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BOUNDED AND PERIODIC SOLUTIONS IN RETARDED DIFFERENCE EQUATIONS USING SUMMABLE DICHOTOMIES

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ABSTRACT. Using the notion of summable dichotomy in ordinary difference equations and the contraction and Schauder fixed point theorem, we obtain the existence of bounded and periodic solutions on $\mathbb{Z}$ under quite general hypotheses for non homogenous retarded difference equations.

Keywords: Difference equations; Summable dichotomies; Bounded solutions on $\mathbb{Z}$; Periodic solutions on $\mathbb{Z}$; Contraction fixed point theorem; Schauder fixed point theorem.

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1. INTRODUCTION

In this paper, we are concerned with a system of homogeneous linear difference equations

$$x(n+1) = A(n)x(n), \; n \in \mathbb{Z}, \tag{1.1}$$

and its general delayed perturbation

$$x(n+1) = A(n)x(n) + F(n, x(n), x(n-\tau(n))), \; n \in \mathbb{Z}, \tag{1.2}$$

under certain conditions for $F : \mathbb{Z} \times \mathbb{C}^r \times \mathbb{C}^r \to \mathbb{C}^r$, where $\mathbb{Z}$ denotes the set of integer numbers, $\mathbb{C}^r$ denotes the r-dimensional Euclidean space with norm $|\cdot|$, and we will consider the initial condition

$$x(0) = \xi \in \mathbb{C}^r. \tag{1.3}$$

Properties of the solution of the linear system (1.1) have well investigated, see for example [1], [9], [11], [12].

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We will denote by $X(n)$ the fundamental matrix of system (1.1), with $X(0) = I$. For a projection matrix P. So, the Green function $G = G_P$ associated the projection P is given by

$$G(n,s) = \begin{cases} X(n)\,PX^{-1}(s+1), & for\ n \geq s+1 \\ -X(n)\,QX^{-1}(s+1), & for\ n < s+1, \end{cases} \tag{1.4}$$

where $Q = I - P$.

The notion of summable dichotomy introduced by Pinto in [19] in the ordinary difference case will be remember below.

Definition 1. We say that system of equations (1.1) has a summable dichotomy on $\mathbb{Z}$ with data (μ, P) if there is a projection P in $\mathbb{C}^r$, such that if $Q = I - P$, we have

$$\mu := \sup_{n\in\mathbb{Z}} \sum_{s=-\infty}^{+\infty} \|G(n,s)\| < +\infty. \tag{1.5}$$

Based on the ordinary dichotomy of a linear system non-autonomous system (1.1) several qualitative properties of the solutions of ordinary difference equations have been investigated (see e.g. [1], [12]). In particular the existence of bounded solutions of linear and nonlinear systems has been considered. Properties of the solutions using the existence of exponential dichotomy have been considered, for example, in [15], [17], [18], [24] and references therein.

As said Pinto in [19], the dichotomies are decomposed in two big groups: the "uniform" dichotomies and the "sumable" dichotomies. The uniform dichotomies are the natural extension of the ordinary dichotomy and the summable dichotomies are an extension of the exponential dichotomy. It is easy to see that if system (1.1) has a summable dichotomy with data (μ, P), then it also has an ordinary dichotomy, and on the other hand, we also note that if the system (1.1) has an exponential dichotomy, thus the system (1.1) has a summable dichotomy.

The objective of this work is considering the existence of a summable dichotomy of the linear system (1.1) on $\mathbb{Z}$ to characterize the bounded and periodic solutions of this linear systems and the nonhomogeneous case. After that we consider the perturbation system (1.2) and our aim is to prove the existence of bounded and periodic solutions of this system under the existence of a summable dichotomy of the linear system (1.1) and some restrictions on the function F in (1.2). Particularly important are the general conditions obtained for the existence of bounded solutions when Schauder fixed point is applied (see Section 4.1) under the use of Schauder's Fixed Point Theorem. Under a suitable condition a new compactness criterion for a summable dichotomy is exhibited (see hypothesis (H5)). We emphasize that the study of summable dichotomy combined with the use of the Schauder Fixed Point Theorem in order to get bounded and periodic solutions has not been considered in the literature, still in the ordinary case. On the other hand, in our approach we considerer the existence of a summable dichotomy on $\mathbb{Z}$.

Our results are important because they improve existing ones in literature. In fact we will see that they are reduced in the assumptions used previously. Preliminary studies in the case of ordinary difference equation were studied in [16] in the functional case and in [7] in the abstract or the phase space case.

In order to get our objective we have organized the paper as follows. In Section 2 under the existence of a summable dichotomy we characterized bounded and periodic solutions of the linear system (1.1). The same problem, but in the non-homogeneous case in Section 3 is considered. In Section 4, we prove the existence of bounded and periodic solutions using the Contraction Fixed Point Theorem under suitable conditions of the perturbed function F in (1.2). After that in Subsection 4.1 we prove the previous result by the use of the Schauder Fixed Point Theorem. We call the attention that in this case it is necessary to verify a compactness criterium. Finally in Section 5 we apply our results to the particular case, namely, the nonlinear ordinary difference equations, and some examples are considered.

