Dynamic Systems and Applications 21 (2012) 93-120

MULTIPLE POSITIVE SOLUTIONS FOR SINGULAR $\phi-$LAPLACIAN BVPS WITH DERIVATIVE DEPENDENCE ON $[0,+\infty)$

SMAÏL DJEBALI AND OUIZA SAIFI

Lab. Fixed Point Theory & Applications, E.N.S., P.B. 92
Kouba, 16050. Algiers (Algeria)
Dept. of Economics, Fac. of Economic and Management Sciences
Algiers University (Algeria)

ABSTRACT. This work deals with the existence of multiple positive solutions for a $\phi-$Laplacian boundary value problem on the half-line. The nonlinearity may exhibit singularities at the solution and its derivative. New existence results are obtained using the fixed point index theory on cones of Banach spaces. The singularity is treated by approximation and sequential arguments. Several examples of applications illustrate the obtained results.

AMS (MOS) Subject Classification. 34B15, 34B18, 34B40, 47H10

1. INTRODUCTION

This paper is devoted to the study of the existence of positive solutions to the following boundary value problem (bvp in short) posed on the positive half-line:

$$(1.1)\qquad \begin{cases} (\phi(x'))'(t)+q(t)f(t,x(t),x'(t))=0, & t\in I,\\ \alpha x(0)-\beta x'(0)=0, & \lim\limits_{t\to+\infty} x'(t)=0 \end{cases}$$

where $\alpha,\beta>0$ are positive constants, $I:=(0,+\infty)$ denotes the set of positive real numbers, and $\mathbb{R}^+:=[0,+\infty)$. The function $q: I\longrightarrow I$ is continuous and the nonlinearity $f:\mathbb{R}^+\times I\times I\longrightarrow\mathbb{R}^+$ is continuous and satisfies $\lim\limits_{x\to0^+} f(t,x,y)=+\infty$ and/or $\lim\limits_{y\to0^+} f(t,x,y)=+\infty$, i.e. $f(t,x,y)$ may be singular at $x=0$ and/or $y=0$. The map $\phi:\mathbb{R}\longrightarrow\mathbb{R}$ is a continuous, increasing homeomorphism such that $\phi(0)=0$, extending the so-called $p-$Laplacian $\varphi_p(s)=|s|^{p-1}s$ $(p>1)$.

Boundary value problems on infinite intervals appear in many phenomena in applied mathematics and physics (see, e.g., [2] and the references therein). Various mathematical results for nonlinear bvps can be found in the recent literature (see [5, 6, 7, 20]) where existence and multiplicity of positive solutions have been obtained.

Received January 30, 2012

In [8], the authors have considered the bvp

$$\begin{cases} x''(t) - k^2 x(t) = q(t)f(t, x(t), x'(t)) = 0, \quad t \in I, \\ x(0) = x(+\infty) = 0. \end{cases}$$

The question of the existence of positive solution was studied when the nonlinearity is sign-changing; the fixed point index and the upper and lower solutions technique were combined to prove some existence results.

In [18], the following bvp is studied

$$\begin{cases} \left(\frac{1}{p(t)}(p(t)(x'(t))\right)'(t) + f(t, x(t)) = 0, \quad t \in I, \\ x(0) = 0, \quad \lim\limits_{t\to+\infty} p(t)x'(t) = b > 0, \end{cases}$$

where p satisfies $\int_0^{+\infty} \frac{dt}{p(t)} < +\infty$. Existence of one or two solutions are proved using index fixed point theory when the nonlinearity $f = f(t, x, px')$ may present singularities at $x = 0$ or/and $px' = 0$.

Recently, Lian *et al.* have studied the following boundary value problem with a $p-$Laplacian operator on the half line

$$\begin{cases} (\varphi_p(x'))'(t) + q(t)f(t, x(t), x'(t)) = 0, \quad t \in I, \\ \alpha x(0) - \beta x'(0) = 0, \quad \lim\limits_{t\to+\infty} x'(t) = 0. \end{cases}$$

The authors have showed the existence of at least three positive solutions using a fixed point theorem due to Avery and Peterson (see [13]). The nonlinearity is assumed to have no singularity.

With a multi-point condition at 0, the same bvp is investigated in [12] with a similar method. Existence results are also obtained for the Sturm-Liouville equation $(p(t)x'(t))'(t) + f(t, x(t), x'(t)) = 0$ in [17]. In [14], Liang and Zhang have considered the equation $(\varphi_p(x'))'(t) + a(t)f(t, x(t)) = 0$ with multi-point condition at the origin and a Neumann condition at positive infinity. However, the nonlinearity does not depend on the first derivative. In [15, 16], the operator of derivation is extended to an increasing homeomorphism ϕ.

In [9], the following singular bvp is considered:

$$\begin{cases} (\phi(x'))'(t) + q(t)f(t, x(t)) = 0, \quad t > 0, \\ x(0) = 0, \quad \lim\limits_{t\to+\infty} x'(t) = 0. \end{cases} \tag{1.2}$$

The authors have showed the existence of multiple positive solutions using the upper and lower solution method combined with the fixed point index theory. When the nonlinearity f also depends on the first derivative, problem (1.2) is discussed in [10]; the authors have proved two existence results: the first one is obtained under a sign condition, the second one when a Nagumo-type growth condition is assumed.

In this work, we aim to investigate the question of existence and multiplicity of positive solutions to problem (1.1) when the nonlinearity depends on the first derivative and may be singular at $x = 0$ and/or $x' = 0$ and when the operator of derivation ϕ is a general increasing homeomorphism. The fixed point index theory on a cone of a suitable Banach space is employed. Existence of single and twin solutions is proved. The paper comprises five sections. In Sect. 2, we define a special norm space, construct a special cone, and give its main properties. In Sect. 3, using the theory of the fixed point index, we prove existence of one and then two positive solutions to problem (1.1) when the nonlinearity is assumed to have no singularities. Similar results are obtained in Sect. 4 when f is singular at $x = 0$ but not at $x' = 0$. The cases when f is singular at both $x = 0$ and at $x' = 0$ are studied in Sect. 5. The singularity is treated by approximating the fixed point operator and then using sequential arguments. Each existence theorem is illustrated by means of an example of application.

2. PRELIMINARIES

In this section, we first gather together some definitions and lemmas we need in the rest of the paper.

