Dynamic Systems and Applications 21 (2012) 83-92

COMPARISON AND OSCILLATION THEOREM FOR SECOND-ORDER NONLINEAR NEUTRAL DIFFERENCE EQUATIONS OF MIXED TYPE

E. THANDAPANI, N. KAVITHA, AND S. PINELAS

Ramanujan Institute for Advanced Study in Mathematics,
University of Madras, Chennai 600 005, India
Departamento de Matemática, Universidade dos Açores,
Ponta Delgada, Portugal

ABSTRACT. In this paper, we establish some comparison theorems for the oscillation of second order neutral difference equations of mixed type

$$\Delta\left(a_n\Delta\left(x_n+b_nx_{n-\sigma_1}+c_nx_{n+\sigma_2}\right)^{\alpha}\right)+q_nx^{\beta}_{n-\tau_1}+p_nx^{\beta}_{n+\tau_2}=0,$$

where α and β are ratio of odd positive integers, σ_1, σ_2, τ_1 and τ_2 are positive integers. Our results are new even if $p_n=c_n=0$. Examples are provided to illustrate the results.

AMS (MOS) Subject Classification. 39A10.

1. INTRODUCTION

In this paper, we shall study the oscillatory behavior of the second order nonlinear neutral difference equation of mixed type

$$\Delta\left(a_n\Delta\left(x_n+b_nx_{n-\sigma_1}+c_nx_{n+\sigma_2}\right)^{\alpha}\right)+q_nx^{\beta}_{n-\tau_1}+p_nx^{\beta}_{n+\tau_2}=0, \tag{1.1}$$

where $n\geq n_0\in\mathbb{N}$, subject to the following conditions:

(H1) $\{a_n\}$ is a positive sequence for all $n\geq n_0$ and $\sum\limits_{n=n_0}^{\infty}\frac{1}{a_n}=\infty$;

(H2) $\{b_n\}$ and $\{c_n\}$ are nonnegative sequences such that $0\leq b_n\leq b$ and $0\leq c_n\leq c$, where b and c are constants;

(H3) $\{p_n\}$ and $\{q_n\}$ are nonnegative real sequences and not eventually zero for many values of n;

(H4) σ_1, σ_2, τ_1 and τ_2 are nonnegative integers and α and β are ratio of odd positive integers.

We put $z_n=\left(x_n+b_nx_{n-\sigma_1}+c_nx_{n+\sigma_2}\right)^{\alpha}$. By a solution of equation (1.1), we mean a real sequence $\{x_n\}$ defined for all $n\geq n_0-\max\{\sigma_1,\tau_1\}$, and satisfies equation (1.1) for

Received January 19, 2012

all $n \geq n_0$. As is customary, a solution $\{x_n\}$ of equation (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, and nonoscillatory otherwise.

Recently, there has been much research activity concerning the oscillation and nonoscillation of solutions of various types of second order difference equations because such equations have applications in various problems of physics, biology, economy and many other fields. We refer the reader to [1,2] and the references cited therein.

In [3, 4] the authors studied the oscillation of mixed type equations of the form

$$\Delta^2 (x_n + ax_{n-\sigma_1} + bx_{n+\sigma_2}) = q_n x_{n-\tau_1} + p_n x_{n+\tau_2} \tag{1.2}$$

and

$$\Delta^2 (x_n + ax_{n-\sigma_1} + bx_{n+\sigma_2}) + q_n x_{n-\tau_1} + p_n x_{n+\tau_2} = 0 \tag{1.3}$$

with $\{q_n\}$ and $\{p_n\}$ are σ_1- periodic functions.

In [10] the author discussed the oscillation of mixed type equations of the form

$$\Delta^2 (x_n + ax_{n-\sigma_1} + bx_{n+\sigma_2}) \pm (q_n x_{n-\tau_1} + p_n x_{n+\tau_2}) = 0 \tag{1.4}$$

In [5, 9] the author studied the oscillatory behavior of higher order mixed type neutral difference equations of the form (1.4). Motivated by the above observation in this paper we establish sufficient conditions for the oscillation of all solutions of equation (1.1). The results obtained here generalize and improve the existing literature [3, 4, 7] . Further when $\alpha = 1$, $c_n \equiv 0$ and $p_n \equiv 0$, our results improve some of the results established for the equation

$$\Delta (a_n \Delta (x_n + b_n x_{n-\sigma_1})) + q_n x^{\beta}_{n-\tau_2} = 0 \tag{1.5}$$

see, for example [6, 8, 11, 13–15].