The existence of bounded and almost periodic solutions for difference equations in the ordinary case have been considered by several authors, for example, [2], [9], [10], [14], [15], [21], [22], [23], [25], [26]. Pinto in [19] using the concept of (h, k)-summable dichotomies in linear difference equations, he proved the existence of convergent and bounded solutions of nonlinear ordinary difference systems for $n \geq n_0$. As it is known the abstract space was introduced by Hale-Kato [13] for studying qualitative theory of functional differential equations with unbounded delay. The investigation of convergent solutions in abstract retarded difference equations in phase phase under the condition of existence of dichotomies has been considered in [3], [5], [6], [7], [16], for example. The similar problem as in our point of view has been considered by Pinto in [20] in the case of nonlinear integro-differential equations with infinite delay.

2. THE HOMOGENOUS LINEAR SYSTEMS

Initially we are going to prove some basic but very importante results associated the linear system (1.1) as consequence of the existence of a sumable dichotomy.

Proposition 2. *Assuming that system* (1.1) *possesses a summable dichotomy, then* $x = 0$ *is the unique bounded solution on* $\mathbb{Z}$ *of the linear homogeneous system* (1.1).

Proof. Note that any solution of (1.1) is given by $x(n) = X(n)\xi = X(n)P\xi + X(n)Q\xi$.

Define $B^0 = \{\xi \in \mathbb{C}^r \,/\, \xi \text{ is a initial condition of a bounded solution on } \mathbb{Z} \text{ of } (1.1)\}$. Take any vector $\xi \in B^0$ and assume first that $P\xi \neq 0$, and consider the auxiliary function $\alpha(n)^{-1} = |X(n) P\xi|$. Then

$$\sum_{s=-\infty}^{n-1} X(n) P\xi\alpha(s+1) = \sum_{s=-\infty}^{n-1} X(n) PX^{-1}(s+1) X(s+1) P\xi\alpha(s+1),$$

so,

$$\sum_{s=-\infty}^{n-1} |X(n) P\xi| \alpha(s+1) \leq \sum_{s=-\infty}^{n-1} \left\|X(n) PX^{-1}(s+1)\right\| |X(s+1) P\xi| \alpha(s+1),$$

i.e.,

$$\alpha(n)^{-1} \sum_{s=-\infty}^{n-1} \alpha(s+1) \leq \sum_{s=-\infty}^{n-1} \left\|X(n) PX^{-1}(s+1)\right\|.$$

Since the dichotomy is summable, we have that

$$\alpha(n)^{-1} \sum_{s=-\infty}^{n-1} \alpha(s+1) \leq \mu,$$

uniformly in n. Thus, the serie $\sum_{s=-\infty}^{n-1} \alpha(s+1)$ is convergent for each n fixed, it follows that the limit of the general term must be zero, in particular $\alpha(n) \to 0$ as $n \to -\infty$, i.e., $|X(n) P\xi| \to +\infty$ as $n \to -\infty$.

On the other hand, if we assume that $Q\xi \neq 0$, then defining the function $\beta(n)^{-1} = |X(n) Q\xi|$, by similar arguments as in the previous case, we obtain that $|X(n) P\xi| \to +\infty$ as $n \to +\infty$.

Thus, the only possibility for boundeness of the solutions of system (1.1) is that $B^0 = \{0\}$, i.e., $x = 0$. □

Remark 3. It is clear by the previous proposition that under the hypothesis of the existence of a summable dichotomy for Eq. (1.1), it follows that the linear system (1.1) does not have non-null periodic solution neither non null almost periodic solution.

It is interesting to observe that, under the hypothesis of summable dichotomy, $X(n) PX^{-1}(n)$ and $X(n) QX^{-1}(n)$ are bounded. In fact, for each $n \in \mathbb{Z}$,

$$\sum_{s=-\infty}^{+\infty} \|G(n,s)\| \leq \mu,$$

and in particular,

$$\left\|X(n) PX^{-1}(n)\right\| = \|G(n,n)\| \leq \sum_{s=-\infty}^{+\infty} \|G(n,s)\| \leq \mu,$$

uniformly in n, and from this $\|X(n) QX^{-1}(n)\|$ is also bounded. Thus, we have

Lemma 4. *If system* (1.1) *has a summable dichotomy, the* $X(n) PX^{-1}(n)$ *and* $X(n) QX^{-1}(n)$ *are uniformly bounded in* $n \in \mathbb{Z}$.