2.1. Auxiliary results.

Definition 2.1. A nonempty subset $\mathbb{P}$ of a Banach space E is called a cone if it is convex, closed, and satisfies the conditions:

(i) $\alpha x \in \mathbb{P}$ for all $x \in \mathbb{P}$ and $\alpha \geq 0$,

(ii) $x \in \mathbb{P}$ and $-x \in \mathbb{P}$ imply $x = 0$.

Definition 2.2. A mapping $A : E \to E$ is said to be completely continuous if it is continuous and maps bounded sets into relatively compact sets.

The following lemmas will be used to prove our main existence results. More details on the theory and the computation of the fixed point index on cones in Banach spaces may be found in [1, 2, 4, 11].

Lemma 2.3. *Let Ω be a bounded open set in a real Banach space E, $\mathbb{P}$ be a cone of E, and $A : \overline{\Omega} \cap \mathbb{P} \to \mathbb{P}$ be a completely continuous map. Suppose that $\lambda Ax \neq x, \forall\, x \in \partial\Omega \cap \mathbb{P}$ and $\forall\, \lambda \in (0, 1]$. Then $i(A, \Omega \cap \mathbb{P}, \mathbb{P}) = 1$.*

Lemma 2.4. *Let Ω be a bounded open set in a real Banach space E, $\mathbb{P}$ be a cone of E, and $A : \overline{\Omega} \cap \mathbb{P} \to \mathbb{P}$ be a completely continuous map. Suppose that $Ax \not\leq x, \forall\, x \in \partial\Omega \cap \mathbb{P}$. Then $i(A, \Omega \cap \mathbb{P}, \mathbb{P}) = 0$.*

Let

$$C_l([0,\infty),\mathbb{R}) = \{x \in C([0,\infty),\mathbb{R}) : \lim_{t\to+\infty} x(t) \text{ exists}\}.$$

For $x \in C_l([0,\infty),\mathbb{R})$, define $\|x\|_l = \sup_{t\in\mathbb{R}^+} |x(t)|\cdot$ This makes C_l a Banach space. However, the basic space to study problem (1.1) is denoted by

$$E = \{x \in C^1([0,\infty),\mathbb{R}),\ \lim_{t\to+\infty} \frac{x(t)}{1+t} \text{ exists},\ \lim_{t\to+\infty} x'(t) = 0\}.$$

It is clear that E is a Banach space when furnished with the norm $\|x\| = \max\{\|x\|_1, \|x\|_2\}$ where $\|x\|_1 = \sup_{t\in\mathbb{R}^+} \frac{|x(t)|}{1+t}$ and $\|x\|_2 = \sup_{t\in\mathbb{R}^+} |x'(t)|$.

Lemma 2.5 ([3, p. 62]). *Let $M \subseteq C_l(\mathbb{R}^+,\mathbb{R})$. Then M is relatively compact in $C_l(\mathbb{R}^+,\mathbb{R})$ if the following three conditions hold:*

(a) *M is uniformly bounded in $C_l(\mathbb{R}^+,\mathbb{R})$.*

(b) *The functions belonging to M are almost equicontinuous on $\mathbb{R}^+$, i.e. equicontinuous on every compact interval of $\mathbb{R}^+$.*

(c) *The functions from M are equiconvergent, that is, given $\varepsilon > 0$, there corresponds $T(\varepsilon) > 0$ such that $|x(t) - x(+\infty)| < \varepsilon$ for any $t \geq T(\varepsilon)$ and $x \in M$.*

Then, we easily deduce

Lemma 2.6. *Let $M \subseteq E$. Then M is relatively compact in E if the following conditions hold:*

(a) *M is bounded in E,*

(b) *the functions belonging to $\{u : u(t) = \frac{x(t)}{1+t}, x \in M\}$ and to $\{z : z(t) = x'(t), x \in M\}$ are almost equicontinuous on $[0,+\infty)$,*

(c) *the functions belonging to $\{u : u(t) = \frac{x(t)}{1+t}, x \in M\}$ and to $\{z : z(t) = x'(t), x \in M\}$ are equiconvergent at $+\infty$.*

2.2. **Related Lemmas.**

Definition 2.7. A function x is said to be a solution of problem (1.1) if $x \in C^1(\mathbb{R}^+,\mathbb{R})$ with $\phi(x') \in AC(\mathbb{R}^+,\mathbb{R})$ and (1.1) is satisfied.

We start with a simple observation:

Lemma 2.8. *Let $x \in C(\mathbb{R}^+,\mathbb{R}^+)$ be a positive concave function. Then x is nondecreasing on $[0,+\infty)$.*

Proof. Let $t, t' \in [0, +\infty)$ be such that $t' \geq t$ and let $\lambda = t' - t$. Since x is positive concave, for all $n \in \mathbb{N}^* = \{1, 2, 3, \dots\}$, we have

$$\begin{aligned} x(t') &= x(t+\lambda) \\ &= x\left((1-\tfrac{1}{n})t + \tfrac{1}{n}(t+n\lambda)\right) \\ &\geq \left(1-\tfrac{1}{n}\right)x(t) + \tfrac{1}{n}x(t+n\lambda) \\ &\geq \left(1-\tfrac{1}{n}\right)x(t). \end{aligned}$$

Therefore

$$x(t') \geq \lim_{n\to+\infty}\left(1-\frac{1}{n}\right)x(t) = x(t).$$

□

Since ϕ is an increasing homeomorphism, it is easy to prove

Lemma 2.9. *If x is a solution of problem* (1.1)*, then x is positive, monotone increasing and concave on* $[0, +\infty)$*.*

Now, define the positive cone

$$\mathbb{P} = \{x \in E : \ x(t) \geq 0, \ \text{concave } [0, +\infty), \ x(0) \geq \frac{\beta}{\alpha+\beta}\|x\|_2 \ \text{and } \alpha x(0) - \beta x'(0) = 0\}.$$

In a series of lemmas, we study the main properties of $\mathbb{P}$.

