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EXISTENCE AND GENERALIZED QUASILINEARIZATION METHODS FOR SINGULAR NONLINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT. We consider the existence problem for the differential equation $\ell u = F(u)$, where ℓ is a formally self-adjoint singular second order differential expression and F is nonlinear. Under certain assumptions on ℓ and F we develop an existence theorem. If the problem has upper and lower solutions these assumptions can be relaxed. A generalized quasilinearization method is then developed for this problem and we obtain a monotonic sequence of approximate solutions converging to a solution of the problem. If F is monotone then the convergence is quadratic.

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1. INTRODUCTION

Let $I = (a, b)$. We consider the nonlinear equation

$$\ell(u(t)) = f(t, u(t)), \quad t \in I, \tag{1.1}$$

where

$$\ell(u) = -(pu')' + qu,$$

$p \geq 0$, $q \geq 0$ and $f : I \times \mathbb{R} \to \mathbb{R}$ with a boundary condition operator $B(\cdot)$ to be specified later.

In this paper we consider the problem (1.1) in the case when the differential expression ℓ is singular and f gives rise to a continuous nonlinear operator in a Hilbert space (see [8], [9] and the references therein). The case when ℓ is a general regular differential expression was considered elsewhere [5].

The setup of the problem takes place in the space $L^2(I)$ and its appropriate subspaces rather than the space of continuous functions which is normally considered when the expression ℓ is regular. The class of nonlinear functions to be dealt with is modified accordingly.

Under certain conditions on the nonlinear function f we will prove a general existence result. The concept of upper and lower solutions will then be extended to the

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context of singular differential expressions. If problem (1) has upper and lower solutions then the conditions on f can be relaxed and problem (1) still has a solution. A generalized quasilinearization method is developed to generate a monotonic sequence of approximate solutions of (1) that converge quadratically to a solution of (1). We also treat the special case when one of the endpoints of the interval I is regular for the expression ℓ.

This paper consists of four sections besides the introduction. In Section 2 we introduce some terminology, state the assumptions, set up the working spaces and formulate problem (1) in these spaces. We then state and prove a general existence theorem. In Section 3 we introduce the definitions of upper and lower solutions in the context of singular differential expressions. We then show that our definitions reduce to the classical ones if the expression ℓ is regular and f is continuous. In Section 4 we show how the existence of upper and lower solutions for (1) can be used to relax the assumptions on f and still obtain existence results. We then develop a quasilinearization method that produces a monotone sequence of approximations to a solution of (1). Under a monotonicity assumption on f we will be able to show that the convergence of this sequence is quadratic. In Section 5 we treat the special case of a singular differential expression with one regular endpoint where we assume nonlinear boundary conditions.

2. **A GENERAL EXISTENCE THEOREM**

The working space will be $L^2(I)$ whose norm and inner product will be denoted by $\|\cdot\|$, $\langle\cdot,\cdot\rangle$, respectively. We assume that $1/p, q \in L^1_{\text{loc}}(I)$.

Define the operator F by

$$Fu(t) = f(t, u(t)) + u(t) \quad t \in I.$$

We assume that $F : L^2(I) \to L^2(I)$.

The expression ℓ is regular if a, b are finite and $1/p, q$ are integrable on I, otherwise it is singular. The end point a(respectively b) is a regular endpoint for ℓ if it is finite and $1/p, q \in L^1(a, c)$ (respectively $L^1(c, b)$) for all $c \in (a, b)$, otherwise, it is a singular end-point for ℓ (see also [13], chapter 18). The maximal subspace of $L^2(I)$ on which to consider the expression ℓ without getting out of $L^2(I)$ is

$$D = \left\{u \in L^2(I) : u, pu' \in AC(I), \ell(u) \in L^2(I)\right\}.$$

The operator $L : L^2(I) \to L^2(I)$ defined by $D(L) = D$ and $Lu = \ell(u)$ is called the maximal operator generated by ℓ. The operator $L_0 = L^*$ (the adjoint of L) is called the minimal operator generated by ℓ. Its domain will be denoted by D_0. The deficiency index d of L_0 is defined as the dimension of the orthogonal complement of the range of $(L_0 - iE)$, where E is the identity operator on $L^2(I)$. In general $0 \le d \le 2$. In this paper, however, we deal only with the case $d = 2$.

Define the formal sesquilinear form

$$a_1(u,v) = \int_I pu'\overline{v'} + qu\overline{v}$$

and let V be the maximal subspace of $L^2(I)$ on which it is defined. To account for the boundary behavior of functions in the various spaces mentioned above we define the "half Lagrangian"

$$\{u,v\}_x = -pu'\overline{v}(x), \ x \in I,$$

the Lagrangian

$$[u,v]_x = \{u,v\}_x - \{\overline{v},\overline{u}\}_x,$$

and the brackets

$$\begin{aligned} \{u,v\}_a^b &= \{u,v\}_b - \{u,v\}_a, \\ [u,v]_a^b &= [u,v]_b - [u,v]_a, \end{aligned}$$

where it is understood that

$$\{u,v\}_a = \lim_{x \to a^+} \{u,v\}_x$$

if the limit exists and so on. For $u, v \in V$ let

$$a(u,v) = a_1(u,v) + \langle u,v \rangle .$$

It is easy to show that $a(\cdot,\cdot)$ is an inner product on V under which V is a Hilbert space. The following lemma is immediate.