Lemma 5. *If system* (1.1) *has a summable dichotomy, then for every* $n_0 \in \mathbb{Z}$, *there exists* $M_0 > 0$, *such that*

$$\|X(n)P\| \leq M_0, \text{ for all } n \geq n_0, \text{ and } \|X(n)Q\| \leq M_0, \text{ for all } n \leq n_0.$$

Proof. Let $\alpha(n) = \|X(n)P\|$ and $u(n) = \sum_{s=n_0}^{n-1} \alpha(s+1)^{-1}$. We have that $u(n)$ is an increasing function, and so $u(n_0+1) \leq u(n)$, for all $n \geq n_0$. Moreover,

$$\begin{aligned}\alpha(n)u(n) &\leq \sum_{s=n_0}^{n-1} \|X(n)P\| \alpha(s+1)^{-1} \\ &\leq \sum_{s=n_0}^{n-1} \left\|X(n)PX^{-1}(s+1)\right\| \|X(s+1)P\| \alpha(s+1)^{-1} \leq \mu.\end{aligned}$$

Then

$$\|X(n)P\| = \alpha(n) \leq \mu u(n)^{-1} \leq \mu u(n_0+1)^{-1} = c_0,$$

for all $n \geq n_0$.

Now, let $\beta(n) = \|X(n)Q\|$ and $v(n) = \sum_{s=n}^{n_0} \beta(s+1)^{-1}$. We have that, for all $n_1 \leq n_2 \leq n_0$, $v(n_2) \leq v(n_1)$. In particular, $v(n_0) \leq v(n)$, for all $n \leq n_0$. Then

$$\begin{aligned}\beta(n)v(n) &\leq \sum_{s=n}^{n_0} \|X(n)Q\| \beta(s+1)^{-1} \\ &\leq \sum_{s=n}^{n_0} \left\|X(n)QX^{-1}(s+1)\right\| \|X(s+1)Q\| \beta(s+1)^{-1} \leq \mu.\end{aligned}$$

Then

$$\|X(n)Q\| = \beta(n) \leq \mu v(n)^{-1} \leq \mu v(n_0)^{-1} = k_0,$$

for all $n \leq n_0$.

Taking $M_0 = \max\{c_0, k_0\}$, we concluded the proof. □

Proposition 6. *If the linear system* (1.1) *has a summable dichotomy with data* (μ, P), *then the projection operator* P *is unique, i.e.,* P *is decided uniquely by the summable dichotomy.*

Proof. Assume that there is another projector $\widetilde{P}$ satisfaying the summable condition. In a similar way to the previous discusion, there exists $\widetilde{M}_0 > 0$, such that $\left\|X(n)\widetilde{P}\right\| \leq \widetilde{M}_0$, for all $n \geq 0$, and $\left\|X(n)\widetilde{Q}\right\| \leq \widetilde{M}_0$, for all $n \leq 0$, where $\widetilde{Q} = I - \widetilde{P}$. If $\xi \in \mathbb{C}^r$ and $n \geq 0$, we have

$$\left|X(n)P\widetilde{Q}\xi\right| \leq \|X(n)P\| \left|\widetilde{Q}\xi\right| \leq M_0 \left|\widetilde{Q}\xi\right|,$$

and if $n \leq 0$,

$$\left|X(n)P\widetilde{Q}\xi\right| \leq \left\|X(n)PX^{-1}(n)\right\| \left\|X(n)\widetilde{Q}\right\| \left|\widetilde{Q}\xi\right| \leq \mu M_0 \left|\widetilde{Q}\xi\right|.$$

Then, $x(n) = X(n) P\widetilde{Q}\xi$ is a bounded solution on $\mathbb{Z}$ of system (1.1). By Proposition 2, we have that $P\widetilde{Q}\xi = 0$. As ξ is arbitrary, it follows that $P\widetilde{Q} = 0$, i.e., $P = P\widetilde{P}$. On the other hand, as $X(n) QX^{-1}(n)$ is also bounded, in a similar way to the previous discussion, we have $Q\widetilde{P} = 0$, i.e., $\widetilde{P} = P\widetilde{P}$. Therefore $\widetilde{P} = P$. □

Lemma 7. *Assuming that the linear system* (1.1) *has a summable dichotomy and* $A(n)$ *is a* p*-periodic matrix in* n*, then* $X(n) PX^{-1}(n)$ *is also a* p*-periodic function in* n*.*

Proof. We note that $X(n+p)$ is also a solution matrix of (1.1). Define $\widetilde{P} = X(p) PX^{-1}(p)$. Then $\widetilde{P}$ is a projection. In addition, we have $X(n)\widetilde{P}X^{-1}(s+1) = X(n+p)PX^{-1}(s+p+1)$ and $X(n)\widetilde{Q}X^{-1}(s+1) = X(n+p)QX^{-1}(s+p+1)$, where $\widetilde{Q} = I - \widetilde{P}$. Then, the dichotomy is also summable with projection $\widetilde{P}$. By Proposition 6, it follows that $\widetilde{P} = P$, i.e., $X(p) PX^{-1}(p) = P$. Thus $X(n+p) PX^{-1}(n+p) = X(n) PX^{-1}(n)$, completing the proof. □

Remark 8. By the previous Lemma, and under the same hypotheses, it follows that the Green function satisfy the identity

$$G(n+p, s+p) = G(n, s),$$

for all $n, s \in \mathbb{Z}$.