Lemma 2.10. *Let $x \in \mathbb{P}$ and $\theta \in (1, +\infty)$. Then*

$$x(t) \geq \frac{1}{\theta}\|x\|_1, \quad \forall\, t \in [1/\theta, \theta].$$

Proof. By definition of $\mathbb{P}$, x is nondecreasing on $[0, +\infty)$. Moreover $x'(\infty) = 0$ implies that the function $\frac{x(t)}{1+t}$ achieves its maximum at some $t_0 \in [0, +\infty)$. So, by the concavity of x, we have for $t \in [1/\theta, \theta]$

$$\begin{aligned} x(t) &\geq \min_{t\in[\frac{1}{\theta},\theta]} x(t) = x(\tfrac{1}{\theta}) = x(\tfrac{\theta-1+\theta t_0}{\theta+\theta t_0}\tfrac{1}{\theta-1+\theta t_0} + \tfrac{1}{\theta+\theta t_0}t_0) \\ &\geq \tfrac{\theta-1+\theta t_0}{\theta+\theta t_0}x(\tfrac{1}{\theta-1+\theta t_0}) + \tfrac{1}{\theta+\theta t_0}x(t_0) \\ &\geq \tfrac{1}{\theta+\theta t_0}x(t_0) = \tfrac{1}{\theta}\tfrac{x(t_0)}{1+t_0} = \tfrac{1}{\theta}\|x\|_1. \end{aligned}$$

□

Lemma 2.11. *Define the function ρ by*

$$\rho(t) = \begin{cases} t, & t \in [0,1] \\ \frac{1}{t}, & t \in (1, +\infty) \end{cases} \tag{2.1}$$

and let $x \in \mathbb{P}$. Then

$$x(t) \geq \rho(t)\|x\|_1, \quad \forall\, t \in [0, +\infty).$$

Proof. Let $t \in [0, +\infty)$ and distinguish between four cases:

(a) If $t = 0$, then $x(0) \geq 0 = \rho(0)\|x\|_1$.

(b) If $t \in (0,1)$, then $\frac{1}{t} \in (1,+\infty)$. By Lemma 2.10, we have that $x(s) \geq t\|x\|_1, \forall\, s \in [t, \frac{1}{t}]$. In particular for $s = t$, $x(t) \geq t\|x\|_1 = \rho(t)\|x\|_1$.

(c) If $t \in (1,+\infty)$, then by Lemma 2.10, we have that $x(s) \geq \frac{1}{t}\|x\|_1, \forall\, s \in [\frac{1}{t}, t]$. In particular for $s = t$, $x(t) \geq \frac{1}{t}\|x\|_1 = \rho(t)\|x\|_1$.

(d) Let $t = 1$ and let $\{t_n\}_n$ be a real sequence such that $t_n > 1$ and $t_n \to 1$, as $n \to +\infty$. By the third case, we have $x(t_n) \geq \frac{1}{t_n}\|x\|_1$, $\forall\, n \geq 1$. Then

$$x(1) = \lim_{n\to+\infty} x(t_n) \geq \lim_{n\to+\infty} \frac{1}{t_n}\|x\|_1 = \|x\|_1 = \rho(1)\|x\|_1.$$

□

Lemma 2.12. *Let $x \in \mathbb{P}$. Then $\|x\|_1 \leq M\|x\|_2$, where $M = \max\{\frac{\beta}{\alpha}, 1\}$. Hence $\|x\| \leq M\|x\|_2$.*

Proof. Since $x \in \mathbb{P}$, then for every $t \in [0,+\infty)$,

$$\frac{x(t)}{1+t} = \frac{1}{1+t}\left(\int_0^t x'(s)ds + \frac{\beta}{\alpha}x'(0)\right) \leq \frac{t+\frac{\beta}{\alpha}}{1+t}\|x\|_2 \leq M\|x\|_2.$$

This implies that $\|x\| = \max\{\|x\|_1, \|x\|_2\} \leq \max\{M\|x\|_2, \|x\|_2\} = M\|x\|_2$. □

Lemma 2.13. *Let $x \in \mathbb{P}$. Then*

$$x(t) \geq \rho(t)\frac{\beta}{\alpha+\beta}\|x\|,\ \forall\, t \in [0,+\infty).$$

Proof. Since $x \in \mathbb{P}$, we have $\|x\|_1 = \sup_{t\in\mathbb{R}^+} \frac{x(t)}{1+t} \geq \frac{x(0)}{1+0} = x(0) \geq \frac{\beta}{\alpha+\beta}\|x\|_2$. Hence $\|x\|_2 \leq \frac{\alpha+\beta}{\beta}\|x\|_1$. As a consequence

$$\|x\| = \max\{\|x\|_1, \|x\|_2\} \leq \max\{\|x\|_1, \frac{\alpha+\beta}{\beta}\|x\|_1\} = \frac{\alpha+\beta}{\beta}\|x\|_1.$$

Finally, Lemma 2.11 implies that $x(t) \geq \rho(t)\|x\|_1 \geq \frac{\beta}{\alpha+\beta}\rho(t)\|x\|$. □

Lemma 2.14. *Let $x \in \mathbb{P}$. Then, for all $t \in \mathbb{R}^+$, $\frac{x(t)}{x'(t)} \geq \frac{\beta}{\alpha}\rho(t)$.*

Proof. Since $x \in \mathbb{P}$, we have that x is nondecreasing and x' is nonincreasing. Hence

$$\frac{x(t)}{x'(t)} \geq \frac{x(0)}{x'(0)} = \frac{\beta}{\alpha}\frac{x(0)}{x(0)} \geq \frac{\beta}{\alpha} \geq \frac{\beta}{\alpha}\rho(t).$$

□

Lemma 2.15. *Let $\delta \in C(\mathbb{R}^+, \mathbb{R}^+)$ be such that $\int_0^{+\infty} \delta(s)ds < +\infty$ and let*

$$x(t) = \frac{\beta}{\alpha}\phi^{-1}\left(\int_0^{+\infty} \delta(\tau)d\tau\right) + \int_0^t \phi^{-1}\left(\int_s^{+\infty} \delta(\tau)d\tau\right)ds.$$

Then

$$\begin{cases} (\phi(x'))'(t) + \delta(t) = 0, & t > 0, \\ \alpha x(0) - \beta x'(0) = 0, & \lim_{t\to+\infty} x'(t) = 0, \end{cases}$$

and hence $x \in \mathbb{P}$.

