Dynamic Systems and Applications 21 (2012) 522-534

LINEAR TRANSITIVITY ON COMPACT CONNECTED HYPERSPACE DYNAMICS

YUHU WU[1], XIAOPING XUE[2], AND DONGHAI JI[1]

[1]Department of Mathematics, Harbin University of Science and Technology, Harbin 150080, PR China

[2]Department of Mathematics, Harbin Institute of Technology, Harbin 150001 PR China

ABSTRACT. Let f be a linear operator on a Banach space X. In this paper we prove that the transitivity of induced compact connected hyperspace dynamical system $(\mathscr{K}\mathscr{K}(X), \overline{f}_{KK})$ is equivalent to weak mixing of the base dynamical system (X, f). Furthermore, we deduce that if X is separable, then (X, f) satisfies the hypercyclicity criterion if and only if $(\mathscr{K}\mathscr{K}(X), \overline{f}_{KK})$ is transitive.

Keywords. topological transitivity; weak mixing; compact connected hyperspace; induced hyperspace dynamical system; hypercyclicity criterion

AMS (MOS) Subject Classification. 37C15, 28B20, 37F15

1. INTRODUCTION

Topological transitivity is a global property of a dynamical system. This concept was introduced by G. D. Birkhoff [1] for characterizing orbital behavior of dynamical systems. Recall that, for a given topological dynamical system (X, f), where X is a topological space and $f : X \to X$ is a continuous map, we say that f is (topologically) transitive if for any pair of non-empty open sets U and V there is a $n \geq 1$ such that $f^n(U) \cap V \neq \emptyset$. Since the 1970s, topological transitivity has become an essential criterion of chaos. According to the popularly accepted definition given by Devaney [2], a dynamical system is chaotic if it satisfies the three conditions: (i) Topological transitivity, (ii) periodic density, and (iii) sensitivity to changes in initial conditions.

We consider, in this article, various induced hyperspaces dynamical systems. The hyperspaces are referred to as subspaces of the power set 2^X with the Vietories or Hausdorff topology. The study on hypersapces could be traced back to the work of Haousdorff, Vietories, Hahn, and Kuratowski [3]. There are certain kinds of hyperspaces which often appear in the literatures [4, 5]: *compact hyperspace* $\mathscr{K}(X)$, *compact convex hyperspace* $(\mathscr{K}\mathscr{C}(X))$, and *compact connected hyperspace* $(\mathscr{K}\mathscr{K}(X))$.

Received April 6, 2011

The precise definition of above hyperspaces are:

$$\mathscr{K}(X) = \{A \subset X : \text{A nonempty and compact}\};$$
$$\mathscr{K}\mathscr{C}(X) = \{A \subset X : \text{A nonempty, compact, and convex}\};$$
$$\mathscr{K}\mathscr{K}(X) = \{A \subset X : \text{A nonempty, compact, and connected}\}.$$

For the sake of completeness and to better understanding of our discussions below, let us now recall the following definitions [5]:

Definition 1.1 (Vietoris Topology)**.** Let (X, τ) be a topological space. Given an $A \in 2^X \backslash \{\emptyset\}$, we define

$$A^- = \{C \in 2^X \backslash \{\emptyset\} : A \cap C \neq \emptyset\}, \ A^+ = \{C \in 2^X \backslash \{\emptyset\} : A \subset C\}.$$

Then the *Vietoris topology* (denoted by τ_V) on $2^X \backslash \{\emptyset\}$ is generated by the subbase $L_{UV} \cup L_{LV}$, where $L_{UV} = \{U^+ : U \in \tau\}$, and $L_{LV} = \{U^- : U \in \tau\}$.

Remark 1.2. • From Definition 1.1, for the Vietoris topology τ_V, a basic element is given by

$$\beta(U_1, U_2, \ldots, U_n) := \{A \in 2^X \backslash \{\emptyset\} : A \subset \bigcup_{i=1}^{n} U_i, A \cap U_i \neq \emptyset\},$$

where $U_1, U_2, \ldots, U_n$ are open subsets of X.

• The Vietoris topology on any hyperspace $\mathscr{H} \subset 2^X \backslash \{\emptyset\}$ is just the subspace topology induced by the Vietories topology on $2^X \backslash \{\emptyset\}$.

Definition 1.3 (Induced Hyperspace Dynamical System)**.** Let (X, f) be a topological dynamical system and $\mathscr{H} \subset 2^X \backslash \{\emptyset\}$ be a hyperspace of X. If f is compatible with $\mathscr{H}$, i.e., $f(A) \in H$ for every $A \in \mathscr{H}$, then we define the induced hyperspace dynamical system associated to $\mathscr{H}$ as $\left(\mathscr{H}, \overline{f}_{\mathscr{H}}\right)$, where $\overline{f}_{\mathscr{H}}$ is the induced map of f to $\mathscr{H}$ as

$$\overline{f}_{\mathscr{H}}(A) = \{f(a) : a \in A\} \ , \ \forall \ A \in \mathscr{H}.$$

A natural problem is to study the relationship of dynamical properties between base system (X, f) and induced hyperspace system $\left(\mathscr{H}, \overline{f}_{\mathscr{H}}\right)$. During the last decade, several researches are dedicated to study the hyperspaces dynamical systems from different angles, e.g., Román-FLores (on transitivity, [6], 2003), Fedeli (on transitivity, dense periodicity and chaos, [7], 2006); Liao (on transitivity, weak mixing and chaos, [8], 2006); Kwietniak and Oprocha (topological entropy and chaos, [9], 2007)); Wang, Wei, and Campbell (on sensitivity, [10], 2009); Sharma and Nagar (on sensitivity, [11], 2010); Wu and Xue (on shadowing property, [12], 2010). The main focus of these authors is on hyperspaces $\mathscr{K}(X)$ and $\mathscr{K}\mathscr{C}(X)$. However, the case of hyperspace $\mathscr{K}\mathscr{K}(X)$ has not been considered yet.