In section 2, we present some new sufficient conditions for the oscillation of all solutions of equation (1.1). In section 3 we provide some examples to illustrate the main results.

2. OSCILLATION RESULTS

In this section, we establish some new oscillation criteria for equation (1.1). Throughout this paper, we denote

$$Q^*_n = Q_n + P_n,$$

$$Q_n = \min \{q_n, q_{n-\sigma_1}, q_{n+\sigma_2}\},$$

$$P_n = \min \{p_n, p_{n-\sigma_1}, p_{n+\sigma_2}\},$$

and

$$R_n = \sum_{s=n_0}^{n-1} \frac{1}{a_s}.$$

To prove our main results, we need the following lemmas.

Lemma 2.1. *Let $A \geq 0,\ B \geq 0,\ \gamma \geq 1$. Then*

$$(A+B)^\gamma \leq 2^{\gamma-1}\left(A^\gamma + B^\gamma\right).$$

Proof. (i) If $A = 0$ or $B = 0$ then the proof is obvious.
(ii) Suppose that $A > 0$, $B > 0$. Define the function by

$$g(u) = u^\gamma, u \in (0,\infty).$$

Then $g''(u) = \gamma(\gamma-1)u^{\gamma-2} \geq 0$ for $u \geq 0$. Thus, g is a convex function. By the definition of convex function, for $\lambda = 1/2,\ A,\ B \in (0,\infty)$, we have

$$g\left(\frac{A+B}{2}\right) \leq \frac{g(A)+g(B)}{2}$$

This completes the proof. □

Lemma 2.2. *Assume $A \geq 0,\ B \geq 0,\ 0 < \gamma \leq 1$. Then*

$$(A+B)^\gamma \leq A^\gamma + B^\gamma.$$

Proof. (i) If $A = 0$ or $B = 0$ then the proof is obvious.
(ii) Assume that $A > 0$, $B > 0$. Define

$$g(A,B) = A^\gamma + B^\gamma - (A+B)^\gamma,\ A,B \in (0,\infty).$$

Fix A. Then for $0 < \gamma < 1$

$$\frac{dg(A,B)}{dB} = \gamma B^{\gamma-1} - \gamma(A+B)^{\gamma-1} = \gamma[B^{\gamma-1} - (A+B)^{\gamma-1}] \geq 0,$$

Thus, f is nondecreasing with respect to B, which yields $g(A,B) \geq 0$. The proof of the lemma is complete. □

Theorem 2.3. *Assume that $\beta \geq 1$ and*

$$\Delta\left(y_n + b^\beta\ y_{n-\sigma_1} + \frac{c^\beta}{2^{\beta-1}}\ y_{n+\sigma_2}\right) + \frac{Q_n^*}{4^{\beta-1}}\ R_{n-\tau_1}^{\beta/\alpha}\ y_{n-\tau_1}^{\beta/\alpha} \leq 0 \tag{2.1}$$

has no eventually positive solution for all sufficiently large $n \geq n_0$. Then every solution of equation (1.1) *is oscillatory.*

Proof. Let $\{x_n\}$ be a nonoscillatory solution of equation (1.1). Without loss of generality, we may assume that there is an integer $n_1 \geq n_0$ such that $x_n > 0,\ x_{n-\sigma_1} > 0,\ x_{n+\sigma_2} > 0,\ x_{n-\tau_1} > 0$, and $x_{n+\tau_2} > 0$ for all $n \geq n_1$. Then $z_n > 0$ for all $n \geq n_1$. Inview of equation (1.1), we obtain

$$\Delta(a_n \Delta z_n) = -q_n x_{n-\tau_1}^\beta - p_n x_{n+\tau_2}^\beta \leq 0, \quad n \geq n_1. \tag{2.2}$$