Proof. It is easy to check that

$$\begin{cases} (\phi(x'))'(t) + \delta(t) = 0, & t > 0, \\ \alpha x(0) - \beta x'(0) = 0, & \lim_{t\to+\infty} x'(t) = 0. \end{cases}$$

By Lemma 2.9, x is positive and concave on $[0, +\infty)$. Moreover, we have that

$$x(0) = \frac{\beta}{\alpha}\phi^{-1}\left(\int_0^{+\infty}(\delta(\tau))d\tau\right) \text{ and } \|x\|_2 = \phi^{-1}\left(\int_0^{+\infty}(\delta(\tau))d\tau\right).$$

Then

$$x(0) = \frac{\beta}{\alpha}\phi^{-1}\left(\int_0^{+\infty}(\delta(\tau))d\tau\right) = \frac{\beta}{\alpha}\|x\|_2 \geq \frac{\beta}{\alpha+\beta}\|x\|_2,$$

and so $x \in \mathbb{P}$. □

3. THE REGULAR CASE

In this section, we suppose that $f : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and there exists $t_0 > 0$ such that $f(t_0, 0, 0) \not\equiv 0$ so that the trivial solution is ruled out. Let $\widetilde{\rho}(t) = \frac{\rho(t)}{1+t}$, $F(t, x, y) = f(t, (1 + t)x, y)$, and assume that

(H$_1$): There exist $m \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $g \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$ such that

$$F(t, x, y) \leq m(t)g(x, y), \quad \forall\, t, x, y \in \mathbb{R}^+, \tag{3.1}$$

where g is a nondecreasing function in each argument with

$$\int_0^{+\infty} q(\tau)m(\tau)d\tau < +\infty$$

and for each $c > 0$

$$\int_0^{+\infty} \phi^{-1}\left(\int_s^{+\infty} g(c, c)q(\tau)m(\tau)d\tau\right) ds < +\infty. \tag{3.2}$$

(H$_2$):

$$\sup_{c>0} \frac{c}{M\phi^{-1}\left(\int_0^{+\infty} q(\tau)m(\tau)g(c, c)d\tau\right)} > 1.$$

(H$_3$): There exist positive numbers $a < b$ such that

$$\lim_{x\to+\infty} \frac{F(t, x, y)}{\phi(x)} = +\infty, \quad \text{uniformly in } t \in [a, b] \text{ and } y \geq 0.$$

For $x \in \mathbb{P}$, define the operator A by

$$\begin{aligned} Ax(t) = {} & \frac{\beta}{\alpha}\phi^{-1}\left(\int_0^{+\infty} q(\tau)f(\tau, x(\tau), x'(\tau))d\tau\right) \\ & + \int_0^t \phi^{-1}\left(\int_s^{+\infty} q(\tau)f(\tau, x(\tau), x'(\tau))d\tau\right) ds. \end{aligned}$$

We have

Lemma 3.1. *Suppose* (**H**$_1$) *holds. Then, the operator* A *sends* $\mathbb{P}$ *into* $\mathbb{P}$ *and* A *is completely continuous.*

Proof. By Lemma 2.15, $A(\mathbb{P}) \subset \mathbb{P}$. We show that A is completely continuous.

Step 1: A is continuous. Let some sequence $\{x_n\}_{n\geq 0} \subseteq \mathbb{P}$ be such that $\lim\limits_{n\to+\infty} x_n = x_0$. Then there exists $r > 0$ such that $\|x_n\| \leq r$, $\forall\, n \geq 0$. By $(\mathbf{H}_1)$, we have

$$\begin{aligned} & q(\tau)|f(\tau, x_n(\tau), x_n'(\tau)) - f(\tau, x_0(\tau), x_0'(\tau))| \\ & = q(\tau)|F(\tau, \frac{x_n(\tau)}{1+\tau}, x_n'(\tau)) - F(\tau, \frac{x_0(\tau)}{1+\tau}, x_0'(\tau))| \\ & \leq 2q(\tau)m(\tau)g(r,r). \end{aligned}$$

The continuity of f and the Lebesgue dominated convergence theorem imply that

$$|\phi((Ax_n(t)))' - \phi((Ax_0(t)))'| \leq \int_0^{+\infty} q(\tau)|f(\tau, x_n(\tau), x_n'(\tau)) - f(\tau, x_0(\tau), x_0'(\tau))|d\tau$$

and the right-hand side tends to 0 as $n \to +\infty$, that is $\|Ax_n - Ax_0\|_2 \to 0$, as $n \to +\infty$; Lemma 2.12 implies that $\|Ax_n - Ax_0\|$ tends to 0 as $n \to +\infty$.

Step 2: Let D be a bounded set. Then there exists $r > 0$ such that $\|x\| \leq r, \forall\, x \in D$. We shall proceed in three steps.

(a) $A(D)$ is uniformly bounded. For $x \in D$, we have

$$\begin{aligned} \|Ax\| & \leq M\|Ax\|_2 \\ & \leq M \sup_{t\in\mathbb{R}^+} |(Ax)'(t)| \\ & \leq M \sup_{t\in\mathbb{R}^+} \phi^{-1}\left(\int_t^{+\infty} q(\tau)f(\tau, x(\tau), x'(\tau))d\tau\right) \\ & \leq M \sup_{t\in\mathbb{R}^+} \phi^{-1}\left(\int_0^{+\infty} q(\tau)F(\tau, \frac{x(\tau)}{1+\tau}, x'(\tau))d\tau\right) \\ & \leq M\phi^{-1}\left(\int_0^{+\infty} q(\tau)m(\tau)g(r,r)d\tau\right) < \infty. \end{aligned}$$

Then $A(D)$ is bounded.