On infinite dimensional spaces, what is very surprising is that certain linear operators can behave chaotically [13, 14, 15, 16]. It shows that chaos is not just a nonlinear phenomenon. Recently, Alfredo [17] proved that *if f is a linear continuous operator on a separable Banach space X, then $\left(\mathscr{K}(X), \overline{f}_K\right)$ and $\left(\mathscr{K}\mathscr{C}(X), \overline{f}_{KC}\right)$ are transitive if and only if (X, f) satisfies the hypercyclicity criterion (H.C).*

Inspired by the above result of Alfredo, we characterize the transitivity of $(\mathscr{K}\mathscr{K}(X), \overline{f}_{KK})$. However, because of the specificity of compact connected hyperspace $\mathscr{K}\mathscr{K}(X)$, it seems impossible to discuss the relationship between transitivity of $\left(\mathscr{K}\mathscr{K}(X), \overline{f}_{KK}\right)$ and that of (X, f) directly. It demands the development of new techniques. We will define a special totally ordered set (Definition 2.6), and use a rearrangement technique to derive our main result.

The remaining of the paper is organized as follows. In section 2, we give some lemmas which are useful for proving our main results. In section 3, we prove that the transitivity of induced compact connected hyperspace dynamical system $(\mathscr{K}\mathscr{K}(X), \overline{f}_{KK})$ is equivalent to weak mixing of the base dynamical system (X, f) (Theorem 3.1). Furthermore, we show that (X, f) satisfies the hypercyclicity criterion (H.C.) if and only if $\left(\mathscr{K}\mathscr{K}(X), \overline{f}_{KK}\right)$ is transitive (Theorem 3.2). Finally, as an application, by using above results, we deduce that the induced hyperspace dynamical systems of weighted right shift on sequence space l^2 is transitive (Example 3.3).

2. PRELIMINARIES

Before stating the main results, we will introduce some notions and lemmas which will be needed later.

Definition 2.1. We say that a continuous map $f : X \to X$ on a topological space X is *weakly mixing*, if $f \times f$ is transitive on $X \times X$.

The following lemma is given by Furstenberg [10, proposition 2.3].

Lemma 2.2. *Let $f : X \to X$ be a continuous map on a topological space X, then the following statements are equivalent:*

(1) f is weakly mixing ;

(2) For $m \geq 2$, the $m-$product map $\underbrace{f \times f \times \cdots \times f}_{m} : X \times X \times \cdots \times X \to X \times X \times \cdots \times X$ is transitive.

According to the definition of the canonical base of the Vietoris topology, we have the following simple fact.

Proposition 2.3. Let $U_1, U_2, \ldots, U_n$ be open subsets of a topological space X. If two open subsets $U_1^1, U_1^2 \subset U_1$, then

$$\beta_{KK}(U_1^1, U_2, \ldots, U_n) \subset \beta_{KK}(U_1, U_2, \ldots, U_n); \tag{2.1}$$

$$\beta_{KK}(U_1^1, U_1^2, U_2, \ldots, U_n) \subset \beta_{KK}(U_1, U_2, \ldots, U_n), \tag{2.2}$$

where $\beta_{KK}(U_1, U_2, \ldots, U_n) = \beta(U_1, U_2, \ldots, U_n) \bigcap \mathscr{K}\mathscr{K}(X)$ is a canonical base element of the Vietories topology on $\mathscr{K}\mathscr{K}(X)$.

The following definition of ε-*dilatation of* A is needed to understand the next lemma (of Heriberro).

Definition 2.4. Let X be a Banach space and $A \subset X$. We define the ε-*dilatation of* A as the set

$$N(A, \varepsilon) = \{x \in X : d(x, A) < \varepsilon\},$$

where $d(x, y) = \|x - y\|$ and $d(x, A) = inf\{d(x, a) : a \in A\}$.

Lemma 2.5 (Heriberro [4])**.** *Let A be a nonempty open set of a Banach space X, if $K \in \mathscr{K}(X)$ and $K \subset A$, then there exists $\varepsilon > 0$ such that $N(K, \varepsilon) \subset A$.*

Definition 2.6. Given an index set:

$$Q_n := \left\{ (i_1, i_2, \ldots, i_m) : \quad \begin{array}{l} 1 \le m \le n, i_1 = 1 \text{ and} \\ i_j \in \{1, 2, \ldots, n-1\}, \text{ for } 1 < j \le m \end{array} \right\},$$

a binary relation '$\prec$' on Q_n is defined as follows:

For any $q_1, q_2 \in Q_n$, where $q_1 \neq q_2, q_1 = (i_1, i_2, \ldots, i_m)$, $q_2 = (j_1, j_2, \ldots, j_l)$ we say that $q_1 \prec q_2$, if one of the following conditions holds:

$$\begin{array}{ll} (I) & m < l, \\ (II) & m = l, i_{k_0} < j_{k_0}, \end{array}$$

where $k_0 = \min\{k \in \{1, 2, \ldots, m\} : i_k \neq j_k\}$.

