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MONOTONE ITERATIVE METHOD TO SECOND ORDER DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS INVOLVING STIELTJES INTEGRAL BOUNDARY CONDITIONS

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ABSTRACT. We use a monotone iterative method for second order differential equations with deviating arguments and boundary conditions involving Stieltjes integrals. We establish sufficient conditions which guarantee that such problems have extremal solutions in the corresponding region bounded by lower and upper solutions. We also discuss the situation when problems of type (1.1) have coupled quasi-solutions. We illustrate our results by three examples.

Key words: Problems with deviating arguments, boundary conditions involving Stieltjes integrals, monotone iterative technique, equations and inequalities with deviating arguments, existence results

AMS (MOS) Subject Classification. 34K10

1. INTRODUCTION

Boundary value problems, using deterministic as well as stochastic approach, are of major concern in engineering applications and mathematical considerations, see for example, [12], [1], [7], [13].

In this paper we shall study boundary value problems for second order differential equations with deviating arguments of the form

$$(1.1)\qquad \begin{cases} x''(t) &= f(t,x(t),x(\alpha(t))) \equiv Fx(t), \quad t \in J=[0,T], \quad T<\infty, \\ x(0) &= \lambda_1[x], \quad x(T)=\lambda_2[x], \end{cases}$$

where $f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $\alpha \in C(J,J)$. Here $\lambda_1[x]$, $\lambda_2[x]$ denote linear functionals on $C(J)$ given by

$$\lambda_1[x] = \int_0^T x(t)dA(t), \quad \lambda_2[x] = \int_0^T x(t)dB(t)$$

involving Stieltjes integrals with suitable functions A and B of bounded variations.

Note that the boundary conditions with functionals $\lambda_1[x]$ and $\lambda_2[x]$ in problem (1.1) cover some nonlocal boundary conditions considered in some papers for second

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order differential equations: for example if $\lambda_1[x] = 0$ see for example [6], [14]; if $\lambda_1[x] = x(T)$ see for example [3], [5], [8]–[10]; if $\lambda_2[x] = \beta x(\gamma)$, see for example paper [6], if $\lambda_2[x] = \sum_{i=1}^{n-2} \beta_i x(\gamma_i)$, see for example [2].

Note that the Boundary Conditions (BCs) in problem (1.1) also cover the following boundary conditions:

$$\begin{aligned}
x(0) &= \sum_{i=1}^{r} \mu_i x(\nu_i), \quad x(T) = \sum_{j=1}^{q} \gamma_j x(\delta_j), \\
x(0) &= \sum_{i=1}^{r} \mu_i x(\nu_i), \quad x(T) = \int_0^T g(t)x(t)dt, \\
x(0) &= \int_0^T h(t)x(t)dt, \quad x(T) = \sum_{j=1}^{q} \gamma_j x(\delta_j), \\
x(0) &= \int_0^T h(t)x(t)dt, \quad x(T) = \int_0^T g(t)x(t)dt
\end{aligned}$$

with corresponding constants $\mu_i,\ \nu_i,\ \gamma_j,\ \delta_j$ and functions h and g.

It is important to indicate that similar nonlocal BCs have also been discussed to guarantee the existence of positive solutions to second order differential equations with no deviating arguments by using corresponding fixed point theorems in cones. For example, for $T = 1$, the Authors in paper [15] studied the BCs including the following:

$$\begin{aligned}
x(0) = 0, &\quad x(1) = \lambda_2[x], \\
x'(0) = 0, &\quad x(1) = \lambda_2[x], \\
x(0) = 0, &\quad x'(1) = \lambda_2[x],
\end{aligned}$$

where the linear functional λ_2 has the same form as in problem (1.1) with a signed measure dB; and in paper [16] for the BCs as in problem (1.1) involving Stieltjes integrals also with signed measures dA, dB; see also paper [4].

Motivated by [15], [16], in this paper, we discuss problems of type (1.1) giving sufficient conditions which guarantee that problem (1.1) has a solution in a corresponding region bounded by lower and upper solutions. To do it we apply a monotone iterative technique, for details see for example [11]. In Section 2, we give some important facts for linear differential inequalities and equations with deviating arguments which are needed to formulate the main results of this paper. For example, we formulate sufficient conditions under which boundary value problems for linear differential equations have a solution (see Lemma 2.4) or a unique solution (see Lemma 2.5). In Section 3, using the notion of lower and upper solutions, we discuss problem (1.1) under such conditions which guarantee that (1.1) has extremal solutions in the region bounded by lower and upper solutions (see Theorem 3.4). Because our results are obtained by using corresponding inequalities, we need to assume that the measures $dA,\ dB$ are positive. Examples 3.6 and 5.3 illustrate the results obtained. To choose lower and upper solutions for problem (1.1), some inequalities must be satisfied for F and BCs connected with functionals λ_1 and λ_2 (see the beginning of Section 3). In Section 4, we establish some relations for chosen lower and upper solutions y_0, z_0 and corresponding functionals λ_1, λ_2. In the next section, we discuss problem (1.1) when the measures dA and dB are negative. Example 5.3 illustrates the results of this part of the paper. It is important to indicate that we investigate problem (1.1) under general BCs with Stieltjes integrals. In this paper, the argument α in problem (1.1) can be both of delayed or advanced type.

2. LEMMAS

To apply the monotone iterative method to problems of type (1.1), we need a fundamental result on differential inequalities.

Lemma 2.1 ([6]). *Assume that:*

$H_1: \ \alpha \in C(J, J),\ M, N \in C(J, [0, \infty)), M(t) > 0,\ t \in (0, T),\ M(0) \geq 0,\ M(T) \geq 0,$

$H_2: \ \rho \equiv \max\left\{\int_0^T s[M(s)+N(s)]ds, \int_0^T (T-s)[M(s)+N(s)]ds\right\} < 1.$

Let $p \in C^2(J, \mathbb{R})$ *and*

$$\begin{cases} p''(t) & \geq & M(t)p(t) + N(t)p(\alpha(t)), \quad t \in J, \\ p(0) & \leq & 0, \quad p(T) \leq 0. \end{cases}$$

Then $p(t) \leq 0$ *on* J.

