text{ for any } l \in (0,1),$$

hence $\frac{l^\gamma - 1}{l^\alpha - l^\gamma} > \beta$, so $Av_0 \leq \beta w \leq \frac{l^\gamma - 1}{l^\alpha - l^\gamma} w$. In virtue of the increasing property of A, we have for any $u \in [\theta, v_0]$, $Au \leq Av_0 \leq \frac{l^\gamma - 1}{l^\alpha - l^\gamma} w$, that is, $l^\alpha Au + w \geq l^\gamma (Au + w)$, hence

$$C(lu) = A(lu) + w \geq l^\alpha Au + w \geq l^\gamma (Au + w) = l^\gamma Cu.$$

Set $v_{n+1} = Cv_n$, $n = 0, 1, 2, \ldots$. By (i), it follows that $v_1 = Cv_0 \leq v_0$. Since A is increasing, C is also increasing, hence $v_{n+1} \leq v_n$, $n = 0, 1, 2, \ldots$. Thus $v_n \in [\theta, v_0]$, $n = 0, 1, 2, \ldots$.

Next we shall prove (a).

Existence. There exists $c > 0$ such that $Cv_0 \geq cv_0$ since $Cv_0, v_0 \in P_h$. Taking a sufficiently small number $l_0 \in (0,1)$ such that $l_0^{\gamma - 1} c > 1$ and setting $u_0 = l_0 v_0$, we have $Cu_0 = C(l_0 v_0) \geq l_0^\gamma Cv_0 \geq l_0^\gamma cv_0 = l_0^{\gamma - 1} cl_0 v_0 \geq l_0 v_0 = u_0$. Set $u_{n+1} = Cu_n$, $n = 0, 1, 2, \ldots$. Since $u_0 \leq Cu_0 = u_1$, $v_1 \leq v_0$, $u_0 = l_0 v_0 \leq v_0$ and C is increasing, by induction, we have $u_n \leq u_{n+1}$, $v_{n+1} \leq v_n$, $u_n \leq v_n$, $n = 0, 1, 2, \ldots$, that is,

$$u_0 \leq u_1 \leq \cdots \leq u_n \leq \cdots \leq v_n \leq \cdots \leq v_1 \leq v_0.$$

We also have

$$u_1 = Cu_0 = C(l_0 v_0) \geq l_0^\gamma Cv_0 = l_0^\gamma v_1,$$

$$u_2 = Cu_1 \geq C(l_0^\gamma v_1) \geq (l_0^\gamma)^\gamma Cv_1 = l_0^{\gamma^2} v_2,$$

and so on and so forth, $u_n \geq l_0^{\gamma^n} v_n$, $n = 3, 4, 5, \ldots$.

Therefore

$$l_0^{\gamma^n} v_n \le u_n \le u_{n+p} \le v_{n+p} \le v_n$$

for any positive integers n and p. So we have

$$\theta \le u_{n+p} - u_n \le v_n - l_0^{\gamma^n} v_n \le (1 - l_0^{\gamma^n}) v_0,$$

hence

$$\|u_{n+p} - u_n\| \le N\|(1 - l_0^{\gamma^n}) v_0\| = N(1 - l_0^{\gamma^n})\|v_0\| \to 0 \text{ as } n \to \infty,$$

which indicates that $\{u_n\}$ is a Cauchy sequence. And we have

$$\theta \le v_n - v_{n+p} \le v_n - l_0^{\gamma^n} v_n \le (1 - l_0^{\gamma^n}) v_0,$$

hence

$$\|v_{n+p} - v_n\| = \|v_n - v_{n+p}\| \le N(1 - l_0^{\gamma^n})\|v_0\| \to 0 \text{ as } n \to \infty,$$

which indicates that $\{v_n\}$ is also a Cauchy sequence. Set $\bar{u} = \lim\limits_{n\to\infty} u_n$ and $\bar{v} = \lim\limits_{n\to\infty} v_n$. Since

$$\theta \le v_n - u_n \le v_n - l_0^{\gamma^n} v_n \le (1 - l_0^{\gamma^n}) v_0,$$

$$\|v_n - u_n\| \le N(1 - l_0^{\gamma^n})\|v_0\| \to 0 \text{ as } n \to \infty.$$