Remark 2.7. It is easy to verify that Q_n has $\sum_{i=0}^{n-1} (n-1)^i$ elements and Q_n is totally ordered set by the relation '$\prec$'.

Lemma 2.8. *Let X be a Banach space and $\mathscr{F} := \{U_i\}_{i=1}^n$ be a family of connected open sets of X. If $\bigcup\limits_{i=1}^{n} U_i$ is connected, then there is a rearranged sets family denoted by $\{T_q : q \in Q_n\}$, which consists of all sets of $\mathscr{F}$ and open sets, with the following properties:*

(PI) *If $U \in \mathscr{F}$, then there exists a unique $q \in Q_n$ such that $T_q = U$;*

(PII) *If $T_q \neq \emptyset$, then, for every $q \in Q_n$, $q \neq (1)$ and $q = (i_1, i_2, \ldots, i_m)$,*

$$T_{(i_1, i_2, \ldots, i_l)} \neq \emptyset, \; 1 \le l \le m-1; \tag{2.3}$$

(PIII) *If $T_q \neq \emptyset$, then, for every $q \in Q_n$, $q \neq (1)$ and $q = (i_1, i_2, \ldots, i_m)$,*

$$T_{(i_1,i_2,\ldots,i_l)} \cap T_q \neq \emptyset, \quad \textit{for } l = m-1, \tag{2.4}$$

$$T_{(i_1,i_2,\ldots,i_l)} \cap T_q = \emptyset, \quad \textit{for } 1 \leq l \leq m-2. \tag{2.5}$$

Proof. For every $i = 1, 2, \ldots, n$, we put

$$I(U_i) := \{j \in \{1, 2, \ldots, n\} \mid j \neq i, U_i \cap U_j \neq \emptyset\}.$$

Since $\bigcup_{i=1}^n U_i$ is connected, it is obvious that $I(U_i) \neq \emptyset$, $(i = 1, 2, \ldots, n)$.

Now we choose the elements of $\{T_q : q \in Q_n\}$ as per the order '$\prec$'. First let $T_{(1)} = U_1$. For $q \in Q_n$, $q \neq (1)$ and $q = (i_1, i_2, \ldots, i_m)$, it is assumed that the elements T_p have been chosen, for every $p \prec q$. To simplify the notation, we denote the index $(i_1, i_2, \ldots, i_{m-1})$ by q' and define $R(q) = \big\{j \in \{1, 2, \ldots, n\} \mid \exists\, p \in Q_n, p \prec q \text{ such that } T_p = U_j\big\}$. Now we take T_q as follows:

$$\begin{cases} T_q = \emptyset, & \text{if } T_{q'} = \emptyset \text{ or } I(T_{q'}) \setminus R(q) = \emptyset, \\ T_q = U_{j_q}, & \text{if } I(T_{q'}) \setminus R(q) \neq \emptyset, \end{cases} \tag{2.6}$$

where $j_q = \min\{I(T_{q'}) \setminus R(q)\}$.

From the above process of choosing elements T_q, the properties (PI) and (PII) are obvious.

If $T_q \neq \emptyset$, for every $q \in Q_n$, $q \neq (1)$, $q = (i_1, i_2, \ldots, i_m)$, then according to (2.6), we observe that $T_q = U_{j_q}$ and $j_q \in I(T_{q'})$. We thus obtain (2.4) in property (PIII). Moreover, we can claim (2.5) in property (PIII) is also true.

In fact, if (2.5) does not hold, there exists some $1 \leq l \leq m-2$ such that $T_{(i_1,i_2,\ldots,i_l)} \cap T_q \neq \emptyset$. We can assume that, $T_q = U_k$, for some $1 \leq k \leq n$. Then, by $T_{(i_1,i_2,\ldots,i_l)} \cap U_k \neq \emptyset$, we note that $k \in I(T_{(i_1,i_2,\ldots,i_l)})$. Since $(i_1, i_2, \ldots, i_l) \prec q' \prec q$, from the above process of choosing the element T_q, there must exist an index $q'' \in Q_n$ satisfying $q'' \prec q$ and $T_{q''} = U_k$. This contradicts Property (PI), and hence our claim. $\square$

Definition 2.9. For given a finite number of points $\{x_1, , x_2, \ldots, x_n\}$ from a Banach space X, we define the *polyline* which connects these points in sequence by

$$l(x_1, x_2, \ldots, x_n) := \{\alpha x_i + (1-\alpha)x_{i+1} | \alpha \in [0, 1], i = 1, 2, \ldots, n-1\}.$$

Lemma 2.10. *Let U be an open set of a Banach space X, and $A \subset U$ be a connected compact subset of X. Then, for any two points $x, y \in A$, there exist some finite set of points $\{x = a_1, a_2, \ldots, a_n = y\} \subset A$ and an $\varepsilon > 0$, such that*

$$l(a_1, a_2, \ldots, a_n) \subset N(A, \varepsilon), \tag{2.7}$$

$$N(l(a_1, a_2, \ldots, a_n), \varepsilon) \subset U. \tag{2.8}$$

Proof. For an open set U and a compact set $A \subset U$, it follows from Lemma 2.1, that there is an $\varepsilon > 0$ such that $N(A, 2\varepsilon) \subset U$. Furthermore, since $\{B(a, \frac{\varepsilon}{2})\}_{a\in A}$ is an open cover of A and A is compact, we find a finite cover $\{B(a'_i, \frac{\varepsilon}{2})\}_{i=1}^{n'}$ of A, where without loss of generality, we assume that $a'_1 = x, a'_{n'} = y$ and $\bigcup_{i=1}^{n} B(a'_i, \frac{\varepsilon}{2})$ is connected. Using Lemma 2.8, we rearrange $\{B(a'_i, \frac{\varepsilon}{2})\}_{i=1}^{n'}$ to a totally ordered set family $\{T_q : q \in Q_{n'}\}$. Then $T_{(1)} = B(x, \frac{\varepsilon}{2})$ and there is an index $(i_1, i_2, \dots, i_n) \in Q_{n'}$, such that $T_{(i_1,i_2,\dots,i_n)} = B(a'_{n'}, \frac{\varepsilon}{2}) = B(y, \frac{\varepsilon}{2})$ (by the property (PI) of Lemma 2.8).