3. THE NON-HOMOGENEOUS LINEAR SYSTEM

In this section we will consider the non-homogeneous linear system

$$x(n+1) = A(n) x(n) + f(n), \tag{3.1}$$

where $f : \mathbb{Z} \to \mathbb{C}^r$ is for instance an arbitrary function.

Initially we will consider the problem of existence of bounded solutions $\mathbb{Z}$ of the initial value problem

$$\begin{aligned} x(n+1) &= A(n) x(n) + f(n), \\ x(0) &= \xi_0 \in \mathbb{C}^r. \end{aligned} \tag{3.2}$$

We denote by $l^\infty := l^\infty(\mathbb{Z}, \mathbb{C}^r)$ the Banach space of all bounded functions $\varphi : \mathbb{Z} \to \mathbb{C}^r$ endowed with the norm

$$\|\varphi\|_\infty := \sup_{n\in\mathbb{Z}} |\varphi(n)|,$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{C}^r$.

Proposition 9. *If system* (1.1) *has a summable dichotomy and* $f \in l^\infty$*, then the system* (3.1) *has exactly one bounded solution* x *on* $\mathbb{Z}$*, which can be represented by*

$$x(n) = \sum_{s=-\infty}^{+\infty} G(n, s) f(s). \tag{3.3}$$

Proof. It is not difficult to check that $x(n)$, given by (3.3) is a bounded solution on $\mathbb{Z}$ of (3.1). If there exists another bounded solution $y(n)$ on $\mathbb{Z}$, then $x(n) - y(n)$ is a bounded solution on $\mathbb{Z}$ of the homogeneous linear system (1.1). By Proposition 2, $x(n) - y(n) \equiv 0$. Therefore the uniqueness of the bounded solution on $\mathbb{Z}$ of (3.1) is proved. □

Corollary 10. *If If system* (1.1) *has a summable dichotomy,* $A(n+p) = A(n)$*, and* $f(n+p) = f(n)$ *for all* $n \in \mathbb{Z}$*, then the solution* x *of system* (3.1) *is* p*-periodic in* n*.*

Proof. It is sufficient to observe that for each $n \in \mathbb{Z}$

$$x(n+p) = \sum_{s=-\infty}^{+\infty} G(n+p, s+p) f(s+p) = \sum_{s=-\infty}^{+\infty} G(n,s) f(s) = x(n).$$

□

4. THE DELAY NONLINEAR SYSTEM

In this section we will consider the nonlinear system

$$x(n+1) = A(n) + g(n, x(n), x(n-\tau(n))), \tag{4.1}$$

where $g : \mathbb{Z} \times \mathbb{C}^r \times \mathbb{C}^r \to \mathbb{C}^r$ function under convenient conditions, and $\tau : \mathbb{Z} \to \mathbb{Z}^+ \cup \{0\}$ is bounded function such that $\tau(n) \leq n$, which involves the bounded delay in (4.1).

For each $\lambda > 0$, we denote by $l^\infty[\lambda]$ the ball $\|\varphi\|_\infty \leq \lambda$ in l^∞.

Using Proposition 9, we define the funcional on l^∞ by

$$(\Gamma\varphi)(n) = \sum_{s=-\infty}^{+\infty} G(n,s) g(s, \varphi(s), \varphi(s-\tau(s))), \tag{4.2}$$

and it is clear that each fixed point of Γ give us a solution of system (4.1). In order, to obtain bounded solutions on $\mathbb{Z}$ we will assume:

(E_1) System (1.1) has a summable dichotomy on $\mathbb{Z}$ with data (μ, P).

(E_2) The function $g(n, \xi, \eta)$ is locally Lipschitz in $\xi, \eta \in \mathbb{C}^r$, i.e., for each positive number R, for all $\xi_1, \xi_2, \eta_1, \eta_2 \in \mathbb{C}^r$, with $\|\xi_j\|, \|\eta_j\| \leq R$, for $j = 1, 2$, it is valid

$$\|g(n, \xi_1, \eta_1) - g(n, \xi_2, \eta_2)\| \leq C_1(R)[|\xi_1 - \xi_2| + |\eta_1 - \eta_2|],$$

where $C : [0, +\infty) \longrightarrow [0, +\infty)$ is a function and there is a positive constant λ, such that $2C(\lambda) < \frac{1}{\mu}$ and $\sup_{n \in \mathbb{Z}}\{|g(n,0,0)|\} < \frac{\lambda}{\mu}[1 - C(\lambda)\,\mu]$.

