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ESTIMATING DYNAMIC GEOMETRIC FRACTIONAL BROWNIAN MOTION AND ITS APPLICATION TO LONG-MEMORY OPTION PRICING

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ABSTRACT. Geometric fractional Brownian motion (GFBM) is an extended dynamic model of the traditional geometric Brownian motion, and has been used in characterizing the long term memory dynamic behavior of financial time series and in pricing long-memory options. A crucial problem in its applications is how the unknown parameters in the model are to be estimated. In this paper, we study the problem of estimating the unknown parameters, which are the drift μ, volatility σ and Hurst index H, involved in the GFBM, based on discrete-time observations. We propose a complete maximum likelihood estimation approach, which enables us not only to derive the estimators of μ and σ^2, but also the estimate of the long memory parameter, H, simultaneously, for risky assets in the dynamic fractional Black-Scholes market governed by GFBM. Simulation outcomes illustrate that our methodology is statistically efficient and reliable. Empirical application to stock exchange index with European option pricing under GFBM is also demonstrated.

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1. INTRODUCTION

The dynamic behavior of financial markets has always intrigued researchers for many decades. Many financial models have been developed, where the underlying dynamic processes are driven by Brownian motion or its extensions. Bachelier (1900)'s famous thesis, for example, considered application of the arithmetic Brownian motion (BM) to stock exchange index for the first time. Samuelson (1965) made further development on Bachelier's fundamental theory by modelling stock prices according to geometric Brownian motions (GBM) so as to better capture the real market dynamic behavior. Black and Scholes (1973) and Merton (1973) are no doubt the two most well known fundamental papers, where BM and GBM are so successfully applied in option pricing theory. In this paper, we are concerned with the estimation and application of dynamic geometric fractional Brownian motion (GFBM) in long-memory option pricing. This model covers the traditional GBM model as a special case.

GFBM is a geometric version of the Fractional Brownian motion (FBM), denoted by $B_H(t)$, which was first studied by Kolmogorov as early as in 1940. However, the

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terminology was suggested in Mandelbrot and Van Ness (1968), where its statistical properties are established and reported. Since then, many works have been carried out in a variety of fields, ranging from network traffic (see, for example, Abry *et al.*. (2000)) to dynamic system (see, for example, Ahmed and Charalambous (2002) and Misiran *et al.*. (2010)) and to economics and finance (see, for example, Mandelbrot and Van Ness (1968), Shiryaev (1999), Cajueiro and Barbachan (2005)). The main emphasis is on the dealing with self-similarity. One of the favorite problems, which is still being debated, in the literature, is to find a good method to estimate the index of self-similarity, i.e., the Hurst parameter H, which is named after the English hydrologist H.E. Hurst, who introduced the index of self-similarity when studying the Nile River in 1951.

Application of FBM and GFBM to dynamic financial option pricing is a natural way of extending the famous Black-Scholes option theory which was based on BM and GBM. Earlier works on FBM based on the pathwise integration theory showed that the mathematical models of markets based on $B_H(t)$ could have arbitrage opportunity and hence are useless in financial modelling (Rogers, 1997). This drawback has discouraged further investigation in this field for many years. Only recently have researchers worked on $B_H(t)$ using ordinary product pathwise as an alternative approach. This approach has produced results, where no arbitrage situation can occur. Consequently, these results have motivated active works, where the underlying dynamic processes of mathematical markets models are driven by $B_H(t)$. Hu and Øksendal (2003) proved that the white noise calculus based on $B_H(t)$ with $\frac{1}{2} < H < 1$, corresponding to Ito type fractional Black Sholes market, has no arbitrage and the market is complete. Elliott and van der Hoek (2003) extended the range of H to $[0, 1]$. They were concerned with option pricing with FBM taken as the driving noise process. Though there is some criticism regarding this approach[1], the option pricing under GFBM has been well developed based on this new framework, covering the Black-Scholes option pricing as a special case ($H = 0.5$). Many researchers have taken into account long-memory dynamic behavior in the option pricing. See, for example, Aldabe et. al (1998) for regularized fractional Brownian motion, Bertrand (2005) for multiscale fractional Brownian motion with European option, Elliott and Chan (2004) for valuation of perpetual American options, and Jumarie (2005) for Merton's optimal portfolio. In this paper, we follow a recent approach reported in Mishura (2008), where a rigourously derived formula for European option under dynamic fractional Black-Scholes market is given.

A crucial problem associated with the real applications of these option pricing formulae in the dynamic fractional Black-Scholes markets is how do we obtain the

[1]Hult and Bjork (2005) criticized on the meaning of self-financing in this framework, but agreed that the method used does not admit arbitrage.

unknown values of the parameters in GFBM. In particular, there are two key parameters, the volatility σ and the long memory parameter H. They play a crucially important role in valuing, say, European option (see Mishura (2008)). See Section 4 for detail. However, in the literature, to the best of our knowledge, it appears that there are very few work devoted to this problem. An exception is the paper by Kukush *et al.* (2005), who developed an incomplete maximum likelihood estimation (IMLE) of the volatility σ, while the long memory parameter H is estimated a priori independently by some estimation methods specially designed for estimating H, such as the R/S analysis, variation analysis, etc. Differently from Kukush *et al.* (2005), in this paper, we will study the problem of estimating the unknown parameters, consisting of the drift μ, volatility σ and Hurst index H in the range of $0 < H < 1$, involved in the GFBM, simultaneously, based on the discrete-time observations. We propose an approach of complete maximum likelihood estimation (CMLE), which enables us not only to derive the estimators of μ and σ^2, but also the estimate of the long memory parameter, H. Our simulation study clearly indicates that our CMLE approach is statistically efficient and reliable for the model of GFBM, while the separating method of estimating σ^2 and H by IMLE together with the widely used R/S analysis may lead to poor estimates of them. Empirical studies using stock exchange indices with long-memory option pricing under GFBM also show that our proposed method gives rise to reasonable outcomes of the European option prices. The traditional Black-Scholes formula tends to undervalue the option, while the IMLE method with R/S analysis for GFBM may lead to overvaluation of the option.

The rest of the paper is organized as follows. A brief background introduction to dynamic FBM and GFBM is given in Section 2. We propose and derive the complete maximum likelihood approach to the estimation problem of GFBM in Section 3. Section 4 presents the simulation results obtained by using the CMLE in estimating GFBM against those obtained by using the separating method. Some empirical work on pricing European call option is reported in Section 5. Finally, we make some concluding remarks in Section 6. Detailed derivation is relegated to Appendix A.