Set $B(a_j, \frac{\varepsilon}{2}) := T_{(i_1,i_2,\dots,i_j)}$, i.e., a_j is the center of $T_{(i_1,i_2,\dots,i_j)}$, for $j = 1, 2, \dots, n$. It is obvious that $a_1 = x, a_n = y$. By the property (PIII) of Lemma 2.8, we get $B(a_i, \frac{\varepsilon}{2}) \cap B(a_{i+1}, \frac{\varepsilon}{2}) \neq \emptyset$, for $i = 1, 2, 3, \dots, n-1$. It implies that $d(a_i, a_{i+1}) < \varepsilon$, for $i = 1, 2, 3, \dots, n-1$.

For any point $z \in l(a_1, a_2, \dots, a_n)$, there exist $i_0 \in \{1, 2, \dots, n-1\}$ and $\alpha \in [0, 1]$ such that $z = \alpha a_{i_0} + (1-\alpha)a_{i_0+1}$. So $d(z, a_{i_0}) = (1-\alpha)d(a_{i_0}, a_{i_0+1}) < \varepsilon$, it implies that (2.7) holds.

For any point $z_1 \in N(l(a_1, a_2, \dots, a_n), \varepsilon)$, we have

$$d(z_1, A) \leq d(z_1, l(a_1, a_2, \dots, a_n)) + d(l(a_1, a_2, \dots, a_n), A) \leq 2\varepsilon.$$

Additionally, by$N(A, 2\varepsilon) \subset U$, we get (2.8). □

Lemma 2.11. *Let X be a Banach space, U, V be two nonempty open sets, and $A \subset U$, $B \subset V$ be two connected compact sets. Let f be a linear operator on X (i.e., $f \in L(X)$) and let f be weakly mixing. Then for any two pairs of points $x_1, x_2 \in A$ and $y_1, y_2 \in B$, there exist finite sets of points $\{x_1 = a_1, a_2, \dots, a_n = x_2\} \subset A$ and $\{y_1 = b_1, b_2, \dots, b_n = y_2\} \subset B$, $\varepsilon > 0$ and $k \in N$, such that*

$$l_1 := l(a_1, a_2, \dots, a_n) \subset \beta_{KK}(U, B(x_1, \varepsilon), B(x_2, \varepsilon)), \tag{2.9}$$

$$l_2 := l(b_1, b_2, \dots, b_n) \subset \beta_{KK}(V, B(y_1, \varepsilon), B(y_2, \varepsilon)), \tag{2.10}$$

$$f^k(l_1) = l_2. \tag{2.11}$$

Proof. By using Lemma 2.10, there exist $\{x_1 = c_1, c_2, \dots, c_n = x_2\} \subset A$, $\{y_1 = d_1, d_2, \dots, d_m = y_2\} \subset B$ and $\varepsilon_1, \varepsilon_2 > 0$ such that

$$l_3 := l(c_1, c_2, \dots, c_n) \subset N(A, \varepsilon_1) \text{ and } N(l_3, \varepsilon_1) \subset U, \tag{2.12}$$

$$l_4 := l(d_1, d_2, \dots, d_m) \subset N(A, \varepsilon_2) \text{ and } N(l_4, \varepsilon_2) \subset V. \tag{2.13}$$

Without loss of generality, we assume $\varepsilon = \varepsilon_1 = \varepsilon_2$ and $m = n$.

Since f is weakly mixing, there exists, by Lemma 2.2, a $k \in N$ such that

$$f^k(B(c_i, \varepsilon)) \cap B(d_i, \varepsilon) \neq \emptyset, \quad (i = 1, 2, \dots, n). \tag{2.14}$$

Now, we choose $a_i \in (B(c_i,\varepsilon))$, which satisfies $b_i := f^k(a_i) \in B(d_i,\varepsilon)$, for $i = 1,2,\ldots,n$. Let $l_1 := l(a_1,a_2,\ldots,a_n)$ and $l_2 := l(b_1,b_2,\ldots,b_n)$. Since $f \in L(X)$, it is clear that $f^k(l_1) = l_2$. Therefore we obtain (2.11).

If $z \in l_1$, then $z = \alpha a_i + (1-\alpha)a_{i+1}$, for some $\alpha \in [0,1]$, $i \in \{1,2,\ldots,n-1\}$. Selecting $z' = \alpha c_i + (1-\alpha)c_{i+1} \in l_3$, we have $d(z,z') \leq \alpha d(a_i,c_i) + (1-\alpha)d(a_{i+1},c_{i+1}) < \varepsilon$. It implies that $l_1 \subset N(l_3,\varepsilon)$. Furthermore, by (2.12), we have $l_1 \subset U$. Therefore we obtain (2.9), since l_1 is connected and compact. The proof of (2.10) is similar to that of (2.9) and so we omit it. □

3. TRANSITIVITY ON $\mathscr{K}\mathscr{K}(X)$

Now we state the main result of this work.