Remark 2.2. Let $M(t) = M > 0$, $N(t) = N \geq 0$ and

$$(M+N)T^2 \leq 2. \tag{2.1}$$

Then Assumption H_2 is satisfied. Indeed, Assumption H_2 is less restrictive than condition (2.1). Put $M(t) = Mt^2$, $N(t) = t^3$ for $J = [0, 1]$. Then $M \leq \frac{16}{5}$, by Assumption H_2, and $M \leq 1$, by condition (2.1).

Lemma 2.3. *Let Assumptions* H_1, H_2 *be satisfied. Let* $y \in C^2(J, \mathbb{R})$, $\sigma \in C(J, \mathbb{R})$ *and*

$$\begin{cases} y''(t) & = & M(t)y(t) + N(t)y(\alpha(t)) + \sigma(t), \quad t \in J, \\ y(0) & = & k_1 \in \mathbb{R}, \quad y(T) = k_2 \in \mathbb{R}. \end{cases} \tag{2.2}$$

Then problem (2.2) *has at most one solution.*

Proof. Suppose problem (2.2) has two distinct solutions $z, w \in C^2(J, \mathbb{R})$. Put $p = z - w$. Then $p(0) = p(T) = 0$ and $p''(t) = M(t)p(t) + N(t)p(\alpha(t))$ on J. In view of Lemma 2.1, $p \leq 0$, so $z(t) \leq w(t)$, $t \in J$. Now putting $p = w - z$, we have $w(t) \leq z(t)$, $t \in J$, by Lemma 2.1 too. Hence $w(t) = z(t)$, $t \in J$ and Lemma 2.3 holds. □

Lemma 2.4. *Let Assumptions H_1, H_2 hold and let $\sigma \in C(J, \mathbb{R})$. Then problem* (2.2) *has a solution $y \in C^2(J, \mathbb{R})$.*

Proof. Note that a solution of problem (2.2) is a fixed point of the operator A defined by

$$Ay(t) = \int_0^T G(t,s)P(s,y)ds + \frac{1}{T}(T-t)k_1 + \frac{t}{T}k_2,$$

with the Green function

$$G(t,s) = -\frac{1}{T}\begin{cases} (T-t)s & \text{if} \quad 0 \le s \le t \le T, \\ (T-s)t & \text{if} \quad 0 \le t \le s \le T, \end{cases}$$

and

$$P(t,y) = M(t)y(t) + N(t)y(\alpha(t)) + \sigma(t).$$

Consider the Banach space $C(J, \mathbb{R})$ with the norm $\|y\| = \max_{t \in J} |y(t)|$. Take a sequence $\{y_n\}$ such that $y_n \in C(J, \mathbb{R})$ and y_n converges to $y \in C(J, \mathbb{R})$. In view of the Lebesque dominated convergence theorem

$$\max_{t \in J} |Ay_n(t) - Ay(t)| \to 0 \quad \text{if} \quad n \to \infty.$$

It shows that A is continuous. Moreover, $|P(t,y)| \le K$, so operator $A : C(J, \mathbb{R}) \to C(J, \mathbb{R})$ is continuous and bounded.

Take $t_1, t_2 \in J$, $t_1 < t_2$ and such that $|t_1 - t_2| \le \dfrac{\epsilon}{4KT + k}$ for $\epsilon > 0$ with $k = \frac{1}{T}|k_2 - k_1|$. Then

$$\begin{aligned} &|Ay(t_1) - Ay(t_2)| \le \frac{1}{T}\left|(t_1 - t_2)\int_0^{t_1} sP(s,y)ds - t_1 \int_{t_1}^{t_2} (T-s)P(s,y)ds \right. \\ &\left. +(T - t_2)\int_{t_1}^{t_2} sP(s,y)ds + (t_2 - t_1)\int_{t_2}^{T} (T-s)P(s,y)ds \right| + k|t_1 - t_2| \\ &\le (4KT + k)|t_1 - t_2| < \epsilon. \end{aligned}$$

Consequently, operator A is compact by the Arzela-Ascoli theorem.

Then, let's put

$$B = \max_{t \in J} \int_0^T |G(t,s)\sigma(t)|ds + |k_1| + |k_2|.$$

Take

$$S = \{y \in C(J, \mathbb{R}) : \ \|y\| \le D\}$$

with $D = \frac{B}{1-\rho}$, where ρ is defined as in Assumption H_2. Let $z \in S$. Then,

$$\begin{aligned}
\|z\| &\le \frac{\|z\|}{T}\max_{t\in J}\left\{\int_0^t (T-t)s[M(s)+N(s)]ds + \int_t^T (T-s)t[M(s)+N(s)]ds\right\}+B\\
&\le \frac{\|z\|}{T}\max_{t\in J}\left\{(T-t)\max\left(\int_0^t s[M(s)+N(s)]ds, \int_t^T (T-s)[M(s)+N(s)]ds\right)\right.\\
&\left.+t\max\left(\int_0^t s[M(s)+N(s)]ds, \int_t^T (T-s)[M(s)+N(s)]ds\right)\right\}+B\\
&= \|z\|\max_{t\in J}\max\left(\int_0^t s[M(s)+N(s)]ds, \int_t^T (T-s)[M(s)+N(s)]ds\right)+B\\
&\le \rho\|z\|+B \le \frac{\rho D}{1-\rho}+B = D.
\end{aligned}$$

It shows that $A : S \to S$. Now, the Schauder fixed point theorem guarantees that operator A has a fixed point $y \in C(J, \mathbb{R})$. Indeed, $y(0) = k_1$, $y(T) = k_2$, y'' exists and $y'' \in C(J, \mathbb{R})$. Moreover, $y \in C^2(J, \mathbb{R})$ is a solution of problem (2.2). This ends the proof. □

We can also obtain the existence result for problem (2.2) under assumptions weaker than in Lemma 2.4. It concerns the next lemma.