Let λ be the constant given by (E_2) satisfying (E_3). We will prove that $\Gamma(\varphi) \in l^\infty[\lambda]$, for all $\varphi \in l^\infty[\lambda]$. In fact,

$$|(\Gamma\varphi)(n)| \leq \sum_{s=-\infty}^{+\infty} \|G(n,s)\| \; |g(s, \varphi(s), \varphi(s-\tau(s)))|$$

$$\leq C(\lambda)\sum_{s=-\infty}^{+\infty}\|G(n,s)\|\ |\varphi(s)|+\sum_{s=-\infty}^{+\infty}\|G(n,s)\||g(s,0,0)|$$
$$\leq C(\lambda)\ \mu\ \|\varphi\|_{\infty}+\mu\sup_{n\in\mathbb{Z}}\{|g(n,0,0)|\}$$
$$=\lambda.$$

This prove that $\Gamma\varphi\in l^{\infty}[\lambda]$, for all $\varphi\in l^{\infty}[\lambda]$.

Now, we will prove that Γ is a contraction. In fact,

$$\begin{aligned}&|(\Gamma\varphi)(n)-(\Gamma\psi)(n)|\\&\leq\sum_{s=-\infty}^{+\infty}\|G(n,s)\|\,|g(s,\varphi(s),\varphi(s-\tau(s))-g(s,\psi(s),\psi(s-\tau(s))|\\&\leq C(\lambda)\textstyle\sum_{s=-\infty}^{+\infty}\|G(n,s)[|\varphi(s)-\psi(s)|+|\varphi(s-\tau(s))-\psi(s-\tau(s))|]\\&\leq 2\mu C(\lambda)\|\varphi-\psi\|_{\infty}.\end{aligned}$$

Now using contraction fixed point theorem we deduce by (E_1)-(E_2), that Γ has a fixed point $\varphi\in l^{\infty}[\lambda]$. Therefore, we have proved

Theorem 11. *Assume that the hypotheses of the previous theorem are satisfied. Then, there exist a unique bounded solution $y(n)$ of equation* (4.1) *on $\mathbb{Z}$.*

If in equation (4.1) we have conditions on periodicity, using Corollary 10 it follows the following result.

Theorem 12. *Assume that all the hypotheses of the previous theorem are satisfied, and that A, g in the first variable and τ are p-periodic. Then, there exist a unique p-periodic solution $y(n)$ of equation* (4.1) *on $\mathbb{Z}$.*

Proof. In this case we define the closed and convex subset of $l^{\infty}[\lambda]$ by $l_p^{\infty}[\lambda]=\{\varphi\in l^{\infty}[\lambda]:\varphi \text{ is } p\text{-periodic}\}$. Under the conditions it is verified that the map Γ given in (4.2) is well defined in $l_p^{\infty}[\lambda]$ onto $l_p^{\infty}[\lambda]$. Thus we conclude the proof of the theorem. □

4.1. **Existence of bounded and periodic solutions on $\mathbb{Z}$ via Schauder's Fixed Point Theorem.** Here we will assume the same notation as in the previous section, but we will suppose that:

(H1) System (1.1) has a summable dichotomy on $\mathbb{Z}$ with data (μ,P).

(H2) g is continuous.

(H3) $|g(n,\xi,\eta)|\leq l(n)f(n,|\xi|,|\eta|)$, where $f:\mathbb{Z}\times\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ is nondecreasing with respect to the second and third variable and $l:\mathbb{Z}\to\mathbb{R}^+$ is a sequence in l^{∞}.

(H4) $\rho_\lambda[f]:=\sup_{n\in\mathbb{Z}}f(n,\lambda,\lambda)\in\mathbb{R}$ and the inequality

$$\|l\|_{\infty}\,\mu\rho_\lambda[f]\leq\lambda$$

holds.

(H5) There is sequence $l \in l^{\infty}$ for which for every $\epsilon > 0$ there exists $N_1 > 0$ such that $\sum_{s=-\infty}^{-N_1} \|G(n,s)\| \, l(s) < \epsilon$ and $\sum_{s=N_1}^{+\infty} \|G(n,s)\| \, l(s) < \epsilon$.

Remark 13. Since system (1.1) has a summable dichotomy on $\mathbb{Z}$ with data (μ, P), it follows that

$$\sum_{s=N_1}^{+\infty} \|G(n,s)\| \, l(s) < \mu \sum_{s=N_1}^{+\infty} l(s).$$

Thus, if the sequence l lies in $l^1(\mathbb{Z})$, then clearly the hypothesis (H5) is satisfied. More generally, (H5) is satisfied if l such that $l\,(s) \to 0$ as $|s| \to \infty$.