2. DYNAMIC MODEL OF GEOMETRIC FRACTIONAL BROWNIAN MOTION

2.1. **Fractional Brownian motion, $B_H(t)$.** The FBM first came to limelight in the financial world due to Mandelbrot and van Ness (1968), who generalized the traditional Brownian motion with $H = \frac{1}{2}$ to FBM with $B_H(t)$ for $0 < H < 1$. $B_H(t)$ is a self-similar Gaussian process, with index $0 < H < 1$ and stationary increments, defined on a probability space. It posses the properties that $B_H(0) = 0$, $E[B_H(t)] = 0$

for every $t \geq 0$, and its covariance is given by

$$C_H(t) = E[B_H(t)B_H(s)] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

The self-similarity means that for any $\alpha > 0$, $B_H(\alpha t)$ has the same law as $\alpha^H B_H(t)$. Clearly, when $H = \frac{1}{2}$, $B_H(t)$ reduces to a standard Brownian motion $B(t)$. For further details, the reader is referred to Biagini, Hu, Øksendal and Zhang (2008).

We need the following property on the increment of the FBM. Set

$$e_j = B_H(j+1) - B_H(j)$$

for $j \in \mathbb{Z}$, where $\mathbb{Z}$ is the set of all integers. Then the covariance of e_j can be expressed as

$$r(k) = Ee_{j+k}e_j = \frac{1}{2}(|k+1|^{2H} + |k-1|^{2H} - 2|k|^{2H}) \tag{2.1}$$

for $k \in \mathbb{Z}$. Note that when $H < \frac{1}{2}$, the increments are negatively correlated whereas $H > \frac{1}{2}$ shows the positive correlation. This increment is a stationary process, which is often referred to as fractional Gaussian noise. It is easily showed that

$$r(k) \sim H(2H-1)k^{2H-2}, \quad \text{as } k \to \infty,$$

which implies that if $H > \frac{1}{2}$ then the summation of correlations diverges, i.e. $\sum_{k=0}^{\infty} r(k) = \infty$, often referred to as long memory or long range dependence property.

2.1.1. *Geometric fractional Brownian motion.* We are concerned with dynamic fractional Black-Scholes markets, in which the dynamic risky asset price process, $S(t)$, driven by FBM is modelled by GFBM, in the form

$$dS(t) = \mu S(t)dt + \sigma S(t)dB_H(t),$$

where $S(0) = s > 0$, and μ and $\sigma > 0$ are the drift and volatility, respectively. The solution to this fractional differential equation (Hu and Øksendal, 2000) is given by

$$S(t) = s\ \exp\left\{\sigma B_H(t) + \mu t - \frac{1}{2}\sigma^2 t^{2H}\right\}. \tag{2.2}$$

The two unknown parameters, i.e., the Hurst index H and the volatility σ, in this model are particularly important in financial asset pricing (see Section 4 below). How to estimate them is what we aim at in next section.

3. METHODOLOGY OF ESTIMATION

3.1. **Model simplification.** We begin with a review of the idea of incomplete likelihood estimation given by Kukush *et al.* (2005) and then put forward a different discrete-time model of GFBM for our proposed complete likelihood estimation.

Let $X(t) = \ln(\frac{S(t)}{s})$. Then it follows from (2.2) that $X(t) = \sigma B_H(t) + \mu t - (\frac{\sigma^2}{2})t^{2H}$, $t \geq 0$. As in Kukush *et al.* (2005), we assume that the historical data are observed at discrete times

$$t_k = \frac{kT}{n}, \quad k = 0, 1, \ldots, n,$$

over the time interval $[0, T]$. By setting $X_k = X(t_k)$ and $B_{Hk} = B_H(t_k)$ and considering $k = 1, \ldots, n$, we have

$$\Delta X_k = \sigma \Delta B_{Hk} + \mu \Delta t_k - \frac{\sigma^2}{2}\Delta(t^{2H})_k, \tag{3.1}$$

where $\Delta X_k = X_k - X_{k-1}$, ΔB_{Hk} and Δt_k are defined similarly, and $\Delta(t^{2H})_k = t_k^{2H} - t_{k-1}^{2H}$.

Kukush *et al.* (2005) develop an incomplete maximum likelihood estimation (IMLE) procedure, which is briefly described as follows. First, they assume that H can be estimated in advance by some existing estimation methods, such as the R/S analysis, variation analysis, etc. Then they define $Y_k = \frac{n^H \Delta X_k}{T^H}$ and write (3.1) as

$$Y_k = \sigma \varepsilon_k + \frac{n^H \mu \Delta t_k}{T^H} - \frac{1}{2}\sigma^2 T^H n^H \Delta \tau_k^{2H} \tag{3.2}$$

for $k = 1, \ldots, n$, where $\Delta \tau_k^{2H} = (\frac{k}{n})^{2H} - (\frac{k-1}{n})^{2H}$, and $\varepsilon_k = \frac{n^H \Delta B_{Hk}}{T^H}$. Simple calculation shows that ε_k is normally distributed with $E\varepsilon_k = 0$, $E\varepsilon_k^2 = 1$ and the covariance of ε_k the same as in (2.1). Using (3.2), Kukush *et al.* (2005) then suggest an IMLE of the volatility σ, based on Y_k, with

$$\hat{\sigma}^2_{\mathrm{IMLE}} = \frac{1}{n}\sum_{k=1}^{n}(Y_k - \bar{Y})^2,$$

where $\bar{Y} = \frac{1}{n}\sum_{k=1}^{n} Y_k$. This estimation method was applied to option pricing by Cajueiro and Barbachan (2005). Note that $\hat{\sigma}^2_{\mathrm{IMLE}}$ is just the usual sample variance of Y_k. It is essentially assumed that the Y_k in the model (3.2) is stationary. However, this is obviously not true in general in the model (3.2) in view of the fact that $\Delta \tau_k^{2H}$ will depend on k if $H \neq 0.5$. Under some restrictive conditions imposed on $n = n(T)$, such as $n/T^{2H} \to \infty$ as $T \to \infty$ for $0.5 < H < 0.75$, which demands a sufficiently larger number of observations to be available over a large time interval $[0, T]$, Theorem 7.1 of Kukush *et al.* (2005, Page 88) showed that $\hat{\sigma}^2_{\mathrm{IMLE}}$ is consistent. However, this requirement of a sufficiently larger number of observations may be violated in practice, especially when the sample size n does not converge to ∞ faster than T in applications. Furthermore, a bad advance estimate of H may lead to a very poor estimate of σ^2. Therefore, it is noticed that this IMLE is not statistically efficient in general.