Thus $\|\bar{u} - \bar{v}\| = \lim\limits_{n\to\infty} \|u_n - v_n\| = 0$, that is, $\bar{u} = \bar{v}$. Write $x^* = \bar{u} = \bar{v}$. Since $u_m \le v_n$ for any positive integers m and n, $x^* \le v_n$ as $m \to \infty$ and $u_m \le x^*$ as $n \to \infty$. Hence $u_n \le x^* \le v_n$ for any positive integer n. By the increasing property of C, we have $u_{n+1} \le Cx^* \le v_{n+1}$. So $x^* \le Cx^* \le x^*$, that is, $Cx^* = x^*$. Thus x^* is a fixed point of C. It is clear that $x^* \in [\theta, v_0]$ and $x^* = Cx^* \in P_h$.

Uniqueness. Suppose that $y^* \in [\theta, v_0]$ is also a fixed point of C. Since $x^*, y^* \in P_h$, there exists $q > 0$ such that $x^* \ge qy^*$. Write $q_0 = \sup\{q | q > 0,\ x^* \ge qy^*\}$. It is clear that $q_0 > 0$ and $x^* \ge q_0 y^*$. If $q_0 < 1$, then $x^* = Cx^* \ge C(q_0 y^*) \ge q_0^\gamma C y^* = q_0^\gamma y^*$, which is contrary to the definition of q_0 since $q_0^\gamma > q_0$. Therefore $q_0 \ge 1$. So we have $x^* \ge y^*$. Similarly it follows that $y^* \ge x^*$. Thus $x^* = y^*$.

We can also obtain that there exists $v_0' \in P_h$ with $v_0' > v_0$ such that C has a unique fixed point in $[\theta, v_0']$. Since $\lim\limits_{s\to 1+} \frac{s-1}{s^\alpha - s} = \frac{1}{\alpha - 1}$, there exists $s_0 > 1$ such that

$$s_0^\alpha \beta < \frac{1}{\alpha - 1} \text{ and } \frac{s_0 - 1}{s_0^\alpha - s_0} > \beta.$$

Write $v_0' = s_0 v_0$. It is clear that $v_0' > v_0$. By (ii), we have $Av_0 \le \beta w \le \frac{s_0 - 1}{s_0^\alpha - s_0} w$, hence $s_0^\alpha A v_0 + w \le s_0(Av_0 + w)$. Again by (i) and the property of A, it follows that

$$\begin{aligned} Cv_0' &= C(s_0 v_0) = A(s_0 v_0) + w = s_0^\alpha \cdot \frac{1}{s_0^\alpha} A(s_0 v_0) + w \le s_0^\alpha A(\frac{1}{s_0} \cdot s_0 v_0) + w \\ &= s_0^\alpha A v_0 + w \le s_0(Av_0 + w) = s_0 C v_0 \le s_0 v_0 = v_0'. \end{aligned}$$

We also have $Av_0' = A(s_0 v_0) = s_0^\alpha \cdot \frac{1}{s_0^\alpha} A(s_0 v_0) \le s_0^\alpha A(\frac{1}{s_0} \cdot s_0 v_0) = s_0^\alpha A v_0 \le (s_0^\alpha \beta) w$. Therefore v_0' satisfies (i) and (ii) which v_0 satisfies. As the above proof, C has a unique fixed point z^* in $[\theta, v_0']$. Since $x^* \in [\theta, v_0] \subset [\theta, v_0']$ is a fixed point of C, we must have $z^* = x^*$. Thus C has no fixed points in $[\theta, v_0'] \backslash [\theta, v_0]$.

At last, we shall prove (b).