Thus, $a_n\Delta z_n$ is nonincreasing, and it is easy to conclude that either $\Delta z_n > 0$ or $\Delta z_n < 0$ eventually. If there exists a $n_2 \geq n_1$ such that $\Delta z_{n_2} < 0$, then from (2.2), we see that

$$a_n\Delta z_n \leq a_{n_2}\Delta z_{n_2} < 0, \quad n \geq n_2.$$

Summing the last inequality from n_2 to $n-1$, we obtain

$$z_n \leq z_{n_2} + a_{n_2}\Delta z_{n_2}\sum_{s=n_2}^{n-1}\frac{1}{a_s}.$$

Letting $n \to \infty$, we obtain $z_n \to -\infty$ due to $(H1)$ which is a contradiction. Thus, there is an integer $n_2 \geq n_1$such that

$$\Delta z_n > 0 \tag{2.3}$$

for all $n \geq n_2$. From equation (1.1) for sufficiently large n, we have

$$\Delta(a_n\Delta z_n) + q_n x^{\beta}_{n-\tau_1} + p_n x^{\beta}_{n+\tau_2} + b^{\beta}\Delta(a_{n-\sigma_1}\Delta z_{n-\sigma_1}) + b^{\beta}q_{n-\sigma_1}x^{\beta}_{n-\tau_1-\sigma_1}$$

$$+b^{\beta}p_{n-\sigma_1}x^{\beta}_{n+\tau_2-\sigma_1} + \frac{c^{\beta}}{2^{\beta-1}}\Delta(a_{n+\sigma_2}\Delta z_{n+\sigma_2}) + \frac{c^{\beta}}{2^{\beta-1}}q_{n+\sigma_2}x^{\beta}_{n-\tau_1+\sigma_2}$$

$$+\frac{c^{\beta}}{2^{\beta-1}}p_{n+\sigma_2}x^{\beta}_{n+\tau_2+\sigma_2} = 0. \tag{2.4}$$

Using Lemma 2.1, we have from (2.4)

$$\Delta(a_n\Delta z_n) + b^{\beta}\Delta\left(a_{n-\sigma_1}\Delta z_{n-\sigma_1}\right) + \frac{c^{\beta}}{2^{\beta-1}}\Delta\left(a_{n+\sigma_2}\Delta z_{n+\sigma_2}\right) + \frac{Q_n}{4^{\beta-1}}z^{\beta/\alpha}_{n-\tau_1}$$

$$+\frac{P_n}{4^{\beta-1}}z^{\beta/\alpha}_{n+\tau_2} \leq 0. \tag{2.5}$$

From (2.3), we have $z_{n+\tau_2} \geq z_{n-\tau_1}$. Then from (2.5), we obtain

$$\Delta(a_n\Delta z_n) + b^{\beta}\Delta\left(a_{n-\sigma_1}\Delta z_{n-\sigma_1}\right) + \frac{c^{\beta}}{2^{\beta-1}}\Delta\left(a_{n+\sigma_2}\Delta z_{n+\sigma_2}\right) + \frac{Q^*_n}{4^{\beta-1}}z^{\beta/\alpha}_{n-\tau_1} \leq 0. \tag{2.6}$$

It follows from (2.2) that

$$z_n = z_{n_2} + \sum_{s=n_2}^{n-1}\frac{a_s\Delta z_s}{a_s} \geq a_n\Delta z_n R_n. \tag{2.7}$$

Set $y_n = a_n\Delta z_n > 0$. From (2.6) and (2.7), we see that $\{y_n\}$is an eventually positive solution of

$$\Delta\left(y_n + b^{\beta}y_{n-\sigma_1} + \frac{c^{\beta}}{2^{\beta-1}}y_{n+\sigma_2}\right) + \frac{Q^*_n}{4^{\beta-1}}\, R^{\beta/\alpha}_{n-\tau_1}\, y^{\beta/\alpha}_{n-\tau_1} \leq 0,$$

which is a contradiction. This completes the proof. □

Theorem 2.4. *Assume that* $\beta \geq 1$ *and*

$$\Delta u_n + \frac{Q_n^* \ R_{n-\tau_1}^{\beta/\alpha}}{4^{\beta-1}\left(1 + b^\beta + \dfrac{c^\beta}{2^{\beta-1}}\right)} \ u_{n+\sigma_1-\tau_1}^{\beta/\alpha} \leq 0 \tag{2.8}$$

has no eventually positive solution for all sufficiently large $n \geq n_0$. *Then every solution of equation* (1.1) *is oscillatory.*