In this paper, we study the problem of estimating the unknown parameters, including the drift μ, volatility σ and Hurst index H, involved in the GFBM based on discrete observations. Unlike Kukush *et al.* (2005), we propose a complete maximum likelihood estimation (CMLE) approach, which enables us to estimate μ, σ^2 and H, simultaneously. We will follow an alternative approach and consider the returns series $Z_k = \Delta X_k$, rather than Y_k, as follows (following from (3.2)):

$$\begin{aligned} Z_k = \Delta X_k &= (\frac{T}{n})^H Y_k \\ &= (\frac{T}{n})^H \sigma \varepsilon_k + \mu \frac{T}{n} - \frac{1}{2}\left\{(\frac{T}{n})^H \sigma\right\}^2 n^{2H} \Delta \tau_k^{2H} \\ &\equiv \sigma_1 \varepsilon_k + \mu_1 - \frac{1}{2}\sigma_1^2 n^{2H} \Delta \tau_k^{2H}, \end{aligned} \tag{3.3}$$

where $\sigma_1 = (\frac{T}{n})^H \sigma$ and $\mu_1 = \frac{\mu T}{n}$. We construct our complete maximum likelihood estimation based on (3.3).

3.2. **Complete maximum likelihood estimation.** In this subsection, we are concerned with the estimation of $\vartheta = (\sigma^2, \mu, H)'$ by using the method of CMLE through the likelihood function of $\theta = (\sigma_1^2, \mu_1, H)'$. Here A' stands for the transpose of a vector or matrix A.

3.2.1. *Likelihood function of* $\boldsymbol{\theta} = (\boldsymbol{\sigma_1^2}, \boldsymbol{\mu_1}, \boldsymbol{H})'$. The following theorem provides the complete likelihood function of $\boldsymbol{\theta}$.

Theorem 3.1. *Suppose our observations are* $Z = (Z_1, \ldots, Z_n)'$ *based on* (3.3)*. Then the CMLE of* $\theta = (\sigma_1^2, \mu_1, H)'$ *is* $\hat\theta = (\hat\sigma_1^2, \hat\mu_1, \hat H)'$ *that maximizes the complete logarithmic likelihood function as follows*

$$\ell_n(\theta) = -\frac{1}{2}(n \log \sigma_1^2 + \log|\Sigma_0|) - \frac{1}{2\sigma_1^2}(Z - \mu_1 \boldsymbol{1} + \frac{1}{2}\sigma_1^2 \mathbf{x}_H)'\Sigma_0^{-1}(Z - \mu_1 \boldsymbol{1} + \frac{1}{2}\sigma_1^2 \mathbf{x}_H),$$

and the estimators of σ^2 *and* μ *are* $\left(\frac{n}{T}\right)^{2\hat H}\widehat{\sigma}_1^2$ *and* $\frac{n}{T}\hat\mu_1$ *respectively, where* $\boldsymbol{1}$ *is a* n*-dimensional vector of components* 1*'s,* $\Sigma_0 = \Sigma_0(H) = (\gamma_{ij})_{n\times n}$ *is given by*

$$\gamma_{ij} = \gamma_{ij}(H) = E\varepsilon_i\varepsilon_j = \frac{1}{2}(|i-j+1|^{2H} - 2|i-j|^{2H} + |i-j-1|^{2H}),$$

and $\mathbf{x}_H = n^{2H}(\Delta\tau_1^{2H}, \ldots, \Delta\tau_n^{2H})'$.

Proof. Based on (3.3), our observations are $Z = (Z_1, \ldots, Z_n)'$, and set $\boldsymbol{\varepsilon} = (\varepsilon_1, \cdots, \varepsilon_n)'$. Then the vector form of (3.3) is as follows

$$Z = \sigma_1 \boldsymbol{\varepsilon} + \mu_1 \mathbf{1} - \frac{1}{2}\sigma_1^2 \mathbf{x}_H. \tag{3.4}$$

Set

$$\Sigma = \mathrm{Var}(Z) = \sigma_1^2(E\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}') = \sigma_1^2\Sigma_0.$$

Since the process is Gaussian, it follows from (3.4) that the log likelihood for Z (c.f., Hamilton, 1994, Chapter 5) is

$$\ell_n(\theta) = -\frac{1}{2}\log|\Sigma| - \frac{1}{2}(Z - \mu_1 \mathbf{1} + \frac{1}{2}\sigma_1^2 \mathbf{x}_H)'\Sigma^{-1}(Z - \mu_1 \mathbf{1} + \frac{1}{2}\sigma_1^2 \mathbf{x}_H)$$

$$= -\frac{1}{2}(n\log\sigma_1^2 + \log|\Sigma_0|) - \frac{1}{2\sigma_1^2}(Z - \mu_1\mathbf{1} + \frac{1}{2}\sigma_1^2\mathbf{x}_H)'\Sigma_0^{-1}(Z - \mu_1\mathbf{1} + \frac{1}{2}\sigma_1^2\mathbf{x}_H). \tag{3.5}$$

Therefore the CMLE of $\theta = (\sigma_1^2, \mu_1, H)'$ is

$$\hat{\theta} = (\hat{\sigma}_1^2, \hat{\mu}_1, \hat{H})' = \arg\max_{\theta\in\Theta}\ \ell_n(\theta), \tag{3.6}$$

where Θ is a compact subset of $\mathbb{R}^+ \times \mathbb{R} \times (0,1)$, which contains the actual parameter vector $\theta_0 = (\sigma_{10}^2, \mu_{10}, H_0)'$.

We finally obtain the estimators of σ^2 and μ as follows:

$$\widehat{\sigma}^2 = \left(\frac{n}{T}\right)^{2\hat{H}}\widehat{\sigma}_1^2, \quad \hat{\mu} = \frac{n}{T}\hat{\mu}_1. \tag{3.7}$$

Hence the proof is finished. □

3.2.2. *Profile likelihood algorithm.* How to calculate $\hat{\theta}$ in (3.6)? Directly maximizing (3.5) is obviously involved. We here suggest a profile likelihood method to simplify the calculation in applications.