Lemma 2.5. *Assume that* $\alpha \in C(J, J)$, $M, N, \sigma \in C(J, \mathbb{R})$ *and let*

$$\rho_1 \equiv \max\left(\int_0^T s(|M(s)|+|N(s)|)ds, \int_0^T (T-s)(|M(s)|+|N(s)|)ds\right) < 1.$$

Then problem (2.2) *has a unique solution* $y \in C^2(J, \mathbb{R})$.

Proof. Indeed, the solution of (2.2) is a fixed point of operator A defined as in the proof of Lemma 2.4. Let $x, y \in C(J, \mathbb{R})$. Then

$$\begin{aligned}
\|Ax-Ay\| &= \max_{t\in J}\left|\int_0^T G(t,s)\left(M(s)[x(s)-y(s)]+N(s)[x(\alpha(s))-y(\alpha(s))]\right)ds\right|\\
&\le \frac{1}{T}\|x-y\|\max_{t\in J}\left\{(T-t)\int_0^t s(|M(s)|+|N(s)|)ds\right.\\
&\left.+t\int_0^T (T-s)(|M(s)|+|N(s)|)ds\right\}\\
&\le \|x-y\|\max\left(\int_0^T s(|M(s)|+|N(s)|)ds, \int_0^T (T-s)(|M(s)|+N(s)|)ds\right)\\
&= \rho_1\|x-y\|.
\end{aligned}$$

This proves that operator A is contractive. Therefore, the Banach fixed point theorem gives the existence of a fixed point of A, i.e. a solution $y \in C^2(J, \mathbb{R})$ of problem (2.2). This ends the proof. □

3. MAIN RESULTS WHEN dA AND dB ARE POSITIVE MEASURES

Let's introduce the following definitions. A function $y_0 \in C^2(J, \mathbb{R})$ is a lower solution of (1.1) if

$$y_0''(t) \geq Fy_0(t), \quad t \in J, \quad y_0(0) \leq \lambda_1[y_0], \quad y_0(T) \leq \lambda_2[y_0].$$

A function $z_0 \in C^2(J, \mathbb{R})$ is an upper solution of problem (1.1) if

$$z_0''(t) \leq Fz_0(t), \quad t \in J, \quad z_0(0) \geq \lambda_1[z_0], \quad z_0(T) \geq \lambda_2[z_0].$$

Put

$$g(t, u, y) = Fu(t) + M(t)[y(t) - u(t)] + N(t)[y(\alpha(t)) - u(\alpha(t))].$$

Theorem 3.1. *Suppose that Assumptions H_1 and H_2 are satisfied. Let $u, v \in C^2(J, \mathbb{R})$ be lower and upper solutions of problem* (1.1)*, respectively, and $u(t) \leq v(t)$, $t \in J$. Moreover, assume that:*

H_3 : *$f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and*

$$f(t, \bar{u}_1, \bar{v}_1) - f(t, u_1, v_1) \geq -M(t)[u_1 - \bar{u}_1] - N(t)[v_1 - \bar{v}_1]$$

for $v(t) \geq u_1 \geq \bar{u}_1 \geq u(t)$, $v(\alpha(t)) \geq v_1 \geq \bar{v}_1 \geq u(\alpha(t))$ on J,

H_4 : *dA and dB are positive measures.*

Then:

(i) *The problem*

$$\left\{ \begin{array}{rcl} y''(t) & = & g(t, u, y), \quad t \in J, \\ y(0) & = & \lambda_1[u], \quad y(T) = \lambda_2[u], \end{array} \right. \tag{3.1}$$

has a unique solution $y \in C^2(J, \mathbb{R})$. Moreover, y is a lower solution of problem (1.1) *and $u(t) \leq y(t) \leq v(t)$, $t \in J$.*

(ii) *The problem*

$$\left\{ \begin{array}{rcl} z''(t) & = & g(t, u, z), \quad t \in J, \\ z(0) & = & \lambda_1[u], \quad z(T) = \lambda_2[u] \end{array} \right.$$

has a unique solution $z \in C^2(J, \mathbb{R})$. Moreover, z is an upper solution of problem (1.1)*) and $u(t) \leq z(t) \leq v(t)$, $t \in J$.*

(iii) *$u(t) \leq y(t) \leq z(t) \leq v(t)$, $t \in J$.*

Proof. Note that problem (3.1) has a unique solution y, by Lemmas 2.3 and 2.4 or by Lemma 2.5. Put $p(t) = u(t) - y(t)$, $t \in J$, so $p(0) \leq 0$, $p(T) \leq 0$. Moreover,

$$\begin{array}{rcl} p''(t) & \geq & Fu(t) - Fu(t) - M(t)[y(t) - u(t)] - N(t)[y(\alpha(t)) - u(\alpha(t))] \\ & = & M(t)p(t) + N(t)p(\alpha(t)). \end{array}$$

Hence $u(t) \le y(t)$, $t \in J$, by Lemma 2.1. Now we put $q(t) = y(t) - v(t)$, $t \in J$, so $q(0) \le 0$, $q(T) \le 0$. In view of Assumption H_3, we get

$$\begin{aligned} q''(t) &\ge f(t, u(t), u(\alpha(t))) - f(t, v(t), v(\alpha(t))) + M(t)[y(t) - u(t)] \\ &\quad + N(t)[y(\alpha(t)) - u(\alpha(t))] \\ &\ge -M(t)[v(t) - u(t)] - N(t)[v(\alpha(t)) - u(\alpha(t))] + M(t)[y(t) - u(t)] \\ &\quad + N(t)[y(\alpha(t)) - u(\alpha(t))] \\ &= M(t)q(t) + N(t)q(\alpha(t)). \end{aligned}$$

Hence, by Lemma 2.1, $y(t) \le v(t)$ on J. This proves that $u(t) \le y(t) \le v(t)$, $t \in J$.