For any $x_0 \in [\theta, v_0]$, since $x_1 = Cx_0 \in P_h,\ v_1 = Cv_0 \in P_h$ and $x^* \in P_h$, there exists $\bar{l} \in (0,1)$ such that $\bar{l}^{\gamma} v_1 \leq x_1 = Cx_0 \leq Cv_0 = v_1$ and $\bar{l}^{\gamma} v_1 \leq x^* = Cx^* \leq Cv_0 = v_1$. By induction, it easily follows that

$$\bar{l}^{\gamma^n} v_n \leq x_n \leq v_n \text{ and } \bar{l}^{\gamma^n} v_n \leq x^* \leq v_n,\ n = 1, 2, \ldots.$$

Because P is normal and $\lim\limits_{n\to\infty} v_n = \lim\limits_{n\to\infty} \bar{l}^{\gamma^n} v_n = x^*$, it follows that $\lim\limits_{n\to\infty} x_n = x^*$ by the property of normal cones. We also have for any positive integer n,

$$\theta \leq v_n - x_n \leq v_n - \bar{l}^{\gamma^n} v_n \leq (1 - \bar{l}^{\gamma^n}) v_0 \text{ and } \theta \leq v_n - x^* \leq v_n - \bar{l}^{\gamma^n} v_n \leq (1 - \bar{l}^{\gamma^n}) v_0,$$

hence

$$\|x_n - x^*\| \leq \|v_n - x_n\| + \|v_n - x^*\| \leq N\|(1 - \bar{l}^{\gamma^n}) v_0\| + N\|(1 - \bar{l}^{\gamma^n}) v_0\| = 2N(1 - \bar{l}^{\gamma^n})\|v_0\|.$$

□

Remark 2.2. In Theorem 2.1, if $E = C[a,b]$ and $P = \{x | x \in E, x(t) \geq 0, t \in [a,b]\}$, then "$\|x_n - x^*\| \leq 2N(1 - \bar{l}^{\gamma^n})\|v_0\|$" in (b) can be changed into "$\|x_n - x^*\| \leq (1 - \bar{l}^{\gamma^n})\|v_0\|$". In fact, we have

$$\bar{l}^{\gamma^n} v_n \leq x_n \leq v_n \text{ and } \bar{l}^{\gamma^n} v_n \leq x_n \leq v_n, n = 1, 2, \ldots.$$

For all $n = 1, 2, \ldots$,

$$x_n - x^* \leq v_n - \bar{l}^{\gamma^n} v_n \leq (1 - \bar{l}^{\gamma^n}) v_0,$$

that is,

$$x_n(t) - x^*(t) \leq (1 - \bar{l}^{\gamma^n}) v_0(t),\ t \in [a,b].$$

Similarly we have

$$x^*(t) - x_n(t) \leq (1 - \bar{l}^{\gamma^n}) v_0(t),\ t \in [a,b].$$

Hence

$$|x_n(t) - x^*(t)| \leq (1 - \bar{l}^{\gamma^n}) v_0(t),\ t \in [a,b].$$

So

$$\|x_n - x^*\| = \sup_{t\in[a,b]} |x_n(t) - x^*(t)| \leq (1 - \bar{l}^{\gamma^n}) \sup_{t\in[a,b]} v_0(t) = (1 - \bar{l}^{\gamma^n}) \sup_{t\in[a,b]} \|v_0\|,\ n = 1, 2, \ldots.$$

3. THIRD ORDER BOUNDARY VALUE PROBLEMS WITH p-LAPLACIAN

Now we shall discuss a class of third order boundary value problems using the conclusions in Section 2.

Let $E = C[0,1],\ P = \{x|x \in E,\ x(t) \geq 0,\ t \in [0,1]\}$ and $h(t) = 1,\ t \in [0,1]$. We have $h > \theta$, where θ is the zero element of E. And let

$$G(t,s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1, \\ s(1-t), & 0 \leq s \leq t \leq 1 \end{cases}$$

and $m = \sup\limits_{r\in[0,1]} \int_0^1 G(r,s)a(s)ds > 0$.

Now we shall discuss a class of third order boundary value problems using the conclusions in Section 2.

Let $E = C[0,1],\ P = \{x|x \in E,\ x(t) \geq 0,\ t \in [0,1]\}$ and $h(t) = 1,\ t \in [0,1]$. We have $h > \theta$, where θ is the zero element of E. And let

$$G(t,s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1, \\ s(1-t), & 0 \leq s \leq t \leq 1 \end{cases}$$

and $m = \sup\limits_{r\in[0,1]} \int_0^1 G(r,s)a(s)ds > 0$.