Given H, we can derive the maximum likelihood estimators for σ_1^2 and μ_1 by maximizing (3.5) with respect to σ_1^2 and μ_1. They are achieved by setting the first order partial derivatives of $\ell_n(\theta)$ with respect to σ_1^2 and μ_1 equal to 0, giving

Theorem 3.2. *The profile complete maximum likelihood estimators of σ_1^2 and μ_1 given H are as follows:*

$$\widehat{\sigma}_1^2(H) = \frac{2Z'\Sigma_1 Z}{\sqrt{n^2 + \mathbf{x}_H'\Sigma_1\mathbf{x}_H Z'\Sigma_1 Z} + n} \tag{3.8}$$

and

$$\hat{\mu}_1(H) = \frac{1}{\boldsymbol{1}'\Sigma_0^{-1}\boldsymbol{1}}(\boldsymbol{1}'\Sigma_0^{-1}Z + \frac{1}{2}\widehat{\sigma}_1^2\boldsymbol{1}'\Sigma_0^{-1}\mathbf{x}_H), \tag{3.9}$$

where

$$\Sigma_1 = \Sigma_1(H) = \Sigma_0^{-1}\left(\boldsymbol{I} - \frac{\boldsymbol{1}\boldsymbol{1}'\Sigma_0^{-1}}{\boldsymbol{1}'\Sigma_0^{-1}\boldsymbol{1}}\right)$$

with $\boldsymbol{I}$ being an $n\times n$ identity matrix.

The derivation of this theorem is deferred to Appendix A for details.

Now in order to estimate H, we replace σ_1^2 and μ_1 in (3.5) by (3.8) and (3.9), respectively, which leads to

$$\begin{aligned}\ell_{1n}(H) &= \ell(\widehat{\sigma}_1^2(H), \hat{\mu}_1(H), H)\\ &= -\frac{1}{2}\{n\log\widehat{\sigma}_1^2(H) + \log|\Sigma_0|\}\\ &\quad - \frac{1}{2\widehat{\sigma}_1^2(H)}\{Z - \hat{\mu}_1(H)\mathbf{1} + \frac{1}{2}\widehat{\sigma}_1^2(H)\mathbf{x}_H\}'\Sigma_0^{-1}\{Z - \hat{\mu}_1(H)\mathbf{1} + \frac{1}{2}\widehat{\sigma}_1^2(H)\mathbf{x}_H\}.\end{aligned} \tag{3.10}$$

This is an involved function of H. Taking the differentiation of $\ell_{1n}(H)$ with respect to H is difficult. However, note that $\ell_{1n}(H)$ is a univariate profile likelihood function of H. Many numerical methods can be used to maximise $\ell_{1n}(H)$ without appealing to differentiation, for example, the golden section search. In this way we easily obtain the estimator $\hat{H}$.

Put together, we suggest our algorithm as follows:

1) Maximize (3.10) numerically to get the estimator of H, $\hat{H}$.
2) Calculate the estimators, $\hat{\sigma}_1^2$ and $\hat{\mu}_1$, by replacing H by $\hat{H}$ in (3.8) and (3.9), respectively.
3) Compute the estimators of σ^2 and μ by (3.7).

We will demonstrate by simulation in next section that the above algorithm works pretty well in calculation and the CMLE is much more statistically efficient than the IMLE.

4. SIMULATION STUDY

In order to examine the performance of the proposed estimators, we did some simulation experiments. Let us first describe how the data is generated. We first consider the model (2.2). As in the last section, we take $t_k = \frac{kT}{n}$. Note that $B_H(t_k)$ has the property of Gaussian distribution with $EB_H(t_k) = 0$ and $E(B_H^2(t_k)) = t_k^{2H} = (\frac{kT}{n})^{2H}$. With this property, equation (2.2) becomes

$$S_k = S(t_k) = s \exp\left[\sigma\left(\frac{T}{n}\right)^H B_H(k) + \mu\left(\frac{kT}{n}\right) - \frac{1}{2}\sigma^2\left(\frac{kT}{n}\right)^{2H}\right].$$

We take the values of the parameters $\mu = 0.2752908$, $\sigma^2 = 0.2554078$, $H = 0.549$ and the initial value of $s = 903.84$. We simulate the time series from this discrete time model and apply our methodology to estimate the parameters $\vartheta = (\sigma^2, \mu, H)$ using the simulated data set. The simulation is repeated one hundred times.

To have an idea on the performance of the estimators suggested by Kukush *et al.* (2005), we also consider the estimation method by Kukush *et al.* as a comparison. No doubt, R/S analysis of Hurst (1951) and Mandelbrot (1972, 1975) is the most widely used method for the estimation of Hurst index in the literature; see also Mandelbrot and Taqqu (1979) and Mandelbrot and Wallis (1968, 1969a-1969c). We apply the Hurst value obtained from R/S analysis in Kukush *et al.*'s method. The simulated outcomes of the average value of estimates based on 100 replications, with bias and variance, are reported in Tables 1-4, for $T = 15$, $T = 30$, $T = 40$ and $T = 50$, respectively. The 5 cases of sample sizes $n = 100, 200, 300, 400, 500$ are considered in each table, where

- $\hat{H}_{CMLE}$ = Hurst index obtained by using the method proposed in this paper;
- $\hat{\mu}_{CMLE} = \mu$ obtained by the method proposed in this paper;
- $\hat{\sigma}^2_{CMLE} = \sigma^2$ obtained by the method proposed in this paper;

- $\hat{H}_{RS}$ = Hurst index obtained by using the method of R/S analysis;
- $\hat{\sigma}^2_{IMLE} = \sigma^2$ obtained by the method of Kukush et. al (2005) with $\hat{H}_{RS}$;
- To compare our proposed $\hat{\sigma}^2_{CMLE}$ with Kukush et. al (2005)'s $\hat{\sigma}^2_{IMLE}$ in terms of statistical efficiency, we also calculated at the end of each table the efficiency of $\hat{\sigma}^2_{IMLE}$ against $\hat{\sigma}^2_{CMLE}$, $\mathrm{Eff}_{\hat{\sigma}^2_{IMLE}:\hat{\sigma}^2_{CMLE}}$, which is defined as the ratio of the simulated variance of $\hat{\sigma}^2_{CMLE}$ to that of $\hat{\sigma}^2_{IMLE}$. Clearly, if $\mathrm{Eff}_{\hat{\sigma}^2_{IMLE}:\hat{\sigma}^2_{CMLE}} < 1$, then our proposed $\hat{\sigma}^2_{CMLE}$ is more statistically efficient than the $\hat{\sigma}^2_{IMLE}$.