Theorem 3.1. *Let X be a Banach space, and $f \in L(X)$. Then the following assertions are equivalent:*

(a) (X,f) is weakly mixing;

(b) $\left(\mathscr{K}\mathscr{K}(X), \overline{f}_{KK}\right)$ is transitive.

Proof. First, the proof of $(b) \Rightarrow (a)$:
By a conclusion of Banks ([19], lemma 5), f is weakly mixing if and only if, for arbitrary nonempty open sets $U, V_1, V_2 \subset X$, there exists a $n_1 \in N$ such that $f^{n_1}(U) \cap V_1 \neq \emptyset$ and $f^{n_1}(U) \cap V_2 \neq \emptyset$. So it is enough to show that, given nonempty open sets $U, V_1, V_2 \subset X$, there exists a $n_1 \in N$ such that $f^{n_1}(U) \cap V_1 \neq \emptyset$, $f^{n_1}(U) \cap V_2 \neq \emptyset$.

If we conside the two basic sets $\beta_{KK}(U), \beta_{KK}(V_1,V_2) \subset \mathscr{K}\mathscr{K}(X)$, obviously, $\beta_{KK}(U) \neq \emptyset$. For any nonempty open sets V_1, V_2, one of the following cases holds:

(I) $\beta_{KK}(V_1,V_2) \neq \emptyset$; $\quad$ (II) $\beta_{KK}(V_1,V_2) = \emptyset$.

In the case of (I), it follows from the transitivity of $\overline{f}_{KK}$ that there exists a $n_1 \in N$ such that

$$\overline{f}^{n_1}_{KK}(\beta_{KK}(U)) \bigcap \beta_{KK}(V_1,V_2) \neq \emptyset.$$

Hence, we can find $A_0 \in \beta_{KK}(U)$ such that $\overline{f}^{n_1}(A_0) \in \beta_{KK}(V_1,V_2)$. So there exist two points $x, y \in A_0 \subset U$ satisfying $f^{n_1}(x) \in V_1, f^{n_1}(y) \in V_2$. It implies that $f^{n_1}(U) \cap V_1 \neq \emptyset$, $f^{n_1}(U) \cap V_2 \neq \emptyset$.

In the case of (II), there is an open set $V_3 \subset X$ (without loss of generality, V_3 can be referred to as X), satisfying $\beta_{WKC}(V_1,V_2,V_3) \neq \emptyset$. Then it is similar to the case of (I) that there is an $n_1 \in N$ such that $f^{n_1}(U) \cap V_1 \neq \emptyset$, $f^{n_1}(U) \cap V_2 \neq \emptyset$.

We now proceed to prove $(a) \Rightarrow (b)$. This proof will be given in three steps.

Step 1. It is enough to show that, given two nonempty canonical base sets $\beta_{KK}(U_1,U_2,\ldots,U_n)$, $\beta_{KK}(V_1,V_2,\ldots,V_n)$ of the Vietoris topology on $\mathscr{K}\mathscr{K}(X)$, there is a $n_0 \in N$ such that

$$\overline{f}_{KK}^{n_0}\left(\beta_{KK}(U_1, U_2, \ldots, U_n)\right) \bigcap \beta_{KK}(V_1, V_2, \ldots, V_n) \neq \emptyset.$$

According to Proposition 2.3 and choosing appropriate component sets of U_i and V_i, without loss of generality, we further assume that $\mathscr{F}_U := \{U_i\}_{i=1}^n$ and $\mathscr{F}_V := \{V_i\}_{i=1}^n$ are two families of connected open sets. Since $\beta_{KK}(U_1, U_2, \ldots, U_n) \neq \emptyset$ and $\beta_{KK}(V_1, V_2, \ldots, V_n) \neq \emptyset$, there exist two connected compact sets $A, B \subset X$ satisfying $A \cap U_i \neq \emptyset$, $B \cap V_i \neq \emptyset$, for $i = 1, 2, \ldots, n$ and $A \subset \bigcup_{i=1}^n U_i$, $B \subset \bigcup_{i=1}^n V_i$. It follows that $\bigcup_{i=1}^n U_i$ and $\bigcup_{i=1}^n V_i$ are connected sets.

Using Lemma 2.8, we rearrange $\mathscr{F}_U = \{U_i\}_{i=1}^n$ and $\mathscr{F}_V = \{V_i\}_{i=1}^n$ to $\{\widetilde{T}_q : q \in Q_n\}$ and $\{\widetilde{S}_q : q \in Q_n\}$ respectively. They satisfy properties (PI), (PII) and $(PIII)$ of Lemma 2.8.

Now we take two new families of connected open sets $\{T_q : q \in Q_n\}$ and $\{S_q : q \in Q_n\}$ as follows. For every $q = (i_1, i_2, \ldots, i_m) \in Q_n$, we choose T_q and S_q in order $\prec$, as follows :

$$\begin{cases} T_q := \widetilde{T}_q, S_q := \widetilde{S}_q, & \text{if } \widetilde{T}_q \neq \emptyset, \widetilde{S}_q \neq \emptyset; \\ T_q := \widetilde{T}_q, S_q := \widetilde{S}_{q'}, & \text{if } \widetilde{T}_q \neq \emptyset, \widetilde{S}_q = \emptyset; \\ T_q := \widetilde{T}_{q'}, S_q := \widetilde{S}_q, & \text{if } \widetilde{T}_q = \emptyset, \widetilde{S}_q \neq \emptyset; \\ T_q := \emptyset, \quad S_q := \emptyset, & \text{if } \widetilde{T}_q = \emptyset, \widetilde{S}_q = \emptyset; \end{cases} \tag{3.1}$$

where $q' = (i_1, i_2, \ldots, i_{m-1})$. Therefore, according to (3.1), for any $q \in Q_n$, $T_q \neq \emptyset$ if and only if $S_q \neq \emptyset$.