Proof. Let $\{x_n\}$ be a nonoscillatory solution of equation (1.1). Without loss of generality, we may assume that there is an integer $n_1 \geq n_0$ such that $x_n > 0$, $x_{n-\sigma_1} > 0$, $x_{n+\sigma_2} > 0$, $x_{n-\tau_1} > 0$, $x_{n+\tau_2} > 0$ for all $n \geq n_1$. Then $z_n > 0$ for all $n \geq n_1$. Proceeding as in the proof of Theorem 2.1, we obtain that $y_n = a_n \Delta z_n > 0$ is nonincreasing and satisfies inequality (2.1). Define

$$u_n = y_n + b^\beta y_{n-\sigma_1} + \frac{c^\beta}{2^{\beta-1}} y_{n+\sigma_2} > 0.$$

Then

$$u_n \leq \left(1 + b^\beta + \frac{c^\beta}{2^{\beta-1}}\right) y_{n-\sigma_1}.$$

Substituting the above inequality into (2.1), we see that $\{u_n\}$ is an eventually positive solution of (2.8). This contradiction completes the proof. □

From Theorem 2.2 and [10] and [12], we establish the following corollaries.

Corollary 2.5. *Assume* $\alpha = \beta \geq 1$, *and* $\sigma_1 - \tau_1 < 0$ *holds. If*

$$\liminf_{n\to\infty} \sum_{s=n+\sigma_1-\tau_1}^{n-1} Q_s^* \ R_{s-\tau_1} > 4^{\beta-1}\left(1 + b^\beta + \frac{c^\beta}{2^{\beta-1}}\right)\left(\frac{\tau_1 - \sigma_1}{1 + \tau_1 - \sigma_1}\right)^{1+\tau_1-\sigma_1} \tag{2.9}$$

then every solution of equation (1.1) *is oscillatory.*

Proof. The proof follows from Theorem 2.2 and a resulting in [10]. □

Corollary 2.6. *Assume* $1 \leq \beta < \alpha$ *and* $\sigma_1 - \tau_1 < 0$ *holds. If*

$$\sum_{n=n_0}^{\infty} Q_s^* \ R_{s-\tau_1}^{\beta/\alpha} = \infty \tag{2.10}$$

then every solution of equation (1.1) *is oscillatory.*

Proof. The proof follows from Theorem 2.2 and [12, Theorem 1]. □

Corollary 2.7. *Assume* $1 \leq \beta < \alpha$ *and* $\sigma_1 - \tau_1 < 0$ *holds. If there exists a* $\lambda > \frac{1}{(\tau_1-\sigma_1)} \log(\beta/\alpha)$ *such that*

$$\liminf_{n\to\infty} \left[Q_n^* \ R_{n-\tau_1}^{\beta/\alpha} \exp\left(-e^{\lambda n}\right)\right] > 0 \tag{2.11}$$

then every solution of equation (1.1) *is oscillatory.*

Proof. The proof follows from Theorem 2.2 and [12, Theorem 2]. □

Theorem 2.8. *Assume that* $\beta \geq 1$ *holds and*

$$\Delta w_n - \frac{Q^*_{n+\sigma_1}}{4^{\beta-1}\left(1+b^\beta+\frac{c^\beta}{2^{\beta-1}}\right)}\left(\sum_{s=n_1}^{n-1+\sigma_1}\frac{1}{a_{s-\sigma_1}}\right) w^{\beta/\alpha}_{n+\sigma_1-\tau_1} \geq 0 \tag{2.12}$$

has no eventually positive solution for sufficiently large $n_1 \geq n_0$*. Then every solution of equation* (1.1) *is oscillatory.*

Proof. Proceeding as in the proof of Theorem 2.1, we obtain (2.2)–(2.6) for all $n \geq n_2 \geq n_1$. Summing (2.6) from n to ∞ yields

$$a_n\Delta z_n + b^\beta a_{n-\sigma_1}\Delta z_{n-\sigma_1} + \frac{c^\beta}{2^{\beta-1}} a_{n+\sigma_2}\Delta z_{n+\sigma_2} \geq \sum_{s=n}^{\infty}\frac{Q^*_s}{4^{\beta-1}}\ z^{\beta/\alpha}_{s-\tau_1}. \tag{2.13}$$