Theorem 3.1. *Suppose that* $f : (0,+\infty) \to [0,+\infty)$ *is an increasing function satisfying that* $f(ku) \geq k^{\frac{\alpha}{q-1}} f(u)$ *for any* $k \in (0,1)$ *and* $u \in (0,+\infty)$ $(\alpha > 1)$. *If there exists* $K \geq \frac{\alpha\lambda}{(\alpha-1)(1-\xi)}$ *such that* $f(K) < \frac{1}{m}[\frac{\lambda}{(\alpha-1)(1-\xi+\xi\eta)}]^{\frac{1}{q-1}}$, *writing* $v_0(t) = K,\ t \in [0,1]$, *then BVP* (1) *has a unique solution* x^* *in* $[\theta, v_0]$, *and* $x^* \in P_h$. *And there exists* $\bar{v}_0 \in P_h$ *with* $\bar{v}_0 > v_0$ *such that BVP (1) has no solutions in* $[\theta, \bar{v}_0] \backslash [\theta, v_0]$. *Moreover, for any* $x_0 \in [\theta, v_0]$, *setting*

$$\begin{aligned} x_{n+1}(t) &= -\int_0^t \phi_q\left(\int_0^1 G(r,s)a(s)f(x_n(s))ds\right)dr \\ &\quad + \frac{\xi}{1-\xi}\int_0^\eta \phi_q\left(\int_0^1 G(r,s)a(s)f(x_n(s))ds\right)dr + \frac{\lambda}{1-\xi}, \end{aligned}$$

$t \in [0,1],\ n = 0,1,2,\ldots,$ *we have the sequence* $\{x_n(t)\}$ *converges uniformly to* $x^*(t)$ *in* $[0,1]$, *and there exist* $l, \gamma \in (0,1)$ *such that* $\max\limits_{t\in[0,1]} |x_n(t) - x^*(t)| \leq (1 - l^{\gamma^n})K$.

Proof. Define the operators A and C on P as follows.

$$\begin{aligned} (Au)(t) &= \int_0^t \phi_q\left(\int_0^1 G(r,s)a(s)f(u(s))ds\right)dr \\ &\quad + \frac{\xi}{1-\xi}\int_0^\eta \phi_q\left(\int_0^1 G(r,s)a(s)f(u(s))ds\right)dr, \\ (Cu)(t) &= (Au)(t) + w(t), \end{aligned}$$

$t \in [0,1]$, $u \in P$, where $w(t) = \frac{\lambda}{1-\xi}$, $t \in [0,1]$.

It is obvious that $w \in P_h$.

We have $Au \in P$ for any $u \in P$. Since $G(r,s) \geq 0$, $(r,s) \in [0,1] \times [0,1]$, A is an increasing operator. For any $u \in P$, $Au \geq A\theta \geq \theta$, that is, $Au \in P$.

It is easily verified that $A(ku) \geq k^\alpha Au$ for any $k \in (0,1)$ any $u \in P$.

We have $Cv_0 \leq v_0$. In fact, for any $t \in [0,1]$,

$$\begin{aligned}
& (Cv_0)(t) = (Av_0)(t) + w(t) \\
= & \int_0^t \phi_q\left(\int_0^1 G(r,s)a(s)f(v_0(s))ds\right)dr \\
& + \frac{\xi}{1-\xi}\int_0^\eta \phi_q\left(\int_0^1 G(r,s)a(s)f(v_0(s))ds\right)dr + \frac{\lambda}{1-\xi} \\
= & \int_0^t \phi_q\left(\int_0^1 G(r,s)a(s)f(K)ds\right)dr \\
& + \frac{\xi}{1-\xi}\int_0^\eta \phi_q\left(\int_0^1 G(r,s)a(s)f(K)ds\right)dr + \frac{\lambda}{1-\xi} \\
= & f(K)^{q-1}\int_0^t \phi_q(m)dr + f(K)^{q-1}\cdot\frac{\xi}{1-\xi}\int_0^\eta \phi_q(m)dr + \frac{\lambda}{1-\xi} \\
\leq & [mf(K)]^{q-1}\left(1 + \frac{\xi\eta}{1-\xi}\right) + \frac{\lambda}{1-\xi} < \frac{\lambda}{(\alpha-1)(1-\xi+\xi\eta)}\cdot\frac{1-\xi+\xi\eta}{1-\xi} + \frac{\lambda}{1-\xi} \\
= & \frac{\alpha}{\alpha-1}\cdot\frac{\lambda}{1-\xi} \leq K = v_0(t).
\end{aligned}$$