Lemma 2.1. *For $u \in D$ and $v \in V$, $|a_1(u,v)| < \infty$ if and only if $\left|\{u,v\}_a^b\right| < \infty$. If either condition holds, then*

$$a_1(u,v) = \{u,v\}_a^b + \langle Lu, v \rangle . \tag{2.1}$$

There is a subspace $\widetilde{D} \subset D \cap V$ which is at least a two dimensional extension of D_0 such that, for $u \in \widetilde{D}$ and $v \in V$, $\left|\{u,v\}_a^b\right| < \infty$ [7].

The boundary condition operator B is taken to be compatible with a so-called Type I operator $\widehat{L}_1$ [3], [6], [7] in the space $L^2(I)$ with separated boundary conditions. It is a special type of self-adjoint operators and its domain $\widehat{D}$ is characterized by the existence of $d = 2$ functions $\varphi_1, \varphi_2 \in \widetilde{D}$ such that

1. φ_1, φ_2 are linearly independent modulo D_0
2. $\{\varphi_i, \varphi_j\}(a) = \{\varphi_i, \varphi_j\}(b) = 0$ for $i, j = 1, 2$
3. $\widehat{D} = D_0 \dotplus \operatorname{span}\{\varphi_1, \varphi_2\}$.

Therefore, we explicitly take

$$B(u) = \left(\{u, \varphi_1\}_a^b , \{u, \varphi_2\}_a^b \right), \quad u \in \widetilde{D}.$$

Notice that, because $d = 2$, for any λ in the resolvent set of the operator $\widehat{L}_1$, $\left(\widehat{L}_1 - \lambda E\right)^{-1}$ is completely continuous as an operator on $L^2(I)$. Define the positive definite Type I operator $\widehat{L}$ by

$$\begin{aligned} D\left(\widehat{L}\right) &= \widehat{D}, \\ \widehat{L} &= \widehat{L}_1 + E. \end{aligned}$$

By (2.1),

$$\left\langle \widehat{L}u, v \right\rangle = a(u, v), \ \forall u, v \in \widehat{D}.$$

For the (self adjoint) square root $\widehat{L}^{1/2}$ of $\widehat{L}$ we have

$$a(u, u) = \left\langle \widehat{L}u, u \right\rangle = \left\| \widehat{L}^{1/2} u \right\|^2, \ u \in \widehat{D}, \tag{2.2}$$

which means that $\widehat{L}^{1/2}$ is continuous on $\widehat{D}$ in the the norm of V. Thus, $\widehat{L}^{1/2}$ extends to a continuous operator on the closure W of $\widehat{D}$ with respect to the norm of V. Furthermore, for all $u, v \in W$,

$$\left\langle \widehat{L}^{1/2}u, \widehat{L}^{1/2}v \right\rangle = a(u, v).$$

Since

$$a(u, u) \geq \|u\|^2 \quad \forall u \in V, \tag{2.3}$$

$\left\| \widehat{L}^{1/2} u \right\| \geq \|u\|$ for all $u \in W$. It follows that, as an operator on $L^2(I)$, $0 \in \rho\left(\widehat{L}^{1/2}\right)$ (the resolvent set of $\widehat{L}^{1/2}$) and $\widehat{L}^{-1/2}$ is completely continuous. Finally we notice that $\widehat{L}^{-1/2}$ has the integral representation

$$\widehat{L}^{-1/2} = \int_0^\infty \frac{1}{\sqrt{\lambda}} \left(\widehat{L} + \lambda E\right)^{-1} d\lambda \tag{2.4}$$

(see [10], page 31).

We can now reformulate problem (1.1) in operator form as

$$\widehat{L}u = Fu. \tag{2.5}$$

Since $\widehat{L}^{1/2} : W \to L^2(I)$ is isometric (see (2.2)), $\widehat{L}^{-1/2} : L^2(I) \to W$ is also isometric.

Introduce the nonlinear operator $N : W \to L^2(I)$ defined by

$$Nu = \widehat{L}^{-1/2} Fu.$$

Then we can rewrite (2.5) in the form

$$\widehat{L}^{1/2} u = Nu. \tag{2.6}$$

In the sequel, M will stand for a generic constant whose value may change from line to line.