First, we prove that Γ as in (4.2) is well defined. In fact, if $\varphi \in l^{\infty}[\lambda]$ we have

$$\begin{aligned} |(\Gamma\varphi)(n)| &\leq \sum_{s=-\infty}^{+\infty} \|G(n,s)\| |g(s,\varphi(s),\varphi(s-\tau(s))| \\ &\leq \sum_{s=-\infty}^{+\infty} \|G(n,s)\| l(s) f(s, \|\varphi\|_{\infty}, \|\varphi\|_{\infty}) \\ &\leq \|l\|_{\infty} \, \mu \rho_{\lambda}[f]. \end{aligned}$$

Let (φ_m) be a sequence in $l^{\infty}[\lambda]$ such that $\varphi_m \to \varphi$ in $l^{\infty}[\lambda]$ as $m \to \infty$. The continuity of the operator Γ follows by the following estimative

$$\begin{aligned} &|(\Gamma\varphi_m)(n) - (\Gamma\varphi)(n)| \leq \sum_{s=-\infty}^{+\infty} \|G(n,s)\| |g(s,\varphi_m(s),\varphi_m(s-\tau(s)))- \\ &g(s,\varphi(s),\varphi(s-\tau(s)))| \\ &= \sum_{s=-\infty}^{-N_1} \|G(n,s)\| |g(s,\varphi_m(s),\varphi_m(s-\tau(s))) - g(s,\varphi(s),\varphi(s-\tau(s)))| + \\ &\sum_{s=-N_1+1}^{N_1-1} \|G(n,s)\| |g(s,\varphi_m(s),\varphi_m(s-\tau(s))) - g(s,\varphi(s),\varphi(s-\tau(s)))| + \\ &\sum_{s=N_1}^{+\infty} \|G(n,s)\| |g(s,\varphi_m(s),\varphi_m(s-\tau(s))) - g(s,\varphi(s),\varphi(s-\tau(s)))| \\ &\leq \sum_{s=-\infty}^{-N_1} \|G(n,s)\| |g(s,\varphi_m(s),\varphi_m(s-\tau(s))) - g(s,\varphi(s),\varphi(s-\tau(s)))| + \qquad (4.3) \\ &\max_{-N_1+1 \leq s \leq N_1-1} |g(s,\varphi_m(s),\varphi_m(s-\tau(s))) - g(s,\varphi(s),\varphi(s-\tau(s)))| \\ &\sum_{s=-N_1+1}^{N_1-1} \|G(n,s)\| + \\ &\sum_{s=N_1}^{+\infty} \|G(n,s)\| |g(s,\varphi_m(s),\varphi_m(s-\tau(s))) - g(s,\varphi(s),\varphi(s-\tau(s)))| \end{aligned}$$

$$\leq 2\ \rho_\lambda[f] \sum_{s=-\infty}^{-N_1} \|G(n,s)\|\ l(s)+$$

$$\max_{-N_1+1\leq s\leq N_1-1} |g(s,\varphi_m(s),\varphi_m(s-\tau(s)))-g(s,\varphi(s),\varphi(s-\tau(s)))|$$

$$\sum_{s=-N_1+1}^{N_1-1} \|G(n,s)\| + 2\rho_\lambda[f] \sum_{s=N_1}^{+\infty} \|G(n,s)\|\ l(s).$$

Now, using the previous argument we prove that the image of Γ is relatively compact or pre-compact. In fact, let $Y=\Gamma(l^\infty[\lambda])$, then $y=\Gamma x$ satisfies

$$\Delta y(n) = (A(n)-I)y + g(n,x(n),x(n-\tau(n))),$$

where

$$\Delta y(n) = y(n+1)-y(n),$$

and it is uniformly bounded on each bounded interval on $\mathbb{Z}$. It is not difficult to check that the sequence $y(n)$ is Lipschitz in each bounded interval $I=[N,M]\subset\mathbb{Z}$. Thus, the set Y is relatively compact in any interval I, i.e., $Y\subset\cup_{i=1}^m B(z_i,\epsilon)$.

Now, let $Z=I_1\cup I_1^c$ with $I_1=[-N_1,N_1]$ and N_1 as in hypothesis $(H5)$, therefore the set Y is relatively compact or totally bounded. In fact, the finite set $\{z_i\}$ defined on I_1 can be completed with constants in I_1^c, i.e., in all $\mathbb{Z}$.

Using the Schauder's fixed point theorem Γ has a fixed point $y\in l^\infty[\lambda]$ which give us a solution of Eq. (4.2). Thus, we have proved

Theorem 14. *Suppose that the conditions* (H1)–(H5) *are satisfied, then there exist a bounded solution $y(n)$ of equation* (4.2) *on $\mathbb{Z}$.*

Remark 15. To our knowledge the compactness condition derived from (H5) is new in the literature. It is an equiconvergence type condition (see [4],[5], [7]) associated to a dichotomy on $\mathbb{Z}$.

In the periodic case, we obtain

Theorem 16. *Assume that the following hypotheses are verified:*

($\tilde{H}1$) System (1.1) *has a summable dichotomy on $\mathbb{Z}$ with data (μ,P).*

($\tilde{H}2$) A, g in the first variable and τ are p-periodic.

($\tilde{H}3$) $g:\mathbb{Z}\times\mathbb{C}^r\times\mathbb{C}^r\to\mathbb{C}^r$ is continuous in the second and third variables, and it it is uniformly continuous in the first variable.

($\tilde{H}4$) $|g(n,\xi,\eta)|\leq f(n,|\xi|,|\eta|)$, where $f:\mathbb{Z}\times\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ is nondecreasing with respect to the second and third variable.