Since $a_n\Delta z_n > 0$ and nonincreasing, we have

$$a_n\Delta z_n + b^\beta a_{n-\sigma_1}\Delta z_{n-\sigma_1} + \frac{c^\beta}{2^{\beta-1}} a_{n+\sigma_2}\Delta z_{n+\sigma_2} \leq \left(1+b^\beta+\frac{c^\beta}{2^{\beta-1}}\right) a_{n-\sigma_1}\Delta\, z_{n-\sigma_1}. \tag{2.14}$$

Inview of (2.13) and (2.14), we have

$$\Delta z_{n-\sigma_1} \geq \frac{1}{\left(1+b^\beta+\frac{c^\beta}{2^{\beta-1}}\right) a_{n-\sigma_1}}\sum_{s=n}^{\infty}\frac{Q^*_s}{4^{\beta-1}}\ z^{\beta/\alpha}_{s-\tau_1}. \tag{2.15}$$

Summing (2.15) from n_2 to $n-1$, we see that

$$\begin{aligned} z_{n-\sigma_1} &\geq \sum_{s=n_2}^{n-1}\frac{1}{\left(1+b^\beta+\frac{c^\beta}{2^{\beta-1}}\right) a_{s-\sigma_1}}\sum_{t=s}^{\infty}\frac{Q^*_t}{4^{\beta-1}}\ z^{\beta/\alpha}_{t-\tau_1} \\ &\geq \frac{1}{4^{\beta-1}\left(1+b^\beta+\frac{c^\beta}{2^{\beta-1}}\right)}\sum_{s=n_2}^{n-1} Q^*_s\ z^{\beta/\alpha}_{s-\tau_1}\sum_{t=n_2}^{s-1}\frac{1}{a_{t-\sigma_1}}. \end{aligned}$$

Thus

$$z_n \geq \frac{1}{4^{\beta-1}\left(1+b^\beta+\frac{c^\beta}{2^{\beta-1}}\right)}\sum_{s=n_2}^{n+\sigma_1-1} Q^*_s\ z^{\beta/\alpha}_{s-\tau_1}\sum_{t=n_2}^{s-1}\frac{1}{a_{t-\sigma_1}}.$$

Let

$$w_n = \frac{1}{4^{\beta-1}\left(1+b^\beta+\frac{c^\beta}{2^{\beta-1}}\right)}\sum_{s=n_2}^{n+\sigma_1-1} Q^*_s\ z^{\beta/\alpha}_{s-\tau_1}\sum_{t=n_2}^{s-1}\frac{1}{a_{t-\sigma_1}} > 0.$$

Then $z_n \geq w_n$, and

$$\begin{aligned} \Delta w_n &= \frac{1}{4^{\beta-1}\left(1+b^\beta+\frac{c^\beta}{2^{\beta-1}}\right)} Q^*_{n+\sigma_1}\ z^{\beta/\alpha}_{n+\sigma_1-\tau_1}\sum_{t=n_2}^{n+\sigma_1-1}\frac{1}{a_{t-\sigma_1}} \\ \Delta w_n &\geq \frac{Q^*_{n+\sigma_1}}{4^{\beta-1}\left(1+b^\beta+\frac{c^\beta}{2^{\beta-1}}\right)}\left(\sum_{t=n_2}^{n+\sigma_1-1}\frac{1}{a_{t-\sigma_1}}\right)\ w^{\beta/\alpha}_{n+\sigma_1-\tau_1}. \end{aligned}$$

Hence we find that $\{w_n\}$ is an eventually positive solution of (2.12). This contradiction completes the proof. ☐

Corollary 2.9. *Assume that* $\beta = \alpha$ *and* $\sigma_1 - \tau_1 > 0$, *and*

$$\liminf_{n\to\infty} \sum_{s=n}^{n+\sigma_1-\tau_1-1} Q^*_{s+\sigma_1} \sum_{t=n_1}^{s+\sigma_1-1} \left(\frac{1}{a_{t+\sigma_1}}\right)$$

$$> 4^{\beta-1}\left(1+b^\beta+\frac{c^\beta}{2^{\beta-1}}\right)\left(\frac{\sigma_1-\tau_1}{1+\sigma_1-\tau_1}\right)^{1+\sigma_1-\tau_1} \tag{2.16}$$

for all sufficiently large $n_1 \geq n_0$. *Then every solution of equation* (1.1) *is oscillatory.*

Next we present oscillation criteria for equation (1.1) when $0 < \beta < 1$.