Next, let $Q_n' := \{q \in Q_n | T_q \neq \emptyset\}$. Then we have the following properties:

(PIV) For every $q \in Q_n'$, there exist $U \in \mathscr{F}_U$ and $V \in \mathscr{F}_V$ such that $T_q = U$ and $S_q = V$.

(PV) For every $q \in Q_n', q \neq (1), q = (i_1, i_2, \ldots, i_m)$, it is true that

$$T_{q'} \neq \emptyset \text{ and } T_{q'} \cap T_q \neq \emptyset, \tag{3.2}$$

$$S_{q'} \neq \emptyset \text{ and } S_{q'} \cap S_q \neq \emptyset. \tag{3.3}$$

Step 2. For every $q \in Q_n'$, $q = (i_1, i_2, \ldots, i_m)$, we choose $x_q \in T_q$, $y_q \in S_q$ as follows :

If $q = (1)$, then $x_q \in T_q$, $y_q \in S_q$;

If $q \neq (1)$, then $x_q \in T_q \cap T_{q'}$, $y_q \in S_q \cap S_{q'}$.

Thus, for every $q \in Q_n'$, $q = (i_1, i_2, \ldots, i_m)$, $q \neq (1)$, we have

$$x_{q'} \text{ and } x_q \in T_{q'}; \tag{3.4}$$

$$y_{q'} \text{ and } y_q \in S_{q'} . \tag{3.5}$$

Since $T_{q'}$ and $S_{q'}$ are open connected subsets of X, it is obvious that there exist connected compact subset $C \subset T_{q'}$ and $D \subset S_{q'}$, with $x_{q'}, x_q \in C$ and $y_{q'}, y_q \in D$.

Hence, from Lemma 2.10, for every $q \in Q'_n, q \neq (1)$, there exist $\varepsilon_q > 0$ and two sequences which contain k_q points respectively,

$$\{x_{q'} = x_1^q, x_2^q, \ldots, x_{k_q}^q = x_q\} \subset T_{q'}, \tag{3.6}$$

$$\{y_{q'} = y_1^q, y_2^q, \ldots, y_{k_q}^q = y_q\} \subset S_{q'}, \tag{3.7}$$

such that

$$N(l(x_1^q, x_2^q, \ldots, x_{k_q}^q), \varepsilon_q) \subset T_{q'}, \tag{3.8}$$

$$N(l(y_1^q, y_2^q, \ldots, y_{k_q}^q), \varepsilon_q) \subset S_{q'}. \tag{3.9}$$

Let $\varepsilon = \min\{\varepsilon_q | q \in Q'_n\}$, and define

$$\Omega_x := \bigcup_{q \in Q'_n, q \neq (1)} \{x_1^q, x_2^q, \ldots, x_{k_q}^q\},$$

$$\Omega_y := \bigcup_{q \in Q'_n, q \neq (1)} \{y_1^q, y_2^q, \ldots, y_{k_q}^q\}.$$

From (3.6) and (3.7), we have that, for every $q \in Q'_n, q = (i_1, i_2, \ldots, i_m), q \neq (1)$,

$$x_{k_{q'}}^{q'} = x_1^q = x_{q'} \in \Omega_x, \quad y_{k_{q'}}^{q'} = y_1^q = y_{q'} \in \Omega_y, \qquad \text{if } 2 < m \leq n, \tag{3.10}$$

$$x_{q'} = x_1^q = x_{(1)} \in \Omega_x, \quad y_{q'} = y_1^q = y_{(1)} \in \Omega_y, \qquad \text{if } m = 2. \tag{3.11}$$

Step 3. Since f is weakly mixing, it follows from Lemma 2.11 that there exists $n_0 \in N$ such that

$$f^{n_0}(B(x_j^q, \varepsilon)) \cap B(y_j^q, \varepsilon) \neq \emptyset, \quad \forall x_j^q \in \Omega_x, \ y_j^q \in \Omega_y.$$

We select now $z_j^q \in (B(x_j^q, \varepsilon))$ satisfying $w_j^q := f^{n_0}(z_j^q) \in (B(y_j^q, \varepsilon))$. Define

$$L_1 := \bigcup_{q \in Q'_n, q \neq (1)} \{l(z_1^q, z_2^q, \ldots, z_{k_q}^q)\},$$

$$L_2 := \bigcup_{q \in Q'_n, q \neq (1)} \{l(w_1^q, w_2^q, \ldots, w_{k_q}^q)\}.$$

By (3.10) and (3.11), we have, for every $q \in Q'_n, q = (i_1, i_2, \ldots, i_m), q \neq (1)$, that

$$z_{k_{q'}}^{q'} = z_{q'} = z_1^q, \ w_{k_{q'}}^{q'} = w_{q'} = w_1^q, \qquad \text{if } 2 < m \leq n,$$
$$z_{q'} = z_{(1)} = z_1^q, \ w_{q'} = w_{(1)} = w_1^q, \qquad \text{if } m = 2.$$