(b) For any $T > 0$ and $t, t' \in [0,T]$ $(t > t')$, we have

$$\begin{aligned} \left|\frac{Ax(t)}{1+t} - \frac{Ax(t')}{1+t'}\right| & \leq \frac{\beta}{\alpha}\phi^{-1}\left(\int_0^{+\infty} q(\tau)f(\tau, x(\tau), x'(\tau))d\tau\right)\left|\frac{1}{1+t} - \frac{1}{1+t'}\right| \\ & + \left|\frac{\int_0^t \phi^{-1}\left(\int_s^{+\infty} q(\tau)f(\tau, x(\tau), x'(\tau))d\tau\right)ds}{1+t}\right. \\ & \left. - \frac{\int_0^{t'} \phi^{-1}\left(\int_s^{+\infty} q(\tau)f(\tau, x(\tau), x'(\tau))d\tau\right)ds}{1+t'}\right| \\ & \leq \frac{\beta}{\alpha}\phi^{-1}\left(\int_0^{+\infty} q(\tau)f(\tau, x(\tau), x'(\tau))d\tau\right)\left|\frac{1}{1+t} - \frac{1}{1+t'}\right| \\ & + \left|\frac{1}{1+t} - \frac{1}{1+t'}\right|\int_0^{+\infty} \phi^{-1}\left(\int_s^{+\infty} q(\tau)f(\tau, x(\tau), x'(\tau))d\tau\right)ds \end{aligned}$$

$$
\begin{aligned}
&+\left|\frac{\int_{t'}^{+\infty}\phi^{-1}\left(\int_s^{+\infty}q(\tau)f(\tau,x(\tau),x'(\tau))d\tau\right)ds}{1+t'}\right.\\
&-\left.\frac{\int_{t}^{+\infty}\phi^{-1}\left(\int_s^{+\infty}q(\tau)f(\tau,x(\tau),x'(\tau))d\tau\right)ds}{1+t}\right|\\
\le\ &\frac{\beta}{\alpha}\phi^{-1}\left(\int_0^{+\infty}q(\tau)f(\tau,x(\tau),x'(\tau))d\tau\right)\left|\frac{1}{1+t}-\frac{1}{1+t'}\right|\\
&+2|\frac{1}{1+t}-\frac{1}{1+t'}|\int_0^{+\infty}\phi^{-1}\left(\int_s^{+\infty}q(\tau)f(\tau,x(\tau),x'(\tau))d\tau\right)ds\\
&+\frac{1}{1+t'}\int_{t'}^{t}\phi^{-1}\left(\int_s^{+\infty}q(\tau)f(\tau,x(\tau),x'(\tau))d\tau\right)ds\\
\le\ &\frac{\beta}{\alpha}\phi^{-1}\left(\int_0^{+\infty}q(\tau)m(\tau)g(r,r)d\tau\right)\left|\frac{1}{1+t}-\frac{1}{1+t'}\right|\\
&+2|\frac{1}{1+t}-\frac{1}{1+t'}|\int_0^{+\infty}\phi^{-1}\left(\int_s^{+\infty}q(\tau)m(\tau)g(r,r)d\tau\right)ds\\
&+\frac{1}{1+t'}\int_{t'}^{t}\phi^{-1}\left(\int_s^{+\infty}q(\tau)m(\tau)g(r,r)d\tau\right)ds.
\end{aligned}
$$

Also we have

$$
|\phi((Ax)'(t))-\phi((Ax)'(t'))|=\left|\int_{t'}^{t}q(\tau)f(\tau,x(\tau),x'(\tau))d\tau\right|\le\int_{t'}^{t}q(\tau)m(\tau)g(r,r)d\tau.
$$

Then, for all $\varepsilon>0$ and $T>0$, there exists $\delta>0$ such that for all $t,t'\in[0,T]$ and $|t-t'|<\delta$, we have

$$
\left|\frac{Ax(t)}{1+t}-\frac{Ax(t')}{1+t'}\right|<\varepsilon\ \text{ and }\ |(Ax)'(t)-(Ax)'(t')|<\varepsilon.
$$

(c) For any $x\in D$, we have $\lim\limits_{t\to+\infty}\frac{Ax(t)}{1+t}=\lim\limits_{t\to+\infty}(Ax)'(t)=0$. Therefore

$$
\begin{aligned}
&\sup_{x\in D}|\frac{Ax(t)}{1+t}-\lim_{t\to+\infty}\frac{Ax(t)}{1+t}|\\
&=\sup_{x\in D}\left|\frac{\frac{\beta}{\alpha}\phi^{-1}\left(\int_0^{+\infty}q(\tau)f(\tau,x(\tau),x'(\tau))d\tau\right)}{1+t}+\frac{\int_0^{t}\phi^{-1}\left(\int_s^{+\infty}q(\tau)f(\tau,x(\tau),x'(\tau))d\tau\right)ds}{1+t}\right|\\
&\le\frac{\frac{\beta}{\alpha}\phi^{-1}\left(\int_0^{+\infty}q(\tau)m(\tau)g(r,r)d\tau\right)}{1+t}+\frac{\int_0^{+\infty}\phi^{-1}\left(\int_s^{+\infty}q(\tau)m(\tau)g(r,r)d\tau\right)ds}{1+t}
\end{aligned}
$$

and

$$
\begin{aligned}
\sup_{x\in D}|(Ax)'(t)|&=\sup_{x\in D}\phi^{-1}\left(\int_t^{+\infty}q(\tau)f(\tau,x(\tau),x'(\tau))d\tau\right)\\
&\le\phi^{-1}\left(\int_t^{+\infty}q(\tau)m(\tau)g(r,r)d\tau\right)
\end{aligned}
$$

which implies that

$$\lim_{t\to+\infty}\sup_{x\in D}\left|\frac{Ax(t)}{1+t}-\lim_{t\to+\infty}\frac{Ax(t)}{1+t}\right|=0 \text{ and } \lim_{t\to+\infty}\sup_{x\in D}|(Ax)'(t)-\lim_{t\to+\infty}(Ax)'(t)|=0.$$

By Lemma 2.6, $A(D)$ is relatively compact in E. Hence $A:\mathbb{P}\longrightarrow\mathbb{P}$ is completely continuous. □

3.1. **Existence of a single solution.**

Theorem 3.2. *Assume that Assumptions* $(\mathbf{H}_1)-(\mathbf{H}_2)$ *hold. Then problem* (1.1) *has at least one positive solution.*

Proof. From condition $(\mathbf{H}_2)$, there exists $R>0$ such that

$$\frac{R}{M\phi^{-1}\left(\int_0^{+\infty}q(\tau)m(\tau)g(R,R)d\tau\right)}>1. \tag{3.3}$$