Theorem 2.10. *Assume that* $0 < \beta < 1$ *and*

$$\Delta\left(y_n + b^\beta y_{n-\sigma_1} + c^\beta y_{n+\sigma_2}\right) + Q^*_n\ R^{\beta/\alpha}_{n-\tau_1}\ y^{\beta/\alpha}_{n-\tau_1} \leq 0 \tag{2.17}$$

has no eventually positive solution for all sufficiently large $n \geq n_0$. *Then every solution of equation* (1.1) *is oscillatory.*

Proof. The proof is exactly the same as in Theorem 2.1 except here we have to use Lemma 2.2 instead of Lemma 2.1, and therefore the details are omitted. ☐

Theorem 2.11. *Assume that* $0 < \beta < 1$ *and*

$$\Delta u_n + \frac{Q^*_n\ R^{\beta/\alpha}_{n-\tau_1}}{1+b^\beta+c^\beta}\ y^{\beta/\alpha}_{n+\sigma_1-\tau_1} \leq 0 \tag{2.18}$$

has no eventually positive solution for all sufficiently large $n \geq n_0$. *Then every solution of equation* (1.1) *is oscillatory.*

Proof. The proof is similar to that of Theorem 2.2 by using Lemma 2.2 instead of Lemma 2.1, and hence the details are omitted. ☐

Similar to Corollaries 2.3 to 2.5, we obtain the following.

Corollary 2.12. *Assume* $\alpha = \beta < 1$, *and* $\sigma_1 - \tau_1 < 0$ *holds. If*

$$\liminf_{n\to\infty} \sum_{s=n+\sigma_1-\tau_1}^{n-1} Q^*_s\ R_{s-\tau_1} > \left(1+b^\beta+c^\beta\right)\left(\frac{\tau_1-\sigma_1}{1+\tau_1-\sigma_1}\right)^{1+\tau_1-\sigma_1} \tag{2.19}$$

then every solution of equation (1.1) *is oscillatory.*

Corollary 2.13. *Assume that* $1 > \beta > \alpha$, *and* $\sigma_1 - \tau_1 < 0$ *holds. If*

$$\sum_{n=n_0}^{\infty} Q^*_n\ R^{\beta/\alpha}_{n-\tau_1} = \infty \tag{2.20}$$

then every solution of equation (1.1) *is oscillatory.*

Corollary 2.14. *Assume that* $1 > \beta > \alpha$, *and* $\sigma_1 - \tau_1 < 0$ *holds. If there exists a* $\lambda > \frac{1}{\tau_1 - \sigma_1} \log(\beta/\alpha)$ *such that*

$$\liminf_{n\to\infty} \left[Q_n^* \, R_{n-\tau_1}^{\beta/\alpha} \exp\left(-e^{\lambda_n}\right) \right] > 0 \tag{2.21}$$

then every solution of equation (1.1) *is oscillatory.*

Theorem 2.15. *Assume that* $0 < \beta < 1$, *holds and*

$$\Delta w_n - \frac{Q_{n+\sigma_1}^*}{(1 + b^\beta + c^\beta)} \left(\sum_{s=n_1}^{n+\sigma_1-1} \frac{1}{a_{s-\sigma_1}} \right) w_{n+\sigma_1-\tau_1}^{\beta/\alpha} \geq 0 \tag{2.22}$$

has no eventually positive solution for sufficiently large $n_1 \geq n_0$. *Then every solution of equation* (1.1) *is oscillatory.*

Proof. The proof is similar to that of Theorem 2.6 and hence the details are omitted. □

Corollary 2.16. *Assume that* $0 < \beta < 1$ *and* $\alpha = \beta$, $\sigma_1 - \tau_1 > 0$ *holds. If*

$$\liminf_{n\to\infty} \sum_{s=n}^{n+\sigma_1-\tau_1-1} Q_{s+\sigma_1}^* \sum_{t=n_1}^{s+\sigma_1-1} \left(\frac{1}{a_{t-\sigma_1}} \right) > \left(1 + b^\beta + c^\beta\right) \left(\frac{\sigma_1 - \tau_1}{1 + \sigma_1 - \tau_1} \right)^{1+\sigma_1-\tau_1} \tag{2.23}$$

for all sufficiently large $n_1 \geq n_0$ *then every solution of equation* (1.1) *is oscillatory.*

Proof. The proof is similar to that of Corollary 2.7 and hence we omit the details. □

3. EXAMPLES

In this section we present some examples to illustrate the main results.