Theorem 2.2. *Suppose the operator $F : L^2(I) \to L^2(I)$ is continuous and bounded on $L^2(I)$. Then* (2.6) *has a solution.*

Proof. We will begin by showing that the operator N is compact. Since by assumption, F is bounded on $L^2(I)$, we have $\|Fu\| \leq M$ for all $u \in L^2(I)$. Furthermore, since $\widehat{L}^{-1/2}$ is continuous as an operator from $L^2(I)$ to $L^2(I)$, N is bounded on $L^2(I)$. To show equicontinuity in the norm of $L^2(I)$ we need to show that, for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$\|\tau_h Nu - Nu\| < \epsilon$$

for all $u \in L^2(I)$ and $|h| < \delta$; here τ_h is the shift operator

$$\tau_h u(t) = u(t+h),$$

and it is understood that functions are extended by 0 outside the interval I. Clearly, for every h, τ_h is a continuous operator on $L^2(I)$. Furthermore, since $\tau_h u \to u$ as $h \to 0$, it follows from the Banach-Steinhaus theorem that $\{\tau_h\}_{|h|\leq 1}$ is uniformly bounded.

It was proven in ([14], Theorem 7.7), that, for $d = 2$ and $-\lambda \in \rho\left(\widehat{L}\right)$,

$$\left(\widehat{L} + \lambda E\right)^{-1} g(t) = \int_I R(t,s,\lambda)\, g(s)\, ds,$$

where the kernel $R(t,s,\lambda)$ is given by

$$R(t,s,\lambda) = \begin{cases} \sum_{i,j=1}^{2} c_{ij}\psi_i(t,\lambda)\,\overline{\psi}_j(s,\lambda), & s < t \\ \sum_{i,j=1}^{2} d_{ij}\psi_i(t,\lambda)\,\overline{\psi}_j(s,\lambda), & s \geq t, \end{cases} \tag{2.7}$$

where $c_{ij}, d_{ij},\ i,j = 1,2$ are complex numbers and $\psi_i(\cdot,\lambda),\ i = 1,2$ are solutions of $(\ell + \lambda)u = 0$ (both are in $L^2(I)$ since $d = 2$). Also, both functions are continuously dependent on λ (see [15], Theorem 3.7). Since τ_h is a continuous operator on $L^2(I)$,

$$\begin{aligned} \tau_h Nu - Nu &= \tau_h \int_0^\infty \frac{1}{\sqrt{\lambda}} \left(\widehat{L} + \lambda E\right)^{-1} Fu d\lambda - \int_0^\infty \frac{1}{\sqrt{\lambda}} \left(\widehat{L} + \lambda E\right)^{-1} Fu d\lambda \\ &= \int_0^\infty \frac{1}{\sqrt{\lambda}} \left(\tau_h \left(\widehat{L} + \lambda E\right)^{-1} - \left(\widehat{L} + \lambda E\right)^{-1}\right) Fu d\lambda. \end{aligned}$$

Now

$$\begin{aligned} &\left|\left(\tau_h \left(\widehat{L} + \lambda E\right)^{-1} - \left(\widehat{L} + \lambda E\right)^{-1}\right) Fu(t)\right| \\ \leq &\left|\sum_{i,j=1}^{2} c_{ij}\left(\psi_i(t+h,\lambda) - \psi_i(t,\lambda)\right) \int_a^t \overline{\psi}_j(s,\lambda)\, Fu(s)\, ds\right| \\ &+ \left|\sum_{i,j=1}^{2} d_{ij}\left(\psi_i(t+h,\lambda) - \psi_i(t,\lambda)\right) \int_a^t \overline{\psi}_j(s,\lambda)\, Fu(s)\, ds\right| \end{aligned}$$

$$\begin{aligned}
&\leq \; cK(\lambda)\|Fu\|\left(\sum_{i,j=1}^{2}|\psi_i(t+h,\lambda)-\psi_i(t,\lambda)|\right) \\
&\leq \; MK(\lambda)\sum_{i=1}^{2}|\psi_i(t+h,\lambda)-\psi_i(t,\lambda)|,
\end{aligned}$$

where $c=\max\{|c_{ij}|,|d_{ij}|:i,j=1,2\}$ and $K(\lambda)=\max\{\|\psi_j(\cdot,\lambda)\|:j=1,2\}$. Notice that $K(\lambda)$ depends continuously on λ. Therefore,

$$\left\|\left(\tau_h\left(\widehat{L}+\lambda E\right)^{-1}-\left(\widehat{L}+\lambda E\right)^{-1}\right)Fu\right\| \leq MK(\lambda)\sum_{i=1}^{2}\|\tau_h\psi_i(\cdot,\lambda)-\psi_i(\cdot,\lambda)\|.$$

Next notice that

$$\begin{aligned}
\|\tau_h Nu-Nu\| &\leq \int_0^\infty \frac{1}{\sqrt{\lambda}}\left\|\left(\tau_h\left(\widehat{L}+\lambda E\right)^{-1}-\left(\widehat{L}+\lambda E\right)^{-1}\right)Fu\right\|d\lambda \\
&= \mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3,
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{I}_1 &= \int_0^c \frac{1}{\sqrt{\lambda}}\left\|\left(\tau_h\left(\widehat{L}+\lambda E\right)^{-1}-\left(\widehat{L}+\lambda E\right)^{-1}\right)Fu\right\|d\lambda, \\
\mathcal{I}_2 &= \int_c^d \frac{1}{\sqrt{\lambda}}\left\|\left(\tau_h\left(\widehat{L}+\lambda E\right)^{-1}-\left(\widehat{L}+\lambda E\right)^{-1}\right)Fu\right\|d\lambda, \\
\mathcal{I}_3 &= \int_d^\infty \frac{1}{\sqrt{\lambda}}\left\|\left(\tau_h\left(\widehat{L}+\lambda E\right)^{-1}-\left(\widehat{L}+\lambda E\right)^{-1}\right)Fu\right\|d\lambda,
\end{aligned}$$

with $c,d\in\mathbb{R}$ to be determined later.