It follows from the results listed in Tables 1–4 that our method performs considerably better. The biases and variances obtained by using our method are in an acceptable tolerance. All of our estimates for H are obviously quite stable and less biased. The performance on our proposed estimation of σ^2 is fairly satisfactory, with $\hat{\sigma}^2_{CMLE}$ much more statistically efficient than Kukush et. al (2005)'s $\hat{\sigma}^2_{IMLE}$. In fact, notice in Tables 1–4 that the statistical efficiency of $\hat{\sigma}^2_{IMLE}$ in comparison with $\hat{\sigma}^2_{CMLE}$ is very low with $\mathrm{Eff}_{\hat{\sigma}^2_{IMLE}:\hat{\sigma}^2_{CMLE}} < 0.2$, which means that the variance of our proposed estimator $\hat{\sigma}^2_{CMLE}$ is less than 20% of the variance of Kukush et. al (2005)'s $\hat{\sigma}^2_{IMLE}$ in all the simulation experiments. In particular, as $n = 500$, $\mathrm{Eff}_{\hat{\sigma}^2_{IMLE}:\hat{\sigma}^2_{CMLE}} < 0.065$ in all 4 Tables, and in Tables 3 and 4 $\hat{\sigma}^2_{IMLE}$ becomes much worse as n becomes larger by noting $\mathrm{Eff}_{\hat{\sigma}^2_{IMLE}:\hat{\sigma}^2_{CMLE}}$ approximately equal to 18% as $n = 100$ with $T = 40$ and $T = 50$, respectively. However, for our estimation method, we can clearly see that the larger the sample size n, the better the estimation performs in general; further, overall, with larger T, the outcomes become better for fixed n.

We can also find that the bias by the incomplete likelihood method of Kukush *et al.* is quite large in comparison to ours. We are able to give estimates not only for σ^2, but also for μ and Hurst exponent, H. To sum up, the simulation outcomes indicate that our method is outstanding in obtaining more statistically efficient estimators for GFBM.

5. **EMPIRICAL APPLICATION TO LONG-MEMORY OPTION PRICING**

We now illustrate the finding using empirical data.

5.1. **Data.**

We used a data set from Kuala Lumpur Composite Index (KLCI) available online at `http://www.econstats.com`. The daily close price data set of KLCI from 3rd January 2005 to 29 December 2006 is examined, with 494 observations. The return series is calculated in logarithm. The figures of the price and return series are presented in Figures 1 and 2. A summary of the return series can be found in Table 5, where the mean of this series is 0.0003915 and the variance is 0.00002584.

5.2. **Estimation based on CMLE method.** We present in this subsection the result of our study of modelling the data of KLCI by GFBM. We try to estimate the parameters of the risky assets model using our proposed complete maximum likelihood estimation method based on daily return series. The estimates are summarized in Table 6.

We can clearly see from Table 6 that the suggested estimates are $H = 0.575$, $\sigma^2 = 0.00002576$ and $\mu = 0.0004510$. This finding agrees with the work by Sadique and Silvapulle (2001), where the presence of weak long memory in Malaysia financial data is reported.

5.3. **Application to European Option Pricing.** It is interesting to note that there has been an active research in pricing the option by using the fractional Black Scholes equations recently in the literature. Elliot and Chan (2004) provides solution to pricing the perpetual American option by considering the log return stock series driven by fractional Brownian motion. In this subsection, we consider the European option pricing. Mishura (2008) showed that the price at time $t_0 \in [0, T_0]$ of a European call option with strike price K and maturity T_0 is given by

$$
\begin{aligned}
C(t_0, S) = S\Phi &\left(\frac{\ln \frac{S}{K} + r(T_0 - t_0) + (T_0^{2H} - t_0^{2H})\frac{\sigma^2}{2}}{\sigma\sqrt{T_0^{2H} - t_0^{2H}}} \right) \\
&- Ke^{-r(T_0 - t_0)}\Phi\left(\frac{\ln \frac{S}{K} + r(T_0 - t_0) - (T_0^{2H} - t_0^{2H})\frac{\sigma^2}{2}}{\sigma\sqrt{T_0^{2H} - t_0^{2H}}} \right),
\end{aligned}
$$

where S is the underlying stock price at time t_0, r is the risk free interest rate and $\Phi(\cdot)$ is the cumulative function of standard normal distribution. Note that it coincides with the solution of the usual Black-Scholes option pricing if $H = \frac{1}{2}$.

By using this formula, we can calculate the appropriate value of European call option. We consider several maturity times for an already traded option as we take $t_0 = 47$ days. The risk-free interest rate is fixed at 3.5% *pa* in regards to the actual Malaysian conventional interest rate on December 29th, 2006, and we are interested in the daily interest rate in this paper. We select the underlying price at time t_0 as MYR1096.24, following the price on December 29th, 2006. The volatility and Hurst exponent are estimated based on our method from the historical daily returns data of KLCI, with estimates listed in Table 6. For comparison, we also calculate the value of European call option using the estimates based on the method of Kukush *et al.* (2005) with R/S analysis and the traditional Black Scholes European option price. The outcomes are listed in Table 7.

From Table 7, we see that all cases exhibit somewhat differently in the call prices. Call prices valued by the traditional Black Scholes provide us with the least values, where the long memory is not taken into account. Method proposed in this paper prices the call in an intermediate value between those obtained by the traditional Black Scholes and the method by Kukush *et al.* with R/S analysis. Call prices valued by Kukush *et al.* with R/S analysis are the highest. Our method is based on rigorous theoretical reasoning (see the results in the previous sections). It gives rise to practically acceptable results, where the long-memory is taken into account. It is seen that the longer the time to expiry, the higher the value of call price becomes. In the case of "in the money", the call price reveals higher value when compared with the case of "out of the money", as expected.