Let

$$\Omega_1=\{x\in E:\ \|x\|<R\}.$$

We claim that $x\neq\lambda Ax$ for all $x\in\partial\Omega_1\cap\mathbb{P}$ and $\lambda\in(0,1]$. On the contrary, suppose that there exist $x_0\in\partial\Omega_1\cap\mathbb{P}$ and $\lambda_0\in(0,1]$ such that $x_0=\lambda_0Ax_0$. By Lemma 2.12, we have

$$\begin{aligned}R&=\|x_0\|=\|\lambda_0Ax_0\|\\&\leq M\|Ax_0\|_2\\&\leq M\sup_{t\geq0}\phi^{-1}\left(\int_t^{+\infty}q(\tau)f(\tau,x(\tau),x'(\tau))d\tau\right),\\&\leq M\phi^{-1}\left(\int_0^{+\infty}q(\tau)m(\tau)g(R,R)\right),\end{aligned}$$

which is a contradiction to (3.3). Owing to Lemma 2.3, we deduce that

$$i(A,\Omega_1\cap\mathbb{P},\mathbb{P})=1. \tag{3.4}$$

Hence there exists an $x_0\in\Omega_1\cap\mathbb{P}$ such that $Ax_0=x_0$. Since $f(t_0,0,0)\not\equiv0$ and $x_0(t)\geq\frac{\beta}{\alpha+\beta}\rho(t)\|x_0\|$, x_0 is a positive solution of (1.1). □

3.2. **Two positive solutions.**

Theorem 3.3. *Assume that* $(\mathbf{H}_1)-(\mathbf{H}_3)$ *hold and suppose that* ϕ^{-1} *is super-multiplicative, that is*

$$\phi^{-1}(xy)\geq\phi^{-1}(x)\phi^{-1}(y),\ \forall\,x,y\geq0. \tag{3.5}$$

Then problem (1.1) *has at least two positive solutions.*

Remark 3.4.

(a) If ϕ is sub-multiplicative, then ϕ^{-1} is super-multiplicative.

(b) The $p-$Laplacian is super-multiplicative and sub-multiplicative, hence multiplicative.

Proof. Choosing the same R as in the proof of Theorem 3.2, we get

$$i(A,\Omega_1\cap\mathbb{P},\mathbb{P})=1, \tag{3.6}$$

and thus there exists x_0 solution of problem (1.1) in Ω_1. Let $0<a<b<+\infty$ be as in $(\mathbf{H}_3)$ and set $N=1+\frac{\phi(\frac{\alpha(\alpha+\beta)}{\beta^2c^2})}{\int_a^b q(s)ds}$ where $c=\min_{t\in[a,b]}\widetilde{\rho}(t)\frac{\beta}{\alpha+\beta}$. By $(\mathbf{H}_3)$, there exists an $R'>\frac{\beta}{\alpha+\beta}R$ such that

$$F(t,x,y)>N\phi(x),\ \forall\, t\in[a,b],\ \forall\, x\geq R',\ \forall\, y\in\mathbb{R}^+.$$

Define the open ball

$$\Omega_2=\{x\in E:\ \|x\|<R'/c\}.$$

We show that $Ax\not\leq x$ for all $x\in\partial\Omega_2\cap\mathbb{P}$. Suppose on the contrary that there exists $x_0\in\partial\Omega_2\cap\mathbb{P}$ such that $Ax_0\leq x_0$. Since $x_0\in\mathbb{P}\cap\partial\Omega_2$, we have

$$\frac{x_0(t)}{1+t}\geq\frac{\beta}{\alpha+\beta}\widetilde{\rho}(t)\|x_0\|\geq\min_{t\in[a,b]}\frac{\beta}{\alpha+\beta}\widetilde{\rho}(t)\frac{R'}{c}=c\frac{R'}{c}\geq R',\quad\forall\, t\in[a,b].$$

Then, for all $t\in[a,b]$, the following estimates hold:

$$\begin{aligned}
\frac{x_0(t)}{1+t} &\geq \frac{Ax_0(t)}{1+t}\\
&\geq \frac{\beta}{\alpha(1+t)}\phi^{-1}\left(\int_0^{+\infty}q(\tau)F(\tau,\tfrac{x_0(\tau)}{1+\tau},x_0'(\tau))d\tau\right)\\
&> \frac{\beta}{\alpha(1+t)}\phi^{-1}\left(\int_a^b q(\tau)N\phi(\tfrac{x_0(\tau)}{1+\tau})d\tau\right)\\
&\geq \frac{\beta}{\alpha(1+t)}\phi^{-1}\left(\int_a^b q(\tau)N\phi(R')d\tau\right)\\
&\geq \frac{\beta}{\alpha}\widetilde{\rho}(t)\phi^{-1}(\phi(R'))\phi^{-1}(N\int_a^b q(\tau)d\tau)\\
&\geq \frac{R'}{c}\frac{\beta^2c^2}{\alpha(\alpha+\beta)}\phi^{-1}\left(N\int_a^b q(\tau)d\tau\right)\\
&> \frac{R'}{c}.
\end{aligned}$$

Passing to the supremum over t yields $\|x_0\|_1>\frac{R'}{c}$. Hence $\|x_0\|>\frac{R'}{c}$, contradicting $\|x_0\|=\frac{R'}{c}$. Finally, Lemma 2.4 yields

$$i(A,\Omega_2\cap\mathbb{P},\mathbb{P})=0, \tag{3.7}$$

while (3.6) and (3.7) imply that

$$i(A,(\Omega_2\setminus\overline{\Omega}_1)\cap\mathbb{P},\mathbb{P})=-1. \tag{3.8}$$

Then A has another fixed point $y_0\in(\Omega_2\setminus\overline{\Omega}_1)\cap\mathbb{P}$. Moreover $y_0(t)\geq\frac{\beta}{\alpha+\beta}\widetilde{\rho}(t)R$ and $R<\|y_0\|<\frac{R'}{c}$. By (3.3) we have $\|x_0\|<R$, which implies that $\|x_0\|<R<\|y_0\|$ and thus x_0 and y_0 are two distinct positive solutions of (1.1). □

Example 3.5. Consider the boundary value problem

$$\begin{cases}((x'(t))^p)'+((x'(t))^r)'+\delta e^{-t}\frac{(x^p(t)+(1+t)^px'^q(t))}{(1+t)^p}=0,\\ \alpha x(0)-\beta x'(0)=0,\quad \lim\limits_{t\to+\infty}x'(t)=0,\end{cases} \tag{3.9}$$

where p and r $(p<r)$ are two odd numbers. $f(t,x,y)=\frac{(x^p+(1+t)^py^r)}{(1+t)^p}$, $\phi(t)=t^p+t^r$, and $q(t)=\delta e^{-t}$ where $0<\delta<1$ is a positive constant. Then ϕ is continuous, increasing, and $\phi(0) = 0$. Moreover $F(t, x, y) = f(t, (1+t)x, y) = x^p + y^r$. Now, we check the main assumptions.