We will establish that $\mathcal{I}_1,\mathcal{I}_2,\mathcal{I}_3$ can be made arbitrarily small for all $u\in L^2(I)$ by choosing c sufficiently small and d sufficiently large and for all sufficiently small h.

For $\mathcal{I}_1$ we have

$$\begin{aligned}
\mathcal{I}_1 &\leq (1+\|\tau_h\|)\|Fu\|\int_0^c \frac{1}{\sqrt{\lambda}}\left\|\left(\widehat{L}+\lambda E\right)^{-1}\right\|d\lambda \\
&= (1+\|\tau_h\|)\|Fu\|\int_0^c \frac{1}{\sqrt{\lambda}(\nu+\lambda)}d\lambda \\
&\leq \frac{1}{\nu}(1+\|\tau_h\|)\|Fu\|\int_0^c \frac{1}{\sqrt{\lambda}}d\lambda,
\end{aligned}$$

where $\nu>0$ is the smallest eigenvalue of $\widehat{L}$. Taking into account the uniform boundedness of the operators $\{\tau_h\}$ and the boundedness of the set $\{Fu\}$, we have

$$\mathcal{I}_1 \leq M\sqrt{c}.$$

Similarly, for $\mathcal{I}_3$ we have

$$\begin{aligned}
\mathcal{I}_3 &\leq (1+\|\tau_h\|)\|Fu\|\int_d^\infty \frac{1}{\sqrt{\lambda}}\left\|\left(\widehat{L}+\lambda\right)^{-1}\right\|d\lambda \\
&= (1+\|\tau_h\|)\|Fu\|\int_d^\infty \frac{1}{\sqrt{\lambda}(\nu+\lambda)}d\lambda
\end{aligned}$$

$$\begin{aligned} &\leq (1+\|\tau_h\|)\,\|Fu\|\int_d^\infty \frac{1}{\lambda^{3/2}}d\lambda \\ &\leq \frac{M}{\sqrt{d}}. \end{aligned}$$

For $\mathcal{I}_2$ we have

$$\begin{aligned} \mathcal{I}_2 &\leq M\sum_{i=1}^{2}\int_c^d K(\lambda)\,\|\tau_h\psi_i(\cdot,\lambda)-\psi_i(\cdot,\lambda)\|\,\frac{1}{\sqrt{\lambda}}d\lambda \\ &\leq M\sum_{i=1}^{2}\int_c^d \|\tau_h\psi_i(\cdot,\lambda)-\psi_i(\cdot,\lambda)\|\,d\lambda. \end{aligned}$$

Since for every $\lambda \in [c,d]$, $\|\tau_h\psi_i(\cdot,\lambda)-\psi_i(\cdot,\lambda)\| \to 0$ as $h \to 0$ and

$$\|\tau_h\psi_i(\cdot,\lambda)-\psi_i(\cdot,\lambda)\| \leq (1+\|\tau_h\|)\,\|\psi_i(\cdot,\lambda)\| \leq M,$$

it follows from the Lebesgue dominated convergence theorem that $\mathcal{I}_2 \to 0$ as $h \to 0$. Thus it can be made arbitrarily small by choosing h sufficiently small.

This establishes the compactness of the operator N. The continuity of N follows from the continuity of F since

$$\begin{aligned} \|Nu-Nv\| &= \left\|\int_0^\infty \frac{1}{\sqrt{\lambda}}\left(\widehat{L}+\lambda E\right)^{-1}(Fu-Fv)\,d\lambda\right\| \\ &\leq \|Fu-Fv\|\int_0^\infty \frac{1}{\sqrt{\lambda}}\left\|\left(\widehat{L}+\lambda E\right)^{-1}\right\|d\lambda \\ &= \|Fu-Fv\|\int_0^\infty \frac{1}{\sqrt{\lambda}\,(\nu+\lambda)}d\lambda = \frac{\pi}{\sqrt{\nu}}\,\|Fu-Fv\|\,. \end{aligned}$$

To prove the existence of solutions, we note that any solution of (1.1) is a fixed point of $\widehat{L}^{-1/2}N$. Since $N : W \to L^2(I)$ is compact and $\widehat{L}^{-1/2} : L^2(I) \to W$ is continuous, $\widehat{L}^{-1/2}N : W \to W$ is compact. By the Schauder fixed point theorem, $\widehat{L}^{-1/2}N$ has a fixed point. □

3. UPPER AND LOWER SOLUTIONS AND EXISTENCE

We begin this section by introducing the definition of upper and lower solutions for Type I operators and then we show that if (1.1) has upper and lower solutions then it has a solution.