Example 3.1. Consider the difference equation

$$\Delta^2 \left(x_n + b x_{n-\sigma_1} + c x_{n+\sigma_2}\right) + \frac{q}{n} x_{n-\tau_1} + \frac{p}{n} x_{n+\tau_2} = 0, \quad n \geq 1, \tag{3.1}$$

where b, c, q and p are positive constants and $\tau_1 - \sigma_1 > 0$. Here $a_n = 1$, $b_n = b$, $c_n = c$, $q_n = \frac{q}{n}$, $p_n = \frac{p}{n}$ and $\alpha = \beta = 1$. Then $Q_n = \frac{q}{(n+\sigma_2)}$, $P_n = \frac{p}{n+\sigma_2}$ and $Q_n^* = \frac{p+q}{(n+\sigma_2)}$. Since

$$\liminf_{n\to\infty} \sum_{s=n+\sigma_1-\tau_1}^{n-1} Q_s^* R_{s-\tau_1} = \liminf_{n\to\infty} \sum_{s=n+\sigma_1-\tau_1}^{n-1} \frac{(p+q)}{(s+\sigma_2)}(s - \tau_1) = (p+q)(\tau_1 - \sigma_1)$$

we conclude that equation (3.1) is oscillatory if

$$(p+q)(\tau_1 - \sigma_1) > (1 + b + c) \left(\frac{\tau_1 - \sigma_1}{1 + \tau_1 - \sigma_1} \right)^{1+\tau_1-\sigma_1}$$

due to Corollary 2.3. Suppose that $\tau_1 < \sigma_1$. Since

$$\liminf_{n\to\infty} \sum_{s=n}^{n+\sigma_1-\tau_1} Q_{s+\sigma_1}^* \left(\sum_{t=n_1}^{s+\sigma_1-1} \frac{1}{a_{t-\sigma_1}} \right) = (p+q)(\sigma_1 - \tau_1)$$

we conclude that equation (3.1) is oscillatory if

$$(p+q)(\sigma_1-\tau_1) > (1+b+c)\left(\frac{\sigma_1-\tau_1}{1+\sigma_1-\tau_1}\right)^{1+\sigma_1-\tau_1}$$

due to Corollary 2.7.

Example 3.2. Consider the difference equation

$$\Delta\left(\frac{1}{n}\Delta\left(x_n+bx_{n-\sigma_1}+cx_{n+\sigma_2}\right)^3\right)+\frac{q}{n}x_{n-\tau_1}+\frac{p}{n}x_{n+\tau_2}=0, \quad n\geq 1, \tag{3.2}$$

where b, c, q and p are positive constants and $\tau_1-\sigma_1>0$. Here $a_n=\frac{1}{n}$, $\alpha=3, \beta=1$. Then $Q_n^*=\frac{p+q}{(n+\sigma_2)}$ and $R_n=\frac{(n-1)n}{2}$. Since

$$\sum_{n=1}^{\infty} Q_n^* R_{n-\tau_1}^{\beta/\alpha} = \sum_{n=1}^{\infty}\frac{(p+q)}{(n+\sigma_2)}\frac{((n-\tau_1-1)(n-\tau_1))^{1/3}}{2^{1/3}}=\infty.$$

Then every solution of equation (3.2) is oscillatory due to Corollary 2.4.