Now, we need to show that y is a lower solution of problem (1.1). We see that

$$\begin{aligned} y(0) &= \lambda_1[u] - \lambda_1[y] + \lambda_1[y] \le \lambda_1[y], \\ y(T) &= \lambda_2[u] - \lambda_2[y] + \lambda_2[y] \le \lambda_2[y]. \end{aligned}$$

Moreover, in view of Assumption H_3, we obtain

$$\begin{aligned} y''(t) &= Fu(t) + M(t)[y(t) - u(t)] + N(t)[y(\alpha(t)) - u(\alpha(t))] - Fy(t) + Fy(t) \\ &\ge Fy(t) - M(t)[y(t) - u(t)] - N(t)[y(\alpha(t)) - u(\alpha(t))] + M(t)[y(t) - u(t)] \\ &\quad + N(t)[y(\alpha(t)) - u(\alpha(t))] = Fy(t). \end{aligned}$$

The above proves that y is a lower solution of problem (1.1)). This ends the proof of part (i).

The proof of part (ii) is similar to the proof of part (i) and therefore it is omitted.

We only need to prove that $y(t) \le z(t)$, $t \in J$. Put $p(t) = y(t) - z(t)$, so $p(0) = 0$, $p(T) = 0$. Using Assumption H_3, we get

$$\begin{aligned} p''(t) &= Fu(t) - Fv(t) + M(t)[y(t) - u(t) - z(t) + v(t)] \\ &\quad + N(t)[y(\alpha(t)) - u(\alpha(t)) - z(\alpha(t)) + v(\alpha(t))] \\ &\ge -M(t)[v(t) - u(t)] - N(t)[v(\alpha(t)) - u(\alpha(t))] \\ &\quad + M(t)[y(t) - u(t) - z(t) + v(t)] \\ &\quad + N(t)[y(\alpha(t)) - u(\alpha(t)) - z(\alpha(t)) + v(\alpha(t))] \\ &= M(t)p(t) + N(t)p(\alpha(t)). \end{aligned}$$

Hence, $y(t) \le z(t)$, $t \in J$, by Lemma 2.1. This ends the proof. □

Remark 3.2. Note that if f is nonincreasing with respect to the last two variables, then Assumption H_3 holds.

By a similar way we can show the following result.

Theorem 3.3. *Let Assumptions H_1–H_4 hold. Let $u, v \in C^2(J, \mathbb{R})$ be lower and upper solutions of problem* (1.1)*, respectively, and $u(t) \le v(t)$, $t \in J$.*

Then the problem

$$\begin{cases} z''(t) = g(t, v, z), \quad t \in J, \\ z(0) = \lambda_1[v], \quad z(T) = \lambda_2[v], \end{cases}$$

has a unique solution $z \in C^2(J, I\!R)$. *Moreover,* z *is an upper solution of problem* (1.1) *and* $u(t) \le z(t) \le v(t)$, $t \in J$.

Theorem 3.4. *Let Assumptions* H_1*–*H_4 *hold. Let* $y_0, z_0 \in C^2(J, I\!R)$ *be lower and upper solutions of problem* (1.1)*, respectively, and* $y_0(t) \le z_0(t)$ *on* J.

Then problem (1.1) *has, in the segment* $[y_0, z_0]$*, the minimal and maximal solutions with* $[y_0, z_0] = \{w \in C^2(J, I\!R) : y_0(t) \le w(t) \le z_0(t), t \in J\}$.

Proof. Let

$$\left\{ \begin{array}{rcl} y_n''(t) & = & g(t, y_{n-1}, y_n), \;\; t \in J, \\ y_n(0) & = & \lambda_1[y_{n-1}], \quad y_n(T) = \lambda_2[y_{n-1}] \end{array} \right. \tag{3.2}$$

$$\left\{ \begin{array}{rcl} z_n''(t) & = & g(t, z_{n-1}, z_n), \;\; t \in J, \\ z_n(0) & = & \lambda_1[z_{n-1}], \quad z_n(T) = \lambda_2[z_{n-1}] \end{array} \right. \tag{3.3}$$

for $n = 1, 2, \ldots$. Function g has been defined earlier.

Note that, for $n = 1$, problems (3.2)) and (3.3) are well defined, and

$$y_0(t) \le y_1(t) \le z_1(t) \le z_0(t), \; t \in J$$

by Theorem 3.1. Also, in view of Theorem 3.1, y_1, z_1 are lower and upper solutions of problem (1.1), respectively. By induction in n, we can prove the relation:

$$y_0(t) \le \cdots \le y_{n-1}(t) \le y_n(t) \le z_n(t) \le z_{n-1}(t) \le \cdots \le z_0(t), \; t \in J, \; n = 1, 2, \ldots.$$

It implies that $\{y_n\}, \{z_n\}$ are uniformly bounded, so

$$A_1 \le y_n(t) \le z_n(t) \le A_2, \quad t \in J, \; n = 0, 1, \ldots.$$

Indeed, y_n, z_n from (3.2) and (3.3) satisfy the integral equations

$$\left\{ \begin{array}{rcl} y_n(t) & = & \displaystyle\int_0^T G(t,s) g(s, y_{n-1}, y_n) ds + \frac{1}{T}(T-t)\lambda_1[y_{n-1}] + \frac{t}{T}\lambda_2[y_{n-1}], \\ z_n(t) & = & \displaystyle\int_0^T G(t,s) g(s, z_{n-1}, z_n) ds + \frac{1}{T}(T-t)\lambda_1[z_{n-1}] + \frac{t}{T}\lambda_2[z_{n-1}], \end{array} \right. \tag{3.4}$$

and

$$\left\{ \begin{array}{rcl} y_n(0) & = & \lambda_1[y_{n-1}], \quad y_n(T) = \lambda_2[y_{n-1}], \\ z_n(0) & = & \lambda_1[z_{n-1}], \quad z_n(T) = \lambda_2[z_{n-1}] \end{array} \right. \tag{3.5}$$

for $n = 1, 2, \ldots$. Here, G is the Green function defined earlier. Note that y_n' exists and

$$\begin{array}{rcl} y_n'(t) & = & \displaystyle\frac{1}{T}\int_0^t s g(s, y_{n-1}, y_n) ds - \frac{1}{T}\int_t^T (T-s) g(s, y_{n-1}, y_n) ds \\ & & \displaystyle -\frac{1}{T}\lambda_1[y_{n-1}] + \frac{1}{T}\lambda_2[t_{n-1}]. \end{array}$$