Definition 3.1. A function $\beta \in \widehat{D}$ ($\alpha \in \widehat{D}$) is called an upper (a lower) solution of (1.1) if $\ell(\beta) \geq f(\cdot,\beta)$ ($\ell(\alpha) \leq f(\cdot,\alpha)$) almost everywhere on I.

Theorem 3.2. *Suppose $F : L^2(I) \to L^2(I)$ is continuous on $L^2(I)$. Suppose further that α, β are lower and upper solutions of* (1.1) *such that $\alpha \leq \beta$ on I and that $F([\alpha,\beta])$ is bounded. Then* (1.1) *has a solution $u \in \widehat{D}$ such that*

$$\alpha \leq u \leq \beta \quad \text{on } I.$$

Proof. Consider the modified problem

$$\widehat{L}u = F^* u \tag{3.1}$$

where $F^* : L^2(I) \to L^2(I)$ is defined by

$$F^* u = F(\beta \wedge u \vee \alpha).$$

Notice that the operators $\vee, \wedge : L^2(I) \times L^2(I) \to L^2(I)$ are continuous. For example, the operator $\vee$ can be written in terms of continuous operators as

$$f \vee g = \frac{|f-g| + (f-g)}{2}.$$

Therefore, F^* is continuous and bounded on $L^2(I)$. By Theorem 2.2, equation (3.1) has a solution $u \in \widehat{D}$. We claim that $u \le \beta$ on I. If not then let $I_1 = (\gamma, \delta)$ be the maximal interval on which $u > \beta$. Let $z = u - \beta$ and $z_1 = z \vee 0$. Then $z \in \widehat{D}$ and

$$\begin{aligned} 0 &< \|z_1\|^2 \le a(z_1, z_1) = a(z, z_1) \\ &= \{z, z_1\}_\gamma^\delta + \left\langle \widehat{L}z, z_1 \right\rangle = \left\langle \widehat{L}z, z_1 \right\rangle \\ &= \left\langle F^* u - \widehat{L}\beta, z_1 \right\rangle = \left\langle F\beta - \widehat{L}\beta, z_1 \right\rangle \le 0, \end{aligned}$$

which is a contradiction. In a similar fashion we can show that $u \ge \alpha$ on I. It follows that $F^* u = Fu$ and u is a solution of (1.1). □

For the next theorem we need a slight generalization of the definition of convex functions.

Definition 3.3. Suppose $\Omega \subset L^2(I)$ is a convex set and $H : \Omega \to L^2(I)$ is real (i.e., $Hu \subset \mathbb{R}$ whenever $u : I \to \mathbb{R}$). We will say that H is convex if

$$H((1-\theta)u + \theta v) \le (1-\theta)Hu + \theta Hv \tag{3.2}$$

for all $\theta \in [0,1]$, $u, v \in \Omega$.

Theorem 3.4. *Let $\Omega \subset L^2(I)$ be convex and suppose that the operator $H : \Omega \to L^2(I)$ has a Gateaux derivative H' on Ω. The following are equivalent*

1. *H is convex*
2. *$Hu \ge Hv + H'u(v-u)$ for all $u, v \in \Omega$.*

Proof. The proof is a simple adaptation of Theorem 4.3.16 in [1]. □

Suppose $F : L^2(I) \to L^2(I)$ is continuous on $L^2(I)$. Suppose further that $\alpha_0, \beta_0 \in \widehat{D}$ are lower and upper solutions of (1.1), respectively such that $\alpha_0 \le \beta_0$ on I and that $F([\alpha_0, \beta_0])$ is bounded. Let $H : [\alpha_0, \beta_0] \to L^2(I)$ be the operator

$$Hu = \gamma(\langle u, u_0 \rangle)^2 w_0,$$

where $u_0, w_0 \in L^2(I), w_0 \geq 0$ and $\gamma > 0$. Then H is continuous, convex and its first and second Fréchet derivatives are

$$\begin{aligned} H'uv &= 2\gamma \langle u, u_0 \rangle \langle v, u_0 \rangle w_0, \\ H''uvw &= 2\gamma \langle v, u_0 \rangle \langle w, u_0 \rangle w_0, \ \forall v, w \in L^2(I). \end{aligned}$$

Clearly

$$\begin{aligned} \|H'u\| &\leq 2\gamma \|u_0\|^2 \|w_0\| \|\beta\|, \\ \|H''u\| &\leq 2\gamma \|u_0\|^2 \|w_0\|, \ \forall u \in [\alpha_0, \beta_0]. \end{aligned}$$

In other words, H' and H'' are uniformly bounded on $[\alpha_0, \beta_0]$. Observe that the representation

$$H(u) = H(v) + H'(v)(u - v) + \int_0^1 \frac{(1-\tau)^2}{2} H''(v + \tau(u - v))(u - v)^2 \, d\tau \tag{3.3}$$

is valid.