6. CONCLUSION

Application of dynamic fractional Brownian motion in financial environment has been a subject of debate by researchers in this field for a number of years since its first appearance in the early 1960s. Extensive works developed in recent years in stochastic integration of this process provide useful tools for its applications to finance, mostly to the problems of option pricing. In this paper, we have proposed a CMLE method and investigated the performance of our method for the dynamic geometric fractional Brownian motion in financial modelling. We have also compared the performance of our method with the previous works in the literature.

From the simulation study, we observed that our method performed significantly better when compared with the previous methods. We also showed that by using this method, we can obtain good estimates of all the parameters involved in the geometric fractional Brownian motion. These parameters are important in modelling the fractional Black Scholes markets. With the values of these parameters obtained, we are able to price the long-memory European option using the fractional Black Scholes models.

Based on the findings in this paper, we conclude that dynamic geometric fractional Brownian motion is a good and promising tool for better understanding on how the financial markets actually behave.

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8. APPENDICES

8.1. **Appendix A: Proof of Theorem 3.2. Derivation of profile complete maximum likelihood estimation.** We will now derive the estimators for μ_1 and σ_1^2 in regards to the log likelihood obtained in (3.5). The partial derivatives of $\ell_n(\theta)$ with respect to μ_1 and σ_1^2 are given by

$$\begin{aligned}\frac{\partial \ell_n}{\partial \mu_1} &= -\frac{1}{2\sigma_1^2}\{(Z-\mu_1\mathbf{1}+\frac{1}{2}\sigma_1^2\mathbf{x}_H)'\Sigma_0^{-1}(-\mathbf{1})+(-\mathbf{1})'\Sigma_0^{-1}(Z-\mu_1\mathbf{1}+\frac{1}{2}\sigma_1^2\mathbf{x}_H)\}\\ &= \frac{1}{\sigma_1^2}\mathbf{1}'\Sigma_0^{-1}(Z-\mu_1\mathbf{1}+\frac{1}{2}\sigma_1^2\mathbf{x}_H),\end{aligned}$$

$$\begin{aligned}\frac{\partial \ell_n}{\partial \sigma_1^2} = -\frac{n}{2\sigma_1^2} - \frac{1}{2}\frac{1}{(\sigma_1^2)^2}\{\sigma_1^2\{(Z-\mu_1\mathbf{1}+\frac{1}{2}\sigma_1^2\mathbf{x}_H)'\Sigma_0^{-1}\frac{1}{2}\mathbf{x}_H+\frac{1}{2}\mathbf{x}_H'\Sigma_0^{-1}(Z-\mu_1\mathbf{1}+\frac{1}{2}\sigma_1^2\mathbf{x}_H)\}\\ -\{(Z-\mu_1\mathbf{1}+\frac{1}{2}\sigma_1^2\mathbf{x}_H)'\Sigma_0^{-1}(Z-\mu_1\mathbf{1}+\frac{1}{2}\sigma_1^2\mathbf{x}_H)\}\}.\end{aligned}$$

Setting them equal to zero yields

$$\mu_1 = \frac{1}{\mathbf{1}'\Sigma_0^{-1}\mathbf{1}}(\mathbf{1}'\Sigma_0^{-1}Z+\frac{1}{2}\sigma_1^2\mathbf{1}'\Sigma_0^{-1}\mathbf{x}_H), \tag{8.1}$$

and

$$\begin{aligned}
\sigma_1^2 &= \frac{1}{n}(Z - \mu_1\mathbf{1} - \frac{1}{2}\sigma_1^2\mathbf{x}_H)'\Sigma_0^{-1}(Z - \mu_1\mathbf{1} + \frac{1}{2}\sigma_1^2\mathbf{x}_H) \\
&= \frac{1}{n}(Z - \frac{1}{2}\sigma_1^2\mathbf{x}_H)'\Sigma_0^{-1}(Z - (\frac{\mathbf{1}'\Sigma_0^{-1}Z + \frac{1}{2}\sigma_1^2\mathbf{1}'\Sigma_0^{-1}\mathbf{x}_H}{\mathbf{1}'\Sigma_0^{-1}\mathbf{1}})\mathbf{1} + \frac{1}{2}\sigma_1^2\mathbf{x}_H) \\
&= \frac{1}{n}(Z - \frac{1}{2}\sigma_1^2\mathbf{x}_H)'\Sigma_0^{-1}(\mathbf{I} - (\frac{\mathbf{1}\mathbf{1}'\Sigma_0^{-1}}{\mathbf{1}'\Sigma_0^{-1}\mathbf{1}}))(Z + \frac{1}{2}\sigma_1^2\mathbf{x}_H),
\end{aligned}$$

respectively. By substituting $\Sigma_1 = \Sigma_0^{-1}(\mathbf{I} - (\frac{\mathbf{1}\mathbf{1}'\Sigma_0^{-1}}{\mathbf{1}'\Sigma_0^{-1}\mathbf{1}}))$, the above equation is simplified to

$$\begin{aligned}
\sigma_1^2 &= \frac{1}{n}(Z - \frac{1}{2}\sigma_1^2\mathbf{x}_H)'\Sigma_1(Z + \frac{1}{2}\sigma_1^2\mathbf{x}_H) \\
&= \frac{1}{n}Z'\Sigma_1 Z - \sigma_1^2(\frac{\mathbf{x}_H'\Sigma_1 Z - Z'\Sigma_1\mathbf{x}_H}{2n}) - \sigma_1^4(\frac{\mathbf{x}_H'\Sigma_1\mathbf{x}_H}{4n}).
\end{aligned}$$

Notice from this equality that the solution for σ^2 can be obtain by solving the quadratic equation

$$\sigma_1^4(\frac{1}{4n}\mathbf{x}_H'\Sigma_1\mathbf{x}_H) + \sigma_1^2 - \frac{1}{n}Z'\Sigma_1 Z = 0,$$

which, when $\mathbf{x}_H'\Sigma_1\mathbf{x}_H \neq 0$, gives

$$\begin{aligned}
\widehat{\sigma}_1^2 &= \frac{\sqrt{1 + \frac{1}{n^2}\mathbf{x}_H'\Sigma_1\mathbf{x}_H Z'\Sigma_1 Z} - 1}{\frac{1}{2n}\mathbf{x}_H'\Sigma_1\mathbf{x}_H} \\
&= \frac{1 + \frac{1}{n^2}\mathbf{x}_H'\Sigma_1\mathbf{x}_H Z'\Sigma_1 Z - 1}{\frac{1}{2n}\mathbf{x}_H'\Sigma_1\mathbf{x}_H(\sqrt{1 + \frac{1}{n^2}\mathbf{x}_H'\Sigma_1\mathbf{x}_H Z'\Sigma_1 Z} + 1)} \\
&= \frac{2Z'\Sigma_1 Z}{\sqrt{n^2 + \mathbf{x}_H'\Sigma_1\mathbf{x}_H Z'\Sigma_1 Z} + n}. \qquad (8.2)
\end{aligned}$$

Note that when $\mathbf{x}_H'\Sigma_1\mathbf{x}_H = 0$, it is obvious that the final equality of (8.2) can still be applied. Therefore, (3.8) and (3.9) in Subsection 3.2 follow from (8.2) and (8.1), respectively.