($\mathbf{H}_1$): Let $g(x, y) = x^p + y^r$ and $m(t) = 1$. Then $F(t, x, y) \leq m(t)g(x, y)$, for all $t, x, y \in \mathbb{R}^+$. Moreover

$$\int_0^{+\infty} q(\tau)m(\tau)d\tau = \int_0^{+\infty} \delta e^{-\tau} d\tau = \delta < +\infty$$

and, for any positive constant c, we have

$$\begin{aligned}
\int_0^{+\infty} \phi^{-1}\left(\int_s^{+\infty} q(\tau)m(\tau)g(c,c)d\tau\right) ds &= \int_0^{+\infty} \phi^{-1}\left(\delta(c^p + c^r)e^{-s}\right) ds \\
&\leq \int_0^{+\infty} \phi^{-1}(c^p + c^r)e^{-s})ds \\
&\leq \int_0^{+\infty} \phi^{-1}((ce^{-\frac{s}{r}})^p + (ce^{-\frac{s}{r}})^r))ds \\
&\leq \int_0^{+\infty} \phi^{-1}(\phi(ce^{-\frac{s}{r}})ds \\
&= \int_0^{+\infty} ce^{-\frac{s}{r}} ds < \infty.
\end{aligned}$$

($\mathbf{H}_2$):

$$\begin{aligned}
\sup_{c>0} \frac{c}{M\phi^{-1}\left(\int_0^{+\infty} q(\tau)m(\tau)g(c,c)d\tau\right)} &= \sup_{c>0} \frac{c}{M\phi^{-1}(\delta(c^p+c^r))} \\
&\geq \sup_{c>0} \frac{c}{M\phi^{-1}((c\delta^{\frac{1}{r}})^p + (c\delta^{\frac{1}{r}})^r))} \\
&= \sup_{c>0} \frac{c}{M\phi^{-1}(\phi(c\delta^{\frac{1}{r}}))} \\
&= \frac{1}{M\delta^{\frac{1}{r}}}.
\end{aligned}$$

If $0 < \delta < (\frac{1}{M})^r$, then all conditions of Theorem 3.2 hold which implies that problem (3.9) has at least one positive solution.

Example 3.6. Consider the boundary value problem

$$\text{(3.10)} \qquad \begin{cases} ((x'(t))^{\frac{1}{5}})' + \delta e^{-t} \frac{(x^2(t) + (1+t)^2 x'^2(t) + (1+t)^2)}{(1+t)^2} = 0, \\ \alpha x(0) - \beta x'(0) = 0, \quad \lim\limits_{t\to+\infty} x'(t) = 0, \end{cases}$$

where $f(t, x, y) = \frac{(x^2 + (1+t)^2 y^2 + (1+t)^2)}{(1+t)^2}$, $\phi(t) = t^{\frac{1}{5}}$, and $q(t) = \delta e^{-t}$ and δ is a positive constant. Then ϕ is continuous, increasing, and $\phi(0) = 0$. Moreover $F(t, x, y) = f(t, (1+t)x, y) = x^2 + y^2 + 1$. We check the main assumptions.

($\mathbf{H}_1$): Let $g(x, y) = x^2 + y^2 + 1$ and $m(t) = 1$. Then $F(t, x, y) \leq m(t)g(x, y)$ for all $t, x, y \in \mathbb{R}^+$,

$$\int_0^{+\infty} q(\tau)m(\tau)d\tau = \int_0^{+\infty} \delta e^{-\tau} d\tau = \delta < +\infty$$

and for each $c > 0$, we have

$$\int_0^{+\infty} \phi^{-1}\left(\int_s^{+\infty} q(\tau)m(\tau)g(c,c)d\tau\right) ds < +\infty.$$

(H$_2$):

$$\sup_{c>0}\frac{c}{M\phi^{-1}\left(\int_0^{+\infty}q(\tau)m(\tau)g(c,c)d\tau\right)}=\sup_{c>0}\frac{c}{M\phi^{-1}(\delta(2c^2+1))}=\sup_{c>0}\frac{c}{M\delta^5(2c^2+1)^5}.$$

If $0<\delta<(\sup\limits_{c>0}\frac{c}{M(2c^2+1)^5})^{\frac{1}{5}}$, then all conditions of Theorem 3.2 hold which implies that problem (3.10) has at least one positive solution.

Example 3.7. Consider the boundary value problem

$$\begin{cases}(a(x'(t))^{\frac{3}{5}})'+e^{-t}\frac{(x^2(t)+(1+t)^2x'^3(t)+(1+t)^2)}{(1+t)^2}=0,\\ \alpha x(0)-\beta x'(0)=0, \quad \lim\limits_{t\to+\infty}x'(t)=0.\end{cases}\tag{3.11}$$

Here $\phi(t)=at^{\frac{3}{5}}$ and $a>\max\{1,(\sup\limits_{c>0}\frac{c}{M(c^2+c^3+1)^{\frac{5}{3}}})^{-1}\}$. Then ϕ is continuous, increasing, $\phi(0)=0$ and, for all $x,y\geq 0$, we have

$$\phi^{-1}(xy)\geq\phi^{-1}(x)\phi^{-1}(y).$$

Moreover $F(t,x,y)=x^2+y^3+1$. Now let $q(t)=e^{-t}, m(t)=1$, and $g(x,y)=x^2+y^3+1$; then it is easy to check (**H$_1$**).

(H$_2$):

$$\begin{aligned}\sup_{c>0}\frac{c}{M\phi^{-1}\left(\int_0^{+\infty}q(\tau)m(\tau)g(c,c)d\tau\right)}&=\sup_{c>0}\frac{c}{M\phi^{-1}(c^2+c^3+1)}\\&=a^{\frac{5}{3}}\sup_{c>0}\frac{c}{M(c^2+c^3+1)^{\frac{5}{3}}}>1.\end{aligned}$$

(H$_3$): It is clear that

$$\lim_{x\to+\infty}\frac{F(t,x,y)}{\phi(x)}=+\infty,\text{ uniformly in t and y.}$$

Then all conditions of Theorem 3.3 are met which implies that problem (3.11) has at least two positive solutions.