Define the operators $\Phi : [\alpha_0, \beta_0] \to L^2(I)$ by

$$\Phi u = Hu - Fu$$

and $G : [\alpha_0, \beta_0] \times [\alpha_0, \beta_0] \to L^2(I)$ by

$$G(u, v) = Fv + H'v(u - v) - [\Phi u - \Phi v]. \tag{3.4}$$

It is easy to check that Φ and G are continuous. Furthermore, by Theorem 3.4,

$$Fu \geq G(u, v)$$

for all $u, v \in [\alpha_0, \beta_0]$. Using the representation (3.3) for H, the operator G of (3.4) can be written as

$$G(u, v) = Fu - \int_0^1 \frac{(1-\tau)^2}{2} H''(v + \tau(u - v))(u - v)^2 \, d\tau.$$

Consider the equation

$$\widehat{L}u = G(u, \alpha_0). \tag{3.5}$$

By Theorem 2.2 and Theorem 3.2, equation (3.5) has a solution $\alpha_1 \in [\alpha_0, \beta_0]$. Furthermore,

$$\widehat{L}\alpha_1 = G(\alpha_1, \alpha_0) \leq F\alpha_1.$$

Therefore, α_1, β_0 are lower and upper solutions of (1.1). Repeating the same step with α_0 replaced by α_1 in (3.5) we obtain a lower solution α_2 of (1.1) with $\alpha_0 \leq \alpha_1 \leq \alpha_2 \leq \beta_0$. Continuing in this manner we obtain a sequence of functions $\{\alpha_n\}$ in $\widehat{D} \cap [\alpha_0, \beta_0]$ such that

$$\begin{aligned} \widehat{L}\alpha_n &= G(\alpha_n, \alpha_{n-1}), \\ \alpha_0 &\leq \alpha_1 \leq \cdots \leq \alpha_n \leq \cdots \leq \beta_0. \end{aligned}$$

Theorem 3.5. *Suppose $F : L^2(I) \to L^2(I)$ is continuous on $L^2(I)$. Suppose further that $\alpha_0, \beta_0 \in \widehat{D}$ are lower and upper solutions of (1.1), respectively such that $\alpha_0 \le \beta_0$ on I and that $F([\alpha_0, \beta_0])$ is bounded. Let $\{\alpha_n\}$ be the monotone sequence generated in the manner described above. Then $\{\alpha_n\}$ converges in V to a solution α of (1.1). Furthermore, if F is also monotone on $[\alpha_0, \beta_0]$; that is*

$$\langle Fu - Fv, u - v\rangle \le 0 \ \forall u, v \in [\alpha_0, \beta_0],$$

then the convergence $\alpha_n \to \alpha$ is quadratic.

Proof. The monotonicity of the sequence $\{\alpha_n\}$ and its boundedness insure the existence of a pointwise limit α. Then $\alpha_0 \le \alpha \le \beta_0$. Therefore, $\alpha \in L^2(I)$. The Lebesgue Dominated Convergence Theorem then implies that $\alpha_n \to \alpha$ in the norm of $L^2(I)$. Since we also have $\alpha \in [\alpha_0, \beta_0]$, and since G is continuous in the norm of $L^2(I)$, we get $G(\alpha_n, \alpha_{n-1}) \to G(\alpha, \alpha) = F\alpha$. Finally, since $\widehat{L}$ is closed, $\alpha \in \widehat{D}$ and $\widehat{L}\alpha = F\alpha$.

Let $e_n = \alpha - \alpha_n$. Then

$$a(e_n, e_n) = \left\langle \widehat{L}e_n, e_n \right\rangle = \langle F\alpha - G(\alpha_n, \alpha_{n-1}), e_n\rangle \to 0.$$

Furthermore, if F is also monotone on $[\alpha_0, \beta_0]$ then

$$\begin{aligned} \widehat{L}e_n &= F\alpha - G(\alpha_n, \alpha_{n-1}) \\ &= F\alpha - F\alpha_n \\ &\quad + \int_0^1 \frac{(1-\tau)^2}{2} H''(\alpha_{n-1} + \tau(\alpha_n - \alpha_{n-1}))(\alpha_n - \alpha_{n-1})^2 \, d\tau. \end{aligned}$$

Therefore, using the monotonicity assumption on F, we get

$$\begin{aligned} \|e_n\|^2 &\le a(e_n, e_n) = \left\langle \widehat{L}e_n, e_n \right\rangle \\ &\le \int_0^1 \frac{(1-\tau)^2}{2} \|H''(\alpha_{n-1} + \tau(\alpha_n - \alpha_{n-1}))\| \, \|\alpha_n - \alpha_{n-1}\|^2 \|e_n\| \, d\tau \\ &\le M \|H''(u)\| \, \|\alpha_n - \alpha_{n-1}\|^2 \|e_n\|. \end{aligned}$$