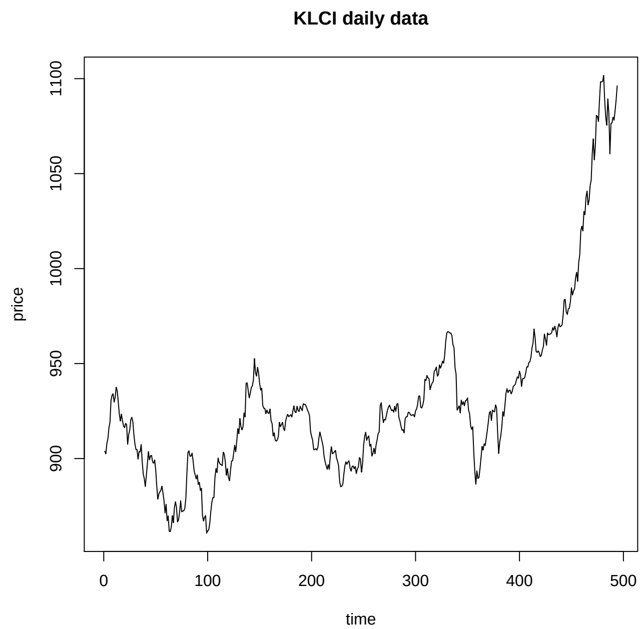

FIGURE 1. Daily close price series of KLCI from 3rd January 2005 to 29 December 2006

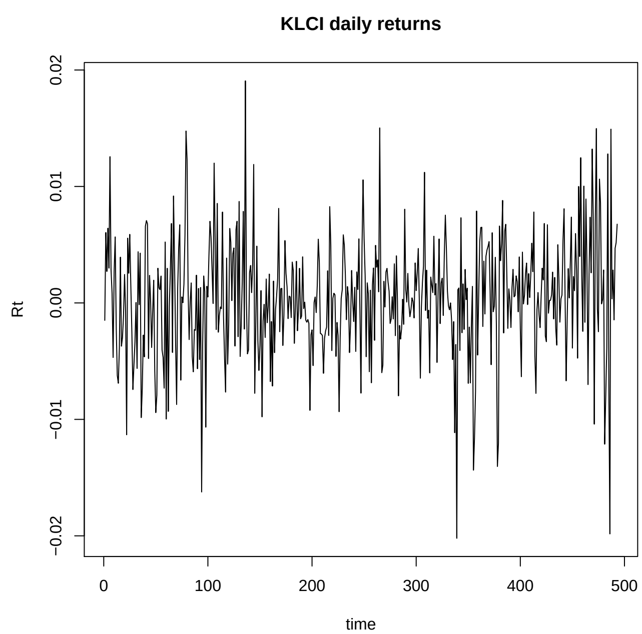

FIGURE 2. Daily return series of KLCI from 3rd January 2005 to 29 December 2006

TABLE 1. Outcome of simulation with $T = 15$: average value of estimates based on 100 replications, with bias in ( ) and variance in [ ]

n	100	200	300	400	500
$\hat{H}_{CMLE}$	0.5378 (−0.0112) [0.0019]	0.5395 (−0.0095) [0.0011]	0.5446 (−0.0044) [0.0012]	0.5438 (−0.0052) [0.0008]	0.5409 (−0.0081) [0.0009]
$\hat{\mu}_{CMLE}$	0.2590 (−0.0163) [0.0321]	0.2593 (−0.0160) [0.1022]	0.3199 (0.0446) [0.0270]	0.2512 (−0.0240) [0.0254]	0.2697 (−0.0056) [0.0159]
$\hat{\sigma}^2_{CMLE}$	0.2439 (−0.0115) [0.0043]	0.2424 (−0.0131) [0.0025]	0.2520 (−0.0034) [0.0037]	0.2500 (−0.0054) [0.0029]	0.2448 (−0.0106) [0.0034]
$\hat{H}_{RS}$	0.6575 (0.1085) [0.0318]	0.6275 (0.0785) [0.0200]	0.6099 (0.0609) [0.0141]	0.6326 (0.0836) [0.0113]	0.5969 (0.0479) [0.0057]
$\hat{\sigma}^2_{IMLE}$	0.4739 (0.2185) [0.1158]	0.5127 (0.2573) [0.2747]	0.4748 (0.2194) [0.1925]	0.5717 (0.3163) [0.2226]	0.4060 (0.1506) [0.0524]
$\mathrm{Eff}_{\hat{\sigma}^2_{IMLE}:\hat{\sigma}^2_{CMLE}}$	0.0371	0.0091	0.0192	0.0130	0.0649

TABLE 2. Outcome of simulation with $T = 30$: average value of estimates based on 100 replications, with bias in ( ) and variance in [ ]

n	100	200	300	400	500
$\hat{H}_{CMLE}$	0.5392 (−0.0098) [0.0017]	0.5374 (−0.0116) [0.0010]	0.5448 (−0.0042) [0.0012]	0.5425 (−0.0065) [0.0009]	0.5454 (−0.0036) [0.0008]
$\hat{\mu}_{CMLE}$	0.2398 (−0.0355) [0.0194]	0.2788 (0.0035) [0.0172]	0.2475 (−0.0278) [0.0152]	0.2691 (−0.0062) [0.0163]	0.2841 (0.0088) [0.0184]
$\hat{\sigma}^2_{CMLE}$	0.2457 (−0.0097) [0.0019]	0.2457 (−0.0097) [0.0018]	0.2520 (−0.0034) [0.0024]	0.2527 (−0.0027) [0.0019]	0.2538 (−0.0016) [0.0021]
$\hat{H}_{RS}$	0.6165 (0.0675) [0.0291]	0.6302 (0.0812) [0.0165]	0.6175 (0.0685) [0.0104]	0.5950 (0.0460) [0.0113]	0.6155 (0.0665) [0.0073]
$\hat{\sigma}^2_{IMLE}$	0.3230 (0.0676) [0.0232]	0.3902 (0.1348) [0.0389]	0.3937 (0.1383) [0.0573]	0.3806 (0.1252) [0.0481]	0.4177 (0.1623) [0.0498]
$\mathrm{Eff}_{\hat{\sigma}^2_{IMLE}:\hat{\sigma}^2_{CMLE}}$	0.0819	0.0463	0.0419	0.0395	0.0422