Indeed, $|g(t,a,b)|$ is bounded by a positive constant W on $J \times [A_1, A_2] \times [A_1, A_2]$. Take $\epsilon > 0$. For $t_1, t_2 \in J$, $|t_1 - t_2| < \frac{\epsilon}{WT}$, we have

$$\begin{aligned}
|y_n(t_1) - y_n(t_2)| &= \left| \int_{t_2}^{t_1} y_n'(\tau) d\tau \right| \\
&= \frac{1}{T} \left| \int_{t_2}^{t_1} \left[\int_0^\tau s g(s, y_{n-1}, y_n) ds - \int_\tau^T (T-s) g(s, y_{n-1}, y_n) ds \right] d\tau \right| \\
&\le WT|t_1 - t_2| < \epsilon.
\end{aligned}$$

By a similar way, we have $|z_n(t_1) - z_n(t_2)| < \epsilon$. It proves that $\{y_n\}, \{z_n\}$ are equicontinuous on J. The Arzeli-Ascoli theorem guarantees the existence of subsequences $\{y_{n_k}\}, \{z_{n_k}\}$ and functions $\bar{y}, \bar{z} \in C(J, \mathbb{R})$ with y_{n_k}, z_{n_k} converging uniformly on J to $\bar{y}$ and $\bar{z}$, respectively, if $n_k \to \infty$. However, since the sequences $\{y_n\}, \{z_n\}$ are monotonic, we conclude that the whole sequences $\{y_n\}, \{z_n\}$ converge uniformly on J to $\bar{y}$ and $\bar{z}$, respectively, if $n \to \infty$. Indeed, y_n, z_n satisfy the integral equations (3.4) and conditions (3.5) and if $n \to \infty$, then we have

$$\begin{cases}
\bar{y}(t) = \int_0^T G(t,s) F\bar{y}(s) ds + \frac{1}{T}(T-t)\lambda_1[\bar{y}] + \frac{t}{T}\lambda_2[\bar{y}], \quad t \in J, \\
\bar{z}(t) = \int_0^T G(t,s) F\bar{z}(s) ds + \frac{1}{T}(T-t)\lambda_1[\bar{z}] + \frac{t}{T}\lambda_2[\bar{z}], \quad t \in J,
\end{cases}$$

and

$$\begin{cases}
\bar{y}(0) = \lambda_1[\bar{y}], \quad \bar{y}(T) = \lambda_2[\bar{y}], \\
\bar{z}(0) = \lambda_1[\bar{z}], \quad \bar{z}(T) = \lambda_2[\bar{z}]
\end{cases}$$

because g is continuous. Finding y'', z'' from the above integral equations, we see that

$$\begin{cases}
y''(t) = F\bar{y}(t), \ t \in J, \quad \bar{y}(0) = \lambda_1[\bar{y}], \ \bar{y}(T) = \lambda_2[\bar{y}], \\
z''(t) = F\bar{z}(t), \ t \in J, \quad \bar{z}(0) = \lambda_1[\bar{z}], \ \bar{z}(T) = \lambda_2[\bar{z}],
\end{cases}$$

so $\bar{y}, \bar{z} \in C^2(J, \mathbb{R})$ are solutions of problem (1.1), and

$$y_0(t) \le \bar{y}(t) \le \bar{z}(t) \le z_0(t), \ \ t \in J.$$

We need to show now that $(\bar{y}, \bar{z})$ are extremal solutions of problem (1.1) in the segment $[y_0, z_0]$. To prove it we assume that $\tilde{y}$ is another solution of problem (1.1), and $y_{n-1}(t) \le \tilde{y}(t) \le z_{n-1}(t)$, $t \in J$ for some positive integer n. Put $p(t) = y_n(t) - \tilde{y}(t)$, $q(t) = \tilde{y}(t) - z_n(t)$, $t \in J$. Hence $p(0) \le 0$, $p(T) \le 0$, $q(0) \le 0$, $q(T) \le 0$. This and Assumption H_3 yield

$$\begin{aligned}
p''(t) &= Fy_{n-1}(t) + M(t)[y_n(t) - y_{n-1}(t)] + N(t)[y_n(\alpha(t)) - y_{n-1}(\alpha(t))] - F\tilde{y}(t) \\
&\ge -M(t)[\tilde{y}(t) - y_{n-1}(t)] - N(t)[\tilde{y}(\alpha(t)) - y_{n-1}(\alpha(t))] + M(t)[y_n(t) - y_{n-1}(t)] \\
&\quad + N(t)[y_n(\alpha(t)) - y_{n-1}(\alpha(t))] \\
&= M(t)p(t) + N(t)p(\alpha(t)), \\
q''(t) &= F\tilde{y}(t) - Fz_{n-1}(t) - M(t)[z_n(t) - z_{n-1}(t)] - N(t)[z_n(\alpha(t)) - z_{n-1}(\alpha(t))] \\
&\ge M(t)q(t) + N(t)q(\alpha(t)).
\end{aligned}$$

By Lemma 2.1, $y_n(t) \le \tilde{y}(t) \le z_n(t)$, $t \in J$. If $n \to \infty$, it yields $y_0(t) \le \bar{y}(t) \le \tilde{y}(t) \le \bar{z}(t) \le z_0(t)$, $t \in J$. It proves that $\bar{y}, \bar{z}$ are extremal solutions of problem (1.1) in the segment $[y_0, z_0]$. This ends the proof. $\square$