($\tilde{H}5$) $\rho_\lambda[f]:=\sup_{n\in\mathbb{Z}} f(n,\lambda,\lambda)\in\mathbb{R}$ and $\frac{\rho_\lambda[f]}{\lambda}\leq\mu^{-1}$.

Then, there exists a p-periodic solution $y(n)$ of equation (4.2) *on $\mathbb{Z}$ satisfying $|y(n)|\leq\lambda$, for all $n\in\mathbb{Z}$.*

5. APPLICATIONS AND EXAMPLES

As a first application we can consider the ordinary nonlinear system

$$x(n+1) = A(n)x\,(n) + F(n, x(n)), \tag{5.1}$$

where F is a $\mathbb{C}^r$-valued function defined on the product space $\mathbb{Z} \times \mathbb{C}^r$ under suitable conditions. According the notation of the previous section, this particular case corresponds to take

$$g(n, x(n), x(n-\tau(n)) = F(x, x(n)),$$

in the equation (4.1). As consequence of Theorem 11 we have the following result:

Theorem 17. *Suppose the following conditions are satisfied*

(D_1) *System* (1.1) *has a summable dichotomy on* $\mathbb{Z}$ *with data* (μ, P).

(D_2) *The function* $F(n,\xi)$ *is locally Lipschitz in* $\xi \in \mathbb{C}^r$, *i.e., for each positive number* R, *for all* ξ_1, $\xi_2 \in \mathbb{C}^r$, *with* $|\xi_1|$, $|\xi_2| \leq R$,

$$|F\,(n,\xi_1) - F\,(n,\xi_2)| \leq C\,(R)\,|\xi_1 - \xi_2|\,,$$

where $C : [0,+\infty) \longrightarrow [0,+\infty)$ *is a function and there is a positive constant* λ, *such that* $C(\lambda) < \frac{1}{\mu}$ *and* $\sup_{n\in\mathbb{Z}}\{|F(n,0)|\} < \frac{\lambda}{\mu}[1 - C(\lambda)\mu]$.

Then, there exist a unique bounded solution $y(n)$ *of equation* (5.1) *on* $\mathbb{Z}$.

Analogously, by Theorem 12 we have

Theorem 18. *Assume that all the hypotheses of the previous theorem are satisfied and* F *is* p*-periodic in the first variable. Then, there exist a unique* p*-periodic solution* $y(n)$ *of equation* (5.1) *on* $\mathbb{Z}$.

Now, as application of the Schauder Fixed Point Theorem, we must assume the same notation as in the previous section, but we will suppose that:

($\overline{H}1$) System (1.1) has a summable dichotomy on $\mathbb{Z}$ with data (μ, P).

($\overline{H}2$) F is continuous.

($\overline{H}3$) $|F(n,\xi)| \leq l(n)f(n,|\xi|)$, where $f : \mathbb{Z} \times \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing with respect to the second variable.

($\overline{H}4$) $\rho_\lambda[f] := \sup_{n\in\mathbb{Z}} f(n,\lambda) \in \mathbb{R}$ and $\|l\|_\infty \mu\rho_\lambda[f] \leq \lambda$.

($\overline{H}5$) There is a sequence $l \in l^\infty$, for which for every $\epsilon > 0$ there exists $N_1 > 0$ such that $\sum_{s=-\infty}^{-N_1} \|G(n,s)\|\; l(s) < \epsilon$ and $\sum_{s=N_1}^{+\infty} \|G(n,s)\|\; l(s) < \epsilon$.

So, by Theorem 14 follows that

Theorem 19. *Suppose that the conditions* $(\overline{H}1) - (\overline{H}5)$ *are satisfied, then there exist a bounded solution* $y(n)$ *of equation* (5.1) *on* $\mathbb{Z}$.

In the periodic case, by Theorem 16 we obtain

Theorem 20. *Under similar hypotheses as in Theorem* 16 *with* F *p-periodic in the first variable. Then, there exist a p-periodic solution* $y(n)$ *of equation* (5.1) *on* $\mathbb{Z}$.

5.1. **Examples.** Let

$$A(n) = A = \begin{pmatrix} 2^{-1} & 0 \\ 0 & 2 \end{pmatrix},$$

and we consider the homogeneous linear system of difference equations in $\mathbb{C}^2$

$$\begin{aligned} x(n+1) &= Ax(n) \quad ; \quad n \in \mathbb{Z}, \\ x(0) &= \xi_0 \in \mathbb{C}^2. \end{aligned} \tag{5.2}$$

If we take the projections

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } Q = I - P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

then, the Green function is

$$G(n,s) = \begin{cases} X(n)PX^{-1}(s+1) = \begin{pmatrix} 2^{-n+(s+1)} & 0 \\ 0 & 0 \end{pmatrix} & ; \quad \text{for } n \geq s+1, \\ -X(n)QX^{-1}(s+1) = \begin{pmatrix} 0 & 0 \\ 0 & 2^{n-(s+1)} \end{pmatrix} & ; \quad \text{for } n < s+1. \end{cases}$$