Next we shall prove that there exists $\beta \in (0, \frac{1}{\alpha-1})$ such that $Av_0 \leq \beta w$. For any $t \in [0,1]$,

$$\begin{aligned}
(Av_0)(t) = & \int_0^t \phi_q\left(\int_0^1 G(r,s)a(s)f(v_0(s))ds\right)dr \\
& + \frac{\xi}{1-\xi}\int_0^\eta \phi_q\left(\int_0^1 G(r,s)a(s)f(v_0(s))ds\right)dr < \frac{1}{\alpha-1}\cdot\frac{\lambda}{1-\xi}.
\end{aligned}$$

Since $(Av_0)(t)$ is continuous on $[0,1]$, $\max\limits_{[0,1]}(Av_0)(t) < \frac{1}{\alpha-1}\cdot\frac{\lambda}{1-\xi}$, hence there exists $\beta \in \left(0, \frac{1}{\alpha-1}\right)$ such that $\max\limits_{[0,1]}(Av_0)(t) \leq \beta\cdot\frac{\lambda}{1-\xi}$, so $(Av_0)(t) \leq \beta\cdot\frac{\lambda}{1-\xi} = \beta w(t)$ for any $t \in [0,1]$, that is, $Av_0 \leq \beta w$.

Now all conditions of Theorem 2.1 are satisfied. Thus C has a unique fixed point x^* in $[\theta, v_0]$, and $x^* \in P_h$; and there exists $\bar{v}_0 \in P_h$ with $\bar{v}_0 > v_0$ such that C has no fixed points in $[\theta, \bar{v}_0]\backslash[\theta, v_0]$. For any $x_0 \in [\theta, v_0]$, writing $x_{n+1} = Cx_n$, $n = 0,1,2,\ldots$, we have $\lim\limits_{n\to\infty} x_n = x^*$. Relating to Remark 2.2, there exist $l, \gamma \in (0,1)$ such that $\|x_n - x^*\| \leq (1 - l^{\gamma^n})\|v_0\| \leq (1 - l^{\gamma^n})K$, $n = 1,2,\ldots$.

It is easily verified that fixed points of C are identical to solutions of BVP (1). So we complete the proof. □

Remark 3.2. We may give an example. For the following third order BVP with p-Laplacian

$$\begin{cases} (\phi_2(u'))'' + u^2 = 0, \ 0 < t < 1, \\ u(0) = \frac{1}{2}u(\frac{1}{2}) + \frac{1}{2}, \ u'(0) = u'(1) = 0, \end{cases}$$

$K = 2$ satisfies conditions in Theorem 3.1.

REFERENCES

[1] D. R. Anderson, Green's function for a third-order generalized right focal problem, *J. Math. Anal. Appl.*, 288:1-14, 2003.

[2] S. L. Clark and J. Henderson, Uniqueness implies existence and uniqueness criterion for nonlocal boundary value problems for third order differential equations, *Proc. Amer. Math. Soc.*, 136:3363-3372, 2006.

[3] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, 1988.

[4] J. R. Greaf and J. R. L. Webb, Third order boundary value problems with nonlocal boundary conditions, *Nonlinear Anal.* 71:1542-1551, 2009.

[5] J. Henderson and W. K. C. Yin, Existence of solutions for third-order boundary value problems on a time scale, *Comput. Math. Appl.*, 45:1101-1111, 2003.

[6] B. Hopkins and N. Kosmatov, Third-order boundary value problems with sign-changing solutions, *Nonlinear Anal.*, 67:126-137, 2007.

[7] L. Kong, D. Piao and L. Wang, Positive solutions for third order boundary value problems with p-Laplacian, *Result Math.*, 55:111-128, 2009.

[8] F. M. Minhós, On some third order nonlinear boundary value problems: existence, location and multiplicity results, *J. Math. Anal. Appl.*, 339:1342-1353, 2008.

[9] Y. Sun, Positive solutions for third-order three-point nonhomogeneous boundary value problems, *Appl. Math. Lett.*, 22:45-51, 2009.