Hence, for $2 \leq m \leq n$, it is true that

$$l(z_1^{q'}, z_2^{q'}, \ldots, z_{k_{q'}}^{q'}) \cap l(z_1^q, z_2^q, \ldots, z_{k_q}^q) \neq \emptyset,$$
$$l(w_1^{q'}, w_2^{q'}, \ldots, w_{k_{q'}}^{q'}) \cap l(w_1^q, w_2^q, \ldots, w_{k_q}^q) \neq \emptyset,$$

which implies that L_1, L_2 are compact and connected sets, i.e.,

$$L_1, L_2 \in \mathscr{K}\mathscr{K}(X). \tag{3.12}$$

Since $l(z_1^q, z_2^q, \ldots, z_{k_q}^q) \subset N(l(x_1^q, x_2^q, \ldots, x_{k_q}^q), \varepsilon)$ and $N(l(x_1^q, x_2^q, \ldots, x_{k_q}^q), \varepsilon) \subset T_{q'}$, by (3.8), we have

$$L_1 \subset \bigcup_{q \in Q'_n} T_q = \bigcup_{i=1}^{n} U_i. \tag{3.13}$$

Similarly, we have

$$L_2 \subset \bigcup_{q \in Q'_n} S_q = \bigcup_{i=1}^{n} V_i. \tag{3.14}$$

From

$$z_{k_q}^q \in B(x_{k_q}^q, \varepsilon) = B(x_q, \varepsilon) \subset T_q,$$
$$w_{k_q}^q \in B(y_{k_q}^q, \varepsilon) = B(y_q, \varepsilon) \subset S_q,$$

we have

$$L_1 \cap U_i \neq \emptyset, i = 1, 2, \ldots, n, \tag{3.15}$$
$$L_2 \cap V_i \neq \emptyset, i = 1, 2, \ldots, n. \tag{3.16}$$

Then from (3.12),(3.13) and (3.15), we get that

$$L_1 \in \beta_{KK}(U_1, U_2, \ldots, U_n) \tag{3.17}$$

Similar to the proof of (3.17), from (3.12),(3.14) and (3.16), we get that

$$L_2 \in \beta_{KK}(V_1, V_2, \ldots, V_n). \tag{3.18}$$

For every $x \in L_1$, there exist, by definition of L_1, $q \in Q'_n$, $q \neq (1)$, $1 \leq j \leq k_q - 1$ and $0 \leq \alpha \leq 1$ so that $x = \alpha z_j^q + (1-\alpha) z_{j+1}^q$. Since $f : X \to X$ is linear,

$$f^{n_0}(x) = \alpha f^{n_0}(z_j^q) + (1-\alpha) f^{n_0}(z_{j+1}^q) = \alpha w_j^q + (1-\alpha) w_{j+1}^q.$$

It implies that

$$f^{n_0}(L_1) = L_2. \tag{3.19}$$

By (3.17),(3.18) and (3.19), we obtain

$$\overline{f}_{KK}^{\,n_0}(\beta_{KK}(U_1, U_2, \ldots, U_n)) \bigcap \beta_{KK}(V_1, V_2, \ldots, V_n) \neq \emptyset,$$

and the proof is completed. □

We shall now consider the relationship of hypercyclicity criterion for the operator f and transitivity for the induced map of f.

Let X be a separable Banach space. An operator $f \in L(X)$ is said to satisfy the *hypercyclicity criterion* provided there exist two dense subsets $X_0, Y_0 \subset X$, a sequence (n_k) of nonnegative integers, and (not necessarily continuous) mappings $S_{n_k} : Y_0 \to X$ so that

$$\begin{array}{lll} (i) & f^{n_k} \to 0 & \text{pointwise on } X_0, \\ (ii) & S_{n_k} \to 0 & \text{pointwise on } Y_0, \\ (iii) & f^{n_k} S_{n_k} \to Id_{Y_0} & \text{pointwise on } Y_0, \end{array}$$

where Id_{Y_0} is the identity restricted to Y_0.

Hypercyclicity criterion is a computable and sufficient condition for the existence of dense orbits for operators on Banach space. The first version of this Criterion was given by Kitai [20], later rediscovered by Gethner and Shapiro [18].

In [21], J. Bes and A. Peris show that if X is a separable Banach space and $f \in L(X)$, then (X, f) satisfies the hypercyclicity criterion if and only if (X, f) is weakly mixing. Then, the following is a consequence of Theorem 3.1.

Theorem 3.2. *Let X be a separable Banach space, $f \in L(X)$. Then the following assertions are equivalent:*

(I) *f satisfies the hypercyclicity criterion (H.C.);*
(II) *$\left(\mathscr{K}(X), \overline{f}_K\right)$ is transitive;*
(III) *$\left(\mathscr{K}\mathscr{C}(X), \overline{f}_{KC}\right)$ is transitive;*
(IV) *$\left(\mathscr{K}\mathscr{K}(X), \overline{f}_{KK}\right)$ is transitive.*

Proof. $(I) \Leftrightarrow (II)$, $(I) \Leftrightarrow (III)$, see [17].
$(I) \Leftrightarrow (IV)$, see Theorem 3.1 and [21]. □

The following Example illustrates our Theorem 3.2.