Example 3.5 (compare [6])**.** Let us consider the problem

$$
\begin{cases} x''(t) &= \dfrac{1}{8}\sin x(t) + \beta x(\alpha t) - \dfrac{1}{8}, \quad t \in J = [0,T], \\ x(0) &= 0, \quad x(T) = rx\left(\dfrac{1}{3}T\right), \end{cases} \tag{3.6}
$$

where $\alpha \in (0,1)$, $\beta \ge \frac{9}{8}$, and

$$
(a)\ \ 1 \le r < 1 + \frac{64}{17+72\beta}, \qquad (b)\ \ \frac{18r-18}{9-r} \le T^2 \le \frac{16}{1+8\beta}.
$$

In this case

$$
\lambda_1[x] = 0, \quad \lambda_2[x] = rx(\gamma) \quad \text{for} \quad \gamma = \frac{T}{3}.
$$

Put $y_0(t) = 0$, $z_0(t) = t^2 + 2$. We can show that y_0, z_0 are lower and upper solutions of problem (3.6). It is easy to see that Assumption H_3 holds with $M(t) = \frac{1}{8}$, $N(t) = \beta$, $t \in J$. In view of (b), we obtain

$$
\left(\frac{1}{8} + \beta\right) T^2 \le \left(\frac{1}{8} + \beta\right) \frac{16}{1+8\beta} = 2,
$$

and therefore (3.6) has, in the segment $[y_0, z_0]$, extremal solutions, by Theorem 3.4.

Example 3.6. Consider the linear problem

$$
\begin{cases} x''(t) &= Mx(t)\sin t + Nx\left(\frac{1}{2}t\right)\cos t - M(t+1)\sin t, \quad t \in J = \left[0, \frac{\pi}{2}\right], \\ x(0) &= a\displaystyle\int_0^{\frac{\pi}{2}} x(s)\cos s ds, \quad x\left(\dfrac{\pi}{2}\right) = bx\left(\dfrac{\pi}{4}\right) \end{cases} \tag{3.7}
$$

for $0 \le a \le \dfrac{2}{\pi}$, $0 \le b \le \dfrac{2(\pi+2)}{\pi+4}$, $M, N > 0$. In this case

$$
\lambda_1[x] = a\int_0^{\frac{\pi}{2}} x(s)\cos s ds, \quad \lambda_2[x] = bx(\gamma), \quad \gamma = \frac{\pi}{4}.
$$

Take $y_0(t) = 0$, $z_0(t) = t + 1$. It is easy to prove that y_0, z_0 are lower and upper solutions of problem (3.7), respectively. If we additionally assume that

$$
\max\left[M + N\left(\frac{\pi}{2} - 1\right), M\left(\frac{\pi}{2} - 1\right) + N\right] < 1,
$$

then problem (3.7) has extremal solutions in the segment $[0, t]$, by Theorem 3.4. For example, if we take $M = \frac{1}{2}$, then $N \le \frac{6-\pi}{4} \approx 0.7146$.

4. SOME COMMENTS

Note that the lower and upper solutions of problem (1.1) depend on the differential equation from (1.1) and the boundary conditions involving the Stieltjes integrals. Assume that A and B are positive measures (see Assumption H_4). Now, we make some comments connected to BCs.

1. Take $y_0(t) = a$, $z_0(t) = b$, $t \in J$ and let $a < b$. Then, for

$$\lambda_1[u] = \int_0^T u(s)dA(s), \quad \lambda_2[u] = \int_0^T u(s)dB(s),$$

the following inequalities hold

$$\begin{cases} y_0(0) \le \lambda_1[y_0], & y_0(T) \le \lambda_2[y_0], \\ z_0(0) \ge \lambda_1[z_0], & z_0(T) \ge \lambda_2[z_0], \end{cases} \tag{4.1}$$

provided that:

(i) $\quad \int_0^T dA(s) = \int_0^T dB(s) = 1$ if $0 < a < b$ or $a < b < 0$,

(ii) $\quad 0 \le \int_0^T dA(s) \le 1$, $\quad 0 \le \int_0^T dB(s) \le 1$ if $a = 0$ or $b = 0$ or $a < 0 < b$.

If we take $dA(s) = \cos s ds$, $dB(s) = \sin s ds$, $T = \frac{\pi}{2}$, then

$$\int_0^{\frac{\pi}{2}} \cos s ds = \int_0^{\frac{\pi}{2}} \sin s ds = 1,$$

so the above conditions (i) and (ii) are satisfied.

2. Let $y_0(t) = t$, $z_0(t) = e^t$, $T = \frac{\pi}{2}$ and

$$\lambda_1[u] = r_1 \int_0^{\frac{\pi}{2}} u(s)ds, \quad \lambda_2[u] = \beta u(\gamma), \;\; 0 < \gamma < \frac{\pi}{2}.$$

In this case, conditions (4.1) hold provided that

$$0 \le (e^{\frac{\pi}{2}} - 1)r_1 \le 1, \quad \frac{\pi}{2\gamma} \le \beta \le e^{\frac{\pi}{2}-\gamma}.$$

Note that as γ we can take, for example, $\gamma = \frac{\pi}{2e}$, then

$$e \le \beta \le e^{\frac{\pi(e-1)}{4}} \approx 3.86.$$

Now, if

$$\lambda_1[u] = \beta u(\gamma), \;\; 0 < \gamma < \frac{\pi}{2}, \quad \lambda_2[u] = r_2 \int_0^{\frac{\pi}{2}} u(s) \sin s ds,$$

then conditions (4.1) hold provided that

$$\frac{\pi}{2} \le r_2 \le \frac{2e^{\frac{\pi}{2}}}{e^{\frac{\pi}{2}} + 1} \approx 1.66, \quad 0 \le \beta \le e^{-\gamma}.$$

3. Let

$$\lambda_1[u] = \sum_{i=1}^{r} \beta_i u(\eta_i), \quad \lambda_2[u] = \sum_{i=1}^{q} \gamma_i u(\delta_i),$$

where $\beta_i, \gamma_j \geq 0$, $\eta_i, \delta_j \in (0,T)$ for $i = 1,2,\dots,r$, $j = 1,2,\dots,q$.