Hence,

$$\|e_n\| \le M \|\alpha_n - \alpha_{n-1}\|^2.$$

Since $\alpha_{n-1} \le \alpha_n \le \alpha$, we have $\alpha_n - \alpha_{n-1} \le \alpha - \alpha_{n-1}$. Therefore, $\|\alpha_n - \alpha_{n-1}\| \le \|e_{n-1}\|$ and so

$$\|e_n\| \le M \|e_{n-1}\|^2.$$

□

4. THE CASE WITH ONE REGULAR ENDPOINT

In this section we consider the case when the point a is a singular endpoint and b is a regular endpoint for ℓ. Then, for any $u \in D$, $u(b)$ and $pu'(b)$ both exist and are finite (see [15], Theorem 2.9). Since assigning boundary conditions at the singular point is not always possible, we will discuss here existence theorems for boundary value problems of the form

$$\begin{aligned} \ell(u) &= f(\cdot, u), \\ g(u(b)) &= 0, \end{aligned} \tag{4.1}$$

where we assume that the function $g : \mathbb{R} \to \mathbb{R}$ is continuous and that there exists a closed and bounded interval $J_0 \subset \mathbb{R}$ such that the interval

$$J_1 = \{t + g(t) : t \in J_0\} \subseteq J_0.$$

Since the function $t \mapsto t + g(t)$ has a fixed point in J_0, g has a zero in J_0. This is obviously a necessary condition for the existence of a solution of (4.1).

The most general separated self adjoint boundary condition [15] at the regular endpoint b is

$$\cos\sigma u(b) + \sin\sigma pu'(b) = 0, \ \sigma \in [0, \pi).$$

For the purposes of this section we will be working with a Type I operator $\widehat{L}$ that corresponds to $\sigma = 0$. Therefore, the functions in its domain $\widehat{D}$ satisfy the boundary conditions $u(b) = 0$ and $\{u, v\}(a) = 0$ for all $u, v \in \widehat{D}$. We will be looking for solutions of (4.1) of the form $v = u + sy$ where $u \in \widehat{D}, s \in R$ and $y \in D$ is suitably chosen. Specifically, we choose $y \in D$ to be the solution of the initial value problem

$$\begin{aligned} (\ell + 1)y = 0, \\ y(b) = 1, py'(b) = 0. \end{aligned} \tag{4.2}$$

This choice is possible since all solutions of $(\ell + 1)y = 0$ are in $L^2(I)$. In what follows we will establish that $y \notin W$.

Lemma 4.1. *Let $y \in D$ be the function given by* (4.2), *then $y \notin W$.*

Proof. Suppose, to the contrary that $y \in W$. Define the space

$$W_0 = \{w \in W : w = 0 \text{ near } a\}.$$

Observe that W_0 is dense in $L^2(I)$ (it contains $C_0^\infty(I)$). Furthermore, $\widehat{L}^{1/2} W_0$ is dense in $L^2(I)$. To see this assume $\theta \in \left(\widehat{L}^{1/2} W_0\right)^\perp$ and let $\theta = \widehat{L}_1^{-1/2}\theta_1$. Then

$$\left\langle \theta, \widehat{L}^{1/2} w \right\rangle = 0 \ \forall w \in W_0,$$

i.e.,

$$\left\langle \widehat{L}_1^{-1/2}\theta_1, \widehat{L}^{1/2} w \right\rangle = 0 \ \forall w \in W_0.$$

Since $\widehat{L}^{-1/2}$ is self-adjoint, then

$$\langle \theta_1, w \rangle = 0 \; \forall w \in W_0.$$

Therefore, $\theta_1 = 0$, and consequently, $\theta = 0$. Now for any $w \in W_0$, using the boundary conditions for y and w we get

$$\left\langle \widehat{L}^{1/2} y, \widehat{L}^{1/2} w \right\rangle = a(y, w) = \{y, w\}_a^b + \langle (L+E) y, w \rangle = 0.$$

Therefore, $\widehat{L}^{1/2} y \in \left(\widehat{L}^{1/2} W_0 \right)^{\perp}$. Thus y is in the kernel of $\widehat{L}^{1/2}$ which is a contradiction since $\widehat{L}^{1/2}$ is invertible. □

Next, define the space W_1 by

$$W_1 = W \dotplus \operatorname{span}\{y\}.$$

Then W_1 can be given the structure of a Hilbert space with the inner product

$$\langle u + sy, v + ty \rangle = \langle u, v \rangle_V + s\bar{t}.$$

Let $s_0 \in I$ be a zero of the function g. Define the operator $\widetilde{F}$ on W_1 by

$$\widetilde{F}(u) = F(u + s_0 y).$$

Then (4.1) can be formalized in the domain $\widehat{D}$ as

$$\widehat{L} u_1 = \widetilde{F}(u_1). \tag{4.3}$$

If (4.3) has a solution u_1 then, letting $u = u_1 + s_0 y$ we get

$$\begin{aligned} (\ell + 1) u &= (L + E)(u_1 + s_0 y) \\ &= \widehat{L} u_1 = \widetilde{F}(u_1) \\ &= F(u_1 + s_0 y) = F(u) \end{aligned}$$

and $u(b) = s_0$ so that $g(u(b)) = 0$. Therefore, u is a solution of (4.1). Conversely, we can show that if u is a solution of (4.1) then $u_1 = u - u(b) y$ is a solution of (4.3)

The definitions of upper and lower solutions are modified as follows.