Example 3.3. We consider the separable Banach space $l^2 := \{\{x_j\}_{j \geq 1} : \sum\limits_{j=1}^{\infty} |x_j|^2 < \infty\}$, with an orthonormal basis $\{e_i = \{0, \dots, 0, 1_i, 0, \dots\}\}_{i \geq 1}$. For constant $w > 1$, the weighted right shift $f : l^2 \to l^2$ is defined by

$$f(e_i) = w e_{i-1}, \quad \text{if} \quad i > 1, \quad \text{and} \quad f(e_1) = 0.$$

Then the induced hyperspace dynamical systems $\left(\mathscr{K}(l^2), \overline{f}_K\right)$, $\left(\mathscr{K}\mathscr{C}(l^2), \overline{f}_{KC}\right)$ and $\left(\mathscr{K}\mathscr{K}(l^2), \overline{f}_{KK}\right)$ are transitive.

To verify the assertion in Example 3.3, we choose $X_0 = Y_0 = span\{e_i : i = 1, 2, \dots\}$, and $S_n = g^n$, where the operator $g : l^2 \to l^2$ is defined by

$$g(e_i) = \frac{1}{w} e_{i+1}, \qquad i \geq 1.$$

Then it is clear that $X_0 = Y_0$ is dense in l^2, and $f^n S_n = f^n g^n = Id$ on l^2 for every $n \geq 1$. So it satisfies the condition (iii) of the hypercyclicity criterion.

For any $x \in X_0$, there exist $m \in N$ and constants $a_1, a_2, \ldots, a_m$ such that $x = \sum_{i=1}^{m} a_i e_i$. So for every $n > m$, it is true that $f^n(x) = 0$. Hence

$$\lim_{n\to\infty} f^n(x) = 0 \quad \text{for} \quad \forall x \in X_0.$$

So it satisfies the condition (i) of the hypercyclicity criterion.

Additionally, for $\forall y \in Y_0, \forall n \in N$, we have $\|g^n(y)\| \leq (1/2)^n \|y\|$. It implies that

$$\lim_{n\to\infty} g^n(y) = 0 \quad \text{for} \quad \forall y \in Y_0.$$

So it satisfies the condition (ii) of the hypercyclicity criterion. Thus, $f : l^2 \to l^2$ satisfies the hypercyclicity criterion. By the Theorem 3.2, the assertions of Example 3.3 are true.

Acknowledgement

The authors would like to thank Professor D. Kannan for stimulating discussion on the subject. This work was supported by the National Natural Science Foundation of China (NNSFC) under Grant No. 11171082.

REFERENCES

[1] G. D. Birkhoff, *Dynamical systems*, Providence, RI: AMS, 1927.

[2] R. L. Deveney, *An introdction to chaotic dynamical systems*, Redwood City: Addison-Wesley, 1989.

[3] R. Engelking, *General Topology*, PWN, Warszawa, 1977.

[4] Beer G., *Topologies on Closed and Closed Convex Sets*, Kluwer Academic Publishers, Dordrecht/Boston/London, 1993.

[5] S. Hu, N. S. Papageorgiou, *Handbook of Multivalued Analysis*, Dordrecht-Boston-London: Kluwer Academic Publishers, 1997.

[6] H. Roman-FLores, A note on transitivity in set-valued discrete systems, *Chaos, Solitons & Fractals* 17 (2003), 99–104.

[7] A. Fedeli, On chaotic set-valued discrete dynamical systems, *Chaos, Solitions & Fractals* 23 (2005), 1381–1384.

[8] G. F. Liao, L. D. Wang, Y. C. Zhang, Transitivity, mixing and chaos for a class of set-valued mappings, *Sci. Sin. Math.* 49(1) (2006), 1–8.

[9] D. Kwietniak, P. Oprocha, Topological entropy and chaos for maps induced on hyperspaces, *Chaos, Solitons & Fractals* , 33 (2007), 76–86.

[10] Y. Wang, G. Wei, W. H. Campbell, Sensitive dependence on initial conditions between dynamical systems and their induced hyperspace dynamical systems, *Topology and its Applications* 156 (2009), 803–811.

[11] P. Sharma, A. Nagar, Inducing sensitivity on hyperspaces, *Topology and its Applications* 157 (2010), 2052–2058.

[12] Y. H. Wu, X. P. Xue, Shadowing property for induced set-valued dynanical systems of some expansive maps, *Dynamic Systems and Applications* 19 (2010), 405–414.

[13] L. Fernando, M. Alfonso, Linear structure of hypercyclic vectors, *J. Funct. Anal.* 148 (1997), 524–545.

[14] G. Gilles, J. H. Shapiro, Operators with dense, invariant, cyclic vector manifolds, *J. Funct. Anal.* 98 (1991), 229–269.

[15] N. S. Hector, Hyperciclic weighted shifts, *Trans. Amer. Math. Soc.* 347 (1995), 993–1004.

[16] G. Herzog, R. Lemmert, On universal subsets of Banach spaces, *Math. Z.* 229 (1998), 615–619.

[17] P. Alfredo, Set-valued discrete chaos, *Chaos, Solitions & Fractals* 26 (2005), 19–23.

[18] R. M. Gethner, J. H. Shapiro, Universal vectors for operator on spaces for holomorphic functions, *Proc. Amer. Math. Soc.* 100(2) (1987), 281–288.

[19] J. Banks, J. Brooks, G. Cairns, P. Stacey, On the Devaney's definition of chaos, *Amer. Math. Monthly* 99 (1992), 332–334.

[20] C. Kitai, Invariant closed sets for linear operator, *Thesis University of Toronto*, 1982.

[21] J. Bes, A. Peris, Hereditarily hypercyclic operators,*J. Funct. Anal.* 167 (1999), 94–112.