Table 3. Outcome of simulation with $T = 40$: average value of estimates based on 100 replications, with bias in ( ) and variance in [ ]

n	100	200	300	400	500
$\hat{H}_{MLE}$	0.539 (−0.0100) [0.0019]	0.5363 (−0.0127) [0.0011]	0.5409 (−0.0081) [0.0012]	0.5455 (−0.0035) [0.0007]	0.5433 (−0.0057) [0.0007]
$\hat{\mu}_{CMLE}$	0.2853 (0.0100) [0.0869]	0.2662 (−0.0091) [0.0130]	0.2841 (0.0089) [0.0153]	0.3083 (0.0330) [0.0155]	0.2799 (0.0046) [0.0133]
$\hat{\sigma}^2_{CMLE}$	0.2504 (−0.0050) [0.0021]	0.2512 (−0.0042) [0.0016]	0.2477 (−0.0077) [0.0016]	0.2510 (−0.0044) [0.0015]	0.2487 (−0.0067) [0.0016]
$\hat{H}_{RS}$	0.6113 (0.0623) [0.0289]	0.6258 (0.0768) [0.0226]	0.6275 (0.0785) [0.0139]	0.6282 (0.0792) [0.0091]	0.6095 (0.0605) [0.0095]
$\hat{\sigma}^2_{IMLE}$	0.3011 (0.0457) [0.0115]	0.3752 (0.1198) [0.0409]	0.3883 (0.1328) [0.0394]	0.4052 (0.1498) [0.0420]	0.3945 (0.1390) [0.0628]
$\mathrm{Eff}_{\hat{\sigma}^2_{IMLE}:\hat{\sigma}^2_{CMLE}}$	0.1826	0.0391	0.0406	0.0357	0.0255

Table 4. Outcome of simulation with $T = 50$: average value of estimates based on 100 replications, with bias in ( ) and variance in [ ]

n	100	200	300	400	500
$\hat{H}_{MLE}$	0.541 (−0.0080) [0.0018]	0.54 (−0.0090) [0.0014]	0.5423 (−0.0067) [0.0012]	0.5428 (−0.0062) [0.0008]	0.5438 (−0.0052) [0.0008]
$\hat{\mu}_{CMLE}$	0.2907 (0.0154) [0.0714]	0.2675 (−0.0078) [0.0150]	0.2867 (0.0114) [0.0141]	0.2690 (−0.0062) [0.0101]	0.2644 (−0.0109) [0.0113]
$\hat{\sigma}^2_{CMLE}$	0.2469 (−0.0085) [0.0013]	0.2478 (−0.0076) [0.0014]	0.2508 (−0.0046) [0.0015]	0.2520 (−0.0034) [0.0012]	0.2526 (−0.0028) [0.0018]
$\hat{H}_{RS}$	0.6249 (0.0759) [0.0327]	0.6167 (0.0677) [0.0127]	0.6134 (0.0644) [0.0100]	0.6212 (0.0722) [0.0095]	0.6008 (0.0518) [0.0101]
$\hat{\sigma}^2_{IMLE}$	0.2881 (0.0327) [0.0072]	0.3210 (0.0656) [0.0116]	0.3417 (0.0863) [0.0148]	0.3784 (0.1230) [0.0278]	0.3614 (0.1060) [0.0283]
$\mathrm{Eff}_{\hat{\sigma}^2_{IMLE}:\hat{\sigma}^2_{CMLE}}$	0.1806	0.1207	0.1014	0.0432	0.0636

Table 5. Summary of the return series of KLCI

Min.	1st Qu.	Median	Mean	Var	3rd Qu.	Max.
-0.0202000	-0.0023850	0.0005159	0.0003915	0.00002584	0.0029860	0.0190700

Table 6. Likelihood value with respect to the model parameters

H value	$\widehat{\sigma^2}$	$\widehat{\mu}$	**likelihood value**
0.500	0.00002573	0.0004035	2357.961
0.570	0.00002571	0.0004470	2361.453
0.573	0.00002574	0.0004494	2361.464
0.574	0.00002575	0.0004502	2361.465
0.575	**0.00002576**	**0.0004510**	**2361.465**
0.576	0.00002577	0.0004518	2361.464
0.600	0.00002613	0.0004740	2361.115

Table 7. Comparison of European call option prices using different methods: C_{CMLE} (this paper), C_{IMLE} (Kukush *et al.* with R/S analysis) and C_{BS} (traditional Black Scholes)

$T_0 - t_0$	**K**	C_{CMLE} ($H = 0.575$) [$\sigma^2 = 0.00002576$]	C_{IMLE} ($H = 0.6551$) [$\sigma^2 = 0.00002590$]	C_{BS} ($H = 0.5$) [$\sigma^2 = 0.00002589$]
15	1070	30.8566	35.2810	28.7439
	1080	23.2219	28.4503	20.2328
	1090	16.6880	22.4382	13.0493
	1100	11.3930	17.2847	7.5809
	1110	7.3561	12.9897	3.9079
30	1070	35.9385	43.3136	31.9585
	1080	28.9344	36.9983	24.1350
	1090	22.7585	31.2702	17.3923
	1100	17.4615	26.1410	11.8932
	1110	13.0511	21.6084	7.6796
40	1070	38.9955	47.9854	34.0120
	1080	32.2057	41.8453	26.4335
	1090	26.1415	36.2119	19.8239
	1100	20.8361	31.0917	14.2966
	1110	16.2947	26.4823	9.8847
50	1070	41.8534	52.3096	35.9756
	1080	35.2107	46.2922	28.5660
	1090	29.2202	40.7253	22.0404
	1100	23.9057	35.6126	16.4850
	1110	19.2709	30.9517	11.9273