Definition 4.2. A function $\widetilde{\beta} \in \widehat{D} \dotplus \operatorname{span}\{y\}$ ($\widetilde{\alpha} \in \widehat{D} \dotplus \operatorname{span}\{y\}$) will be called an upper (a lower) solution of (4.1) if $\ell \widetilde{\beta} \geq f(\cdot, \widetilde{\beta})$ ($\ell \widetilde{\alpha} \leq f(\cdot, \widetilde{\alpha})$) almost everywhere in I.

If $\widetilde{\alpha}$ is a lower solution for (4.1) and if we put $\widetilde{\alpha} = \alpha + s_0 y$ with $\alpha \in \widehat{D}$ then

$$\widehat{L} \alpha = (L + E)(\widetilde{\alpha} - s_0 y) = (L + E) \widetilde{\alpha} \leq F(\widetilde{\alpha}) = \widetilde{F}(\alpha).$$

Therefore, α is a lower solution of (4.3) in the sense of Definition 3.1. A similar remark holds for upper solutions.

With these remarks we will now work out the equivalents of Theorem 2.2, Theorem 3.2 and Theorem 3.5. It can be easily checked that the mapping $T : W_1 \to L^2(I) \times \mathbb{C}$ defined by

$$T(u + sy) = \left(\widehat{L}^{1/2}u, s\right) \tag{4.4}$$

is an onto isometry. Define the operator $\widetilde{N} : W \dotplus J_0 y \to L^2(I) \times J_0$ by

$$\widetilde{N}(u + sy) = (N(u + sy), s + g(s))$$

Proposition 4.3. *Suppose that the operator equation*

$$T(u + sy) = \widetilde{N}(u + sy) \tag{4.5}$$

has a solution $v \in W_1$. Then v is a solution of (4.1).

Proof. Write $v = u + sy$. Then

$$\widehat{L}^{1/2}u = \widehat{L}^{-1/2}F(u + sy)$$

and

$$s = s + g(s).$$

The first equation above gives that $u \in \widehat{D}$ and the second equation gives

$$g(v(b)) = g(s) = 0.$$

Furthermore,

$$\begin{aligned} (\ell + 1)v &= (L + E)v = (L + E)(u + sy) \\ &= \widehat{L}u = F(u + sy) = Fv. \end{aligned}$$

Thus, v is a solution of (4.1). □

Theorem 4.4. *Suppose the operator $F : L^2(I) \to L^2(I)$ is continuous and bounded. Then the operator $\widetilde{N}$ is compact and continuous. Consequently equation* (4.5) *has a solution.*

Proof. Since N is compact and continuous and $(u + sy) \mapsto s + g(s)$ is continuous and bounded, $\widetilde{N}$ is compact and continuous. Therefore, the operator $T^{-1}\widetilde{N} : W \dotplus J_0 y \to W \dotplus J_0 y$ is compact and continuous. By the Schauder fixed point theorem it has a fixed point. □

Theorem 4.5. *Suppose $F : L^2(I) \to L^2(I)$ is continuous on $L^2(I)$. Suppose further that*

1. *$\widetilde{\alpha}, \widetilde{\beta}$ are lower and upper solutions of* (4.1) *such that $\widetilde{\alpha} \le \widetilde{\beta}$on I*
2. *$g^{-1}(0) \subseteq \left[\widetilde{\alpha}(b), \widetilde{\beta}(b)\right]$.*
3. *$F\left(\left[\widetilde{\alpha}, \widetilde{\beta}\right]\right)$ is bounded.*

Then (4.1) *has a solution* $u \in W_1$ *such that*

$$\widetilde{\alpha} \leq u \leq \widetilde{\beta} \quad \text{on } I.$$

Proof. Define the operator F^* by

$$F^* u = F\left(\widetilde{\alpha} \vee u \wedge \widetilde{\beta}\right)$$

and consider the operator equation

$$T_1 u = \widetilde{N} u,$$

where, in the definition of $\widetilde{N}$, F is replaced by F^*. By Theorem 4.4 this problem has a solution $u \in W \dot{+} J_0 y$. We may write $u = v + u(b)\,y$ with $v \in W$ and observe that $g(u(b)) = 0$. Assumption 2 implies that $\widetilde{\alpha}(b) \leq u(b) \leq \widetilde{\beta}(b)$. Using an argument similar to that of Theorem 3.2 we can show that $u \in \left[\widetilde{\alpha}, \widetilde{\beta}\right]$. Furthermore,

$$(L + E)\,u = \widehat{L} v = F^*\left(\widetilde{\alpha} \wedge u \vee \widetilde{\beta}\right) = F u.$$

Hence, u is a solution of (4.1). □

Finally we notice that the parallel of Theorem 3.5 is also obtainable in the case (4.1) under similar assumptions and the appropriate modification of Assumption 2 in Theorem 4.5, namely, something like $g^{-1}(0) \in \left[\widetilde{\alpha}_0(b), \widetilde{\beta}_0(b)\right]$.

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