Now, we need to choose y_0 and z_0 such that

$$\left\{ \begin{array}{l} y_0(0) \leq \displaystyle\sum_{i=1}^{r} \beta_i y_0(\eta_i), \quad y_0(T) \leq \sum_{i=1}^{q} \gamma_i y_0(\delta_i), \\ z_0(0) \geq \displaystyle\sum_{i=1}^{r} \beta_i z_0(\eta_i), \quad z_0(T) \geq \sum_{i=1}^{q} \gamma_i z_0(\delta_i), \end{array} \right. \tag{4.2}$$

Assume that $y_0(t) = a$, $z_0(t) = b$, $t \in J$ and $a < b$. Then conditions (4.2) hold provided that:

$$(i) \qquad \sum_{i=1}^{r} \beta_i = \sum_{i=1}^{q} \gamma_i = 1 \qquad \text{if} \quad 0 < a < b \quad \text{or} \quad a < b < 0,$$

$$(ii) \qquad \sum_{i=1}^{r} \beta_i \leq 1, \quad \sum_{i=1}^{q} \gamma_i \leq 1 \qquad \text{if} \quad a = 0 \quad \text{or} \quad b = 0 \quad \text{or} \quad a < 0 < b.$$

Now, we take $y_0(t) = t$, $z_0(t) = t^2 + 2$, $t \in J$. Then $y_0(t) < z_0(t)$, $t \in J$ and conditions (4.2) hold if

$$T \leq \sum_{i=1}^{q} \gamma_i \delta_i, \quad 2 \geq \sum_{i=1}^{r} \beta_i(\eta_i^2 + 2), \quad T^2 + 2 \geq \sum_{i=1}^{q} \gamma_i(\delta_i^2 + 2). \tag{4.3}$$

For example, if

$$T = \frac{1}{2},\ q = 1,\ r = 2,\ 0 \leq \beta_1 \leq \frac{25}{38},\ \beta_2 = \frac{1}{4},\ \gamma_1 = 1,\ \delta_1 = \frac{1}{2},\ \eta_1 = \frac{1}{3},\ \eta_2 = \frac{2}{3}$$

then conditions (4.3) are satisfied.

5. MAIN RESULTS WHEN dA AND dB ARE NEGATIVE MEASURE

A pair of functions $y_0, z_0 \in C^2(J, \mathbb{R})$ are called weakly coupled (wc) lower and upper solutions of problem (1.1) if

$$\left\{ \begin{array}{lcl} y_0''(t) & \geq & Fy_0(t),\ t \in J, \quad y_0(0) \leq \lambda_1[z_0],\ \ y_0(T) \leq \lambda_2[z_0], \\ z_0''(t) & \leq & Fz_0(t),\ t \in J, \quad z_0(0) \geq \lambda_1[y_0],\ \ z_0(T) \geq \lambda_2[y_0]. \end{array} \right.$$

A pair (U,V), $U,V \in C^2(J,\mathbb{R})$ is called a weakly coupled quasi-solution of problem (1.1) if

$$\left\{ \begin{array}{lcl} U''(t) & = & FU(t),\ t \in J, \quad U(0) = \lambda_1[V],\ \ U(T) = \lambda_2[V], \\ V''(t) & = & FV(t),\ t \in J, \quad V(0) = \lambda_1[U],\ \ V(T) = \lambda_2[U]. \end{array} \right.$$

A weakly coupled quasi-solution $(\bar{U}, \bar{V})$, $\bar{U}, \bar{V} \in C^2(J, \mathbb{R})$ is called the weakly coupled minimal and maximal quasi-solution of problem (1.1) if for any weakly coupled quasi-solution (U,V) of (1.1) we have $\bar{U}(t) \leq U(t)$, $V(t) \leq \bar{V}(t)$ on J.

Theorem 5.1. *Suppose that Assumptions H_1, H_2 and H_3 are satisfied. Let $u, v \in C^2(J, I\!R)$ be wc lower and upper solutions of problem (1.1), and $u(t) \le v(t)$, $t \in J$. In addition, we assume that Assumption H_5 holds with*

H_5 : *dA and dB are negative measures.*

Then:

(i) *The problems*

$$\begin{cases} y''(t) &= g(t,u,y), \quad t \in J, \\ y(0) &= \lambda_1[v], \quad y(T) = \lambda_2[v], \end{cases}$$

$$\begin{cases} z''(t) &= g(t,v,z), \quad t \in J, \\ z(0) &= \lambda_1[u], \quad z(T) = \lambda_2[u], \end{cases}$$

have their unique solutions $y, z \in C^2(J, I\!R)$, respectively; y, z are wc lower and upper solutions of (1.1) and $u(t) \le y(t) \le v(t)$, $u(t) \le z(t) \le v(t)$, $t \in J$.

ii) *$u(t) \le y(t) \le z(t) \le v(t)$ on J.*

Proof. The proof of part (i) is similar to the proof of Theorem 3.1 (parts (i)–(ii)) and therefore it is omitted.

To prove part (ii) we put $q(t) = y(t) - z(t)$, so $q(0) \le 0$, $q(T) \le 0$. Moreover,

$$q''(t) = g(t,u,y) - g(t,v,z) \ge M(t)q(t) + N(t)q(\alpha(t))$$

in view of Assumption H_3. This and Lemma 2.1 prove that part (ii) holds. This ends the proof. □

Theorem 5.2. *Let Assumptions H_1, H_2, H_3, H_5 hold. Let $y_0, z_0 \in C^2(J, I\!R)$ be wc lower and upper solutions of problem (1.1) and $y_0(t) \le z_0(t)$ on J.*

Then problem (1.1) has, in the segment $[y_0, z_0]$ the wc minimal and maximal quasi-solutions.