Therefore, the system (5.2) have a summable dichotomy with data (μ, P), where

$$\mu := \sup_{n\in\mathbb{Z}} \sum_{s=-\infty}^{+\infty} \|G(n,s)\| = 3.$$

Let $\{b(n)\}$ and $\{c(n)\}$ be nonzero bounded sequences in $\mathbb{Z}$, with $b = \sup_{n\in\mathbb{Z}} |b(n)|$ and $\lambda > 0$ a positive constant such that $b\lambda < \frac{1}{6}$. Let $\tau(n) = n$ and $g(n, x(n), x(n - \tau(n))) = b(n)(|x(n)|x(n) + |x(0)|x(0)) + c(n)$. Consider the system

$$x(n+1) = A(n)x(n) + g(n, x(n), x(n-\tau(n))), \ n \in \mathbb{Z}, \tag{5.3}$$

Let ξ_1, ξ_2, η_1, $\eta_2 \in \mathbb{C}^2$, $|\xi_1|$, $|\xi_2|$, $|\eta_1|$, $|\eta_2| < \lambda$, we have

$$|g(n,\xi_1,\xi_2) - g(n,\eta_1,\eta_2)| \leq C(\lambda)(|\xi_1 - \xi_2| + |\eta_1 - \eta_2|),$$

where $C(t) = 2bt$, and

$$C(\lambda) = 2b\lambda < \frac{1}{\mu}; \quad c(n) < \frac{\lambda}{\mu}(1 - C(\lambda)\mu).$$

Thus, we see that if the Eq. (5.3) satisfies the hypothesis of Theorem 11, then there exist a unique bounded solution of (5.3), $y(n)$, $n \in \mathbb{Z}$.

Similarly, if we consider the delay difference system (5.3) with

$$|g(n,\xi,\eta)| \leq \alpha(n)\omega_1(|\xi|) + \beta(n)\omega_2(|\eta|) + \gamma(n),$$

we have

$$|g(n,\xi,\eta)| \leq l(n)(\omega_1(|\xi|)+\omega_2(|\eta|)+1),$$

where $l(n)=\alpha(n)+\beta(n)+\gamma(n)$.

Assume that

(i) $l(n)\to 0$ as $|n|\to\infty$.

(ii) ω_i, $i=1,2$, are increasing functions such that there exists λ verifying

$$\|l\|_\infty \frac{\omega_1(\lambda)+\omega_2(\lambda)+1}{\lambda} < 3^{-1}. \tag{5.4}$$

Note that several cases satisfy (5.4), $\omega_1(\lambda)=\omega_2(\lambda)=\lambda^\nu$, with $\nu\geq 1$ for $\|l\|_\infty$ small enough. For $0\leq\nu<1$ Eq. (5.4) holds for λ big enough.

Then by Theorem 14, there exists a bounded solution y on $\mathbb{Z}$ satisfying $|y|\leq\lambda$.

The existence of periodic solutions of system (5.3) can be studied in a similar way.

Elaydi in [12] consider the application of difference equations to study the distribution of heat through a thin bar composed of a homogeneous material. $x_1,\ldots,x_k$ denote k equidistant points on the bar and $T_i(n)$ be the temperature at time $t_n=(\Delta)n$ at the point x_i, $i=1,\ldots,n$. Using the Newton law of Cooling the correlation of change in the temperature can be written in the compact form

$$T(n+1)=AT(n)+f(n), \tag{5.5}$$

where

$$A=\begin{pmatrix} (1-2\alpha) & \alpha & 0 & \cdots & 0\\ \alpha & (1-2\alpha) & \alpha & \cdots & 0\\ 0 & \alpha & (1-2\alpha) & \ddots & 0\\ \vdots & \vdots & \vdots & \vdots & \alpha\\ 0 & 0 & 0 & \cdots & (1-2\alpha)\end{pmatrix}, \tag{5.6}$$

and $f(n)=\alpha(b(n),0,0,\ldots,0,c(n))^T$, with α a real parameter associated the problem and $b(n)$ and $c(n)$ are sequences that depend on n and the temperature. Now, taking attention to the case $k=2$, firstly we observe that the eigenvalues of A are $\lambda_1=1-2\alpha+|\alpha|$ and $\lambda_2=1-2\alpha-|\alpha|$. In particular, if $\alpha>0$ the eigenvalues are $\lambda_1=1-\alpha$ and $\lambda_2=1-3\alpha$. Considering $\alpha=1.1$ we have $\lambda_1=-0.1$ and $\lambda_2=-2.3$, so $|\lambda_1|<1$ and $|\lambda_2|>1$. In particular the homogeneous linear system associated to (5.5) has a summable dichotomy, in particular has a exponential dichotomy. Here $f\in l^\infty$ so the results obtained in Section 3 can be applied. On the other hand, we can modify the perturbation f in order to apply our results obtained for more general perturbations as in Section 4.

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