Example 3.3. Consider the difference equation

$$\Delta\left(\frac{1}{n}\Delta\left(x_n+bx_{n-1}+cx_{n+2}\right)\right)+q\exp(e^{2(n+1)})x_{n-2}^3+px_{n+3}^3=0, \tag{3.3}$$

where $n\geq 1$, b, c, q and p are positive constants. Here $\alpha=1$, $\beta=3$, $a_n=\frac{1}{n}$, $q_n=q\exp(e^{2(n+1)})$, $p_n=p$, $\sigma_1=1$, $\sigma_2=2$, $\tau_1=2$, $\tau_2=3$. Choose $\lambda=2$, then $\lambda>\frac{1}{\tau_1-\sigma_1}\log(\beta/\alpha)$ and

$$\liminf_{n\to\infty}\left[Q_n^* R_{n-\tau_1}^{\beta/\alpha}\exp(-e^{\lambda_n})\right]=\liminf_{n\to\infty}\left[\left(qe^{e^{2n}}+p\right)\frac{(n-2)^3(n-1)^3}{2^3}e^{-e^{2n}}\right]>0.$$

Hence by Corollary 2.5, every solution of equation (3.3) is oscillatory.

Remark 3.4. The results presented in this paper are new. It is remarkable that our results possibly valid either $p_n\equiv 0$ or $q_n\equiv 0$ (but not $p_n\equiv q_n\equiv 0$) provided that either $b_n\equiv 0, c_n\equiv 0$ or $b_n\equiv c_n\equiv 0$. Here we omit the details.

Remark 3.5. It would be interesting to obtain results similar to those presented here for equations of the type

$$\Delta\left(a_n\left(\Delta\left(x_n+b_nx_{n-\sigma_1}+c_nx_{n+\sigma_2}\right)\right)^{\alpha}\right)+q_nx_{n-\tau_1}^{\beta}+p_nx_{n+\tau_2}^{\gamma}=0,$$

where α,β,γ are ratio of odd positive integers and either $\sum_{n=n_0}^{\infty}\frac{1}{a_n}=\infty$ or $\sum_{n=n_0}^{\infty}\frac{1}{a_n}<\infty$.

REFERENCES

[1] R. P. Agarwal, *Difference Equations and Inequalities*, Marcel Dekker, New York, 2000.

[2] R. P. Agarwal, M. Bohner, S. R. Grace and D. O'Regan, *Discrete Oscillation Theory,* Hindawi Publ. Corp. New York, 2005.

[3] R. P.Agarwal and S. R. Grace, The oscillation of certain difference equations, *Math.Comp. Modelling,* 30 (1999), 53–66.

[4] R. P. Agarwal and S. R. Grace, Oscillation theorems for certain difference equations, *Dynam. Syst. Appl.* 9 (2000), 299–308.

[5] R. P. Agarwal, S. R. Grace and E. A. Bohner, On the oscillation of higher order neutral difference equations of mixed type, *Dynam. Syst. Appl.* 11 (2002), 459–470.

[6] S. Elizabeth, J. R. Graef, P. Sundaram and E. Thandapani, Classifying nonoscillatory solutions and oscillation of neutral difference equations, *J.Diff. Eqns. Appl.* 11 (2005), 605–618.

[7] S. R. Grace, Oscillation of certain difference equations of mixed type, *J. Math. Anal. Appl.* 224 (1998), 241–254.

[8] S. R. Grace and H. A. EL-Morshedy, Oscillation criteria of comparison type for second order difference equations, *J. Appl. Anal.*6 (2000), 87–103.

[9] S. R. Grace and S. Dontha, Oscillation of a higher order neutral difference equations of mixed type, *Dynam. Syst. Appl.* 12 (2003), 521–532.

[10] G. Ladas, Ch. G. Philos and Y. G. Sficas, Sharp conditions for the oscillation of delay difference equations, *J. Appl. Math. Simulations* 2 (1989), 101–112.

[11] B. S. Lalli and S. R. Grace, Oscillation theorems of second order delay and neutral difference equations, *Utilitas Math.* 45 (1994), 197–212.

[12] X. H. Tang and Y. Liu, Oscillation for nonlinear delay difference equations, *Tamkang J. Math.* 32 (2001), 275–280.

[13] E. Thandapani and M. M. S. Manuel, Asymptotic and oscillatory behavior of second order neutral delay difference equations, *Engg. Simulation,* 15 (1998), 423–430.

[14] E. Thandapani and P. Mohankumar, Oscillation and nonoscillation of nonlinear neutral delay difference equations, *Tamkang J. Kath.* 38 (2007), 323–333.

[15] B. G. Zhang and S. H. Saker, Kamanev-type oscillation criteria for nonlinear delay difference equations, *Indian J. Pure Appl. Math* 34 (2003), 1571–1584.