Proof. Let

$$\begin{cases} y_n''(t) &= g(t, y_{n-1}, y_n), \quad t \in J, \\ y_n(0) &= \lambda_1[z_{n-1}], \quad y_n(T) = \lambda_2[z_{n-1}], \end{cases} \tag{5.1}$$

$$\begin{cases} z_n''(t) &= g(t, z_{n-1}, z_n), \quad t \in J, \\ z_n(0) &= \lambda_1[y_{n-1}], \quad z_n(T) = \lambda_2[y_{n-1}], \end{cases} \tag{5.2}$$

for $n = 1, 2, \ldots$. Note that, for $n = 1$, problems (5.1) and (5.2) are well defined, and

$$y_0(t) \le y_1(t) \le z_1(t) \le z_0(t), \ t \in J$$

by Theorem 5.1. Also, in view of Theorem 5.1, y_1, z_1 are wc lower and upper solutions of problem (1.1). By induction in n, we can prove relation (i). It yields that (y_n, z_n)

converge uniformly and monotonically on J to $(\bar{y}, \bar{z})$. Indeed, functions $\bar{y}$, $\bar{z}$ are wc quasi-solutions of problem (1.1), and

$$y_0(t) \le \bar{y}(t) \le \bar{z}(t) \le z_0(t), \quad t \in J.$$

We show that $(\bar{y}, \bar{z})$ is wc maximal and minimal quasi-solution of problem (1.1). Let $(\tilde{y}, \tilde{z})$ be another wc quasi-solution of (1) such that $y_0(t) \le \tilde{y}(t)$, $\tilde{z}(t) \le z_0(t)$, $t \in J$. We have to show that $\bar{y}(t) \le \tilde{y}(t)$, $\tilde{z}(t) \le \bar{z}(t)$, $t \in J$. To do this we assume that $y_{m-1}(t) \le \tilde{y}(t)$, $\tilde{z}(t) \le z_{m-1}(t)$, $t \in J$ for some positive m. Put $p(t) = y_m(t) - \tilde{y}(t)$, $q(t) = \tilde{z}(t) - z_m(t)$, $t \in J$, so $p(0) \le 0$, $p(T) \le 0$, $q(0) \le 0$, $q(T) \le 0$. Moreover, in view of Assumption H_3, we get

$$\begin{aligned} p''(t) &= g(t, y_{m-1}, y_m) - F\tilde{y}(t) \ge M(t)p(t) + N(t)p(\alpha(t)), \\ q''(t) &= F\tilde{z}(t) - g(t, z_{m-1}, z_m) \ge M(t)q(t) + N(t)q(\alpha(t)). \end{aligned}$$

Hence, by Lemma 2.1, we obtain $y_n(t) \le \tilde{y}(t)$, $\tilde{z}(t) \le z_n(t)$, for all n, by mathematical induction. Now if $n \to \infty$, this shows that $(\bar{y}, \bar{z})$ is wc maximal and minimal quasi-solution of problem (1.1). This ends the proof. $\square$

Example 5.3 (compare [6]). Now we consider the problem

$$(5.3) \quad \begin{cases} x''(t) = \beta_1(t)\sin^2 x(t) + \beta_2(t)\sin^2 x\left(\dfrac{1}{2}t\right) + \beta_3(t)x\left(\dfrac{1}{2}t\right) + h(t), \quad t \in J, \\ x(0) = 0, \quad x(T) = -x\left(\dfrac{1}{2}T\right), \end{cases}$$

where $J = [0, T]$, $h \in C(J, \mathbb{R})$, $\beta_i \in C(J, [0, \infty))$, $i = 1, 2, 3$, $\beta_1(0) \ge 0$, $\beta_1(T) \ge 0$, $\beta_1(t) > 0$ for $t \in (0, T)$,

$$\begin{aligned} &(a) \quad (\beta_1(t) + \beta_2(t))0.709 - \beta_3(t) + h(t) \le 0, \ t \in J, \\ &(b) \quad (\beta_1(t) + \beta_2(t))0.708 + \beta_3(t) + h(t) \ge 0, \ t \in J, \end{aligned}$$

and

$$(5.4) \qquad \max\left\{\int_0^T \left(\int_s^T \beta(t)dt\right) ds, \ \int_0^T \left(\int_0^s \beta(t)dt\right) ds\right\} < 1$$

for $\beta(t) = \beta_1(t) + \beta_2(t) + \beta_3(t)$, $t \in J$. In this case

$$\lambda_1[x] = 0, \quad \lambda_2[x] = -x\left(\frac{T}{2}\right).$$

Take $y_0(t) = -1$, $z_0(t) = 1$. Then $y_0(0) = -1 < 0$, $y_0(T) + z_0\left(\frac{1}{2}T\right) = 0$, $z_0(0) = 1 > 0$, $z_0(T) + y_0\left(\frac{1}{2}T\right) = 0$, and

$$\begin{aligned} Fy_0(t) &= (\beta_1(t) + \beta_2(t))\sin^2 1 - \beta_3(t) + h(t) \le 0 = y_0''(t), \\ Fz_0(t) &= (\beta_1(t) + \beta_2(t))\sin^2 1 + \beta_3(t) + h(t) \ge 0 = z_0''(t), \end{aligned}$$

by conditions (a) and (b). This shows that y_0, z_0 are wc lower and upper solutions of problem (5.3). Note that Assumption H_3 holds with $M(t) = \beta_1(t)$, $N(t) = \beta_2(t) +$

$\beta_3(t)$. This and (5.4) guarantee that (y_0, z_0) is wc minimal and maximal quasi-solution of problem (5.3), by Theorem 5.2.

For example, if we take $\beta_i(t) = \frac{1}{3}$, $t \in J$, $i = 1, 2, 3$, $h(t) = -0.3$, then conditions (a), (b) and (5.4) hold if $T \leq \sqrt{2}$. If we take $\beta_i(t) = e^{-t}$, $t \in J = [0, 0.9]$, $i = 1, 2, 3$ and $-2.416e^{-t} \leq h(t) \leq -0.418e^{-t}$, $t \in J$, then conditions $(a), (b)$ and (5.4) are satisfied.

Remark 5.4. We can also discuss a similar problem to (1.1) having more deviating arguments α.

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