4. SINGULARITIES AT $x=0$ BUT NOT AT $x'=0$

In this section, we suppose that $f:\mathbb{R}^+\times I\times\mathbb{R}^+\to\mathbb{R}^+$ is continuous and assume that

(H$_4$): There exist $m,\psi\in C(\mathbb{R}^+,\mathbb{R}^+)$ and $g,h\in C(I,I)$ such that h is a decreasing function and ψ, $\frac{g}{h}$ are increasing functions with

$$F(t,x,y)\leq m(t)g(x)\psi(y),\quad\forall\, t,y\in\mathbb{R}^+,\ \forall\, x\in I\tag{4.1}$$

and for each $c,c'>0$,

$$\int_0^{+\infty}q(\tau)m(\tau)h(c\widetilde{\rho}(\tau))d\tau<+\infty,\tag{4.2}$$

$$\int_0^{+\infty} \phi^{-1}\left(\frac{g(c')}{h(c')}\psi(c')\int_s^{+\infty} q(\tau)m(\tau)h(c\widetilde{\rho}(\tau))d\tau\right) ds < +\infty. \tag{4.3}$$

(H$_5$): For any $c > 0$, there exists $\psi_c \in C(\mathbb{R}^+, \mathbb{R}^+)$ and there exists an interval $J \subset (1, +\infty)$ such that $\psi_c(t) > 0$ on J and

$$F(t,x,y) \geq \psi_c(t), \quad \forall\, t, y \in \mathbb{R}^+,\ \forall\, x \in (0, c]$$

with

$$\int_0^{+\infty} q(\tau)\psi_c(\tau)d\tau < +\infty. \tag{4.4}$$

(H$_6$):

$$\sup_{c>0} \frac{c}{M\phi^{-1}\left(\frac{g(c)}{h(c)}\psi(c)\int_0^{+\infty} q(\tau)m(\tau)h(c\frac{\beta}{\alpha+\beta}\widetilde{\rho}(\tau))d\tau\right)} > 1.$$

(H$_7$): There exist positive numbers $a < b$ such that

$$\lim_{x\to+\infty} \frac{F(t,x,y)}{\phi(x)} = +\infty, \quad \text{uniformly in } t \in [a,b] \text{ and } y \geq 0.$$

Given $f \in C(\mathbb{R}^+ \times I \times \mathbb{R}^+, \mathbb{R}^+)$, define a sequence of approximating functions $\{f_n\}_{n\geq 1}$ by

$$f_n(t,x,y) = f(t, \max\{(1+t)/n, x\}, y), \quad n \in \{1, 2, \ldots\}$$

and for $x \in \mathbb{P}$, define a sequence of operators by

$$\begin{aligned} A_n x(t) = {} & \frac{\beta}{\alpha}\phi^{-1}\left(\int_0^{+\infty} q(\tau)f_n(\tau, x(\tau), x'(\tau))d\tau\right) \\ & + \int_0^t \phi^{-1}\left(\int_s^{+\infty} q(\tau)f_n(\tau, x(\tau), x'(\tau))d\tau\right) ds. \end{aligned}$$

We have

Lemma 4.1. *Suppose* ($\mathbf{H}_4$) *holds. Then, for each $n \geq 1$, the operator A_n sends $\mathbb{P}$ into $\mathbb{P}$ and is completely continuous.*

Proof. First, we check the integrability of the function δ in Lemma 2.15. For all $n \geq 1$, we have:

$$\begin{array}{rcl} \int_0^\infty \delta(\tau)d\tau & = & \int_0^\infty q(\tau)f_n(\tau, x(\tau), x'(\tau))d\tau \\ & = & \int_0^\infty q(\tau)f(\tau, \max\{\frac{1+\tau}{n}, x(\tau)\}, x'(\tau))d\tau \\ & = & \int_0^\infty q(\tau)F(\tau, \max\{\frac{1}{n}, \frac{x(\tau)}{1+\tau}\}, x'(\tau))d\tau \\ & \leq & \int_0^\infty q(\tau)m(\tau)h(\max\{\frac{1}{n}, \frac{x(\tau)}{1+\tau}\})\frac{g(\max\{\frac{1}{n}, \frac{x(\tau)}{1+\tau}\})}{h(\max\{\frac{1}{n}, \frac{x(\tau)}{1+\tau}\})}\psi(x'(\tau))d\tau \\ & \leq & \int_0^\infty q(\tau)m(\tau)h(\frac{1}{n})\frac{g(\|x\|)}{h(\|x\|)}\psi(\|x\|)d\tau \\ & \leq & \frac{g(\|x\|)}{h(\|x\|)}\psi(\|x\|)\int_0^\infty q(\tau)m(\tau)h(\frac{1}{n}\widetilde{\rho}(\tau))d\tau. \end{array}$$

By (4.2), the right-hand side is finite. Therefore $A_n\mathbb{P} \subseteq \mathbb{P}$. The proof that A_n is completely continuous is similar to that of the operator A in Theorem 3.2 and is omitted. $\square$

4.1. **Existence of a single solution.**

Theorem 4.2. *Assume that Assumptions* $(\mathbf{H}_4)-(\mathbf{H}_6)$ *hold. Then problem* (1.1) *has at least one positive solution.*

Proof.

Step 1: An approximating solution. From condition $(\mathbf{H}_6)$, there exists $R>0$ such that

$$\frac{R}{M\phi^{-1}\left(\frac{g(R)}{h(R)}\psi(R)\int_0^{+\infty}q(\tau)m(\tau)h(\frac{\beta}{\alpha+\beta}R\widetilde{\rho}(\tau))d\tau\right)}>1. \tag{4.5}$$

Let

$$\Omega_1=\{x\in E:\ \|x\|<R\}.$$

We claim that $x\neq\lambda A_nx$ for any $x\in\partial\Omega_1\cap\mathbb{P},\lambda\in(0,1]$, and $n\geq n_0>1/R$. On the contrary, suppose that there exists $n_1\geq n_0$, $x_1\in\partial\Omega_1\cap\mathbb{P}$ and $\lambda_1\in(0,1]$ such that $x_1=\lambda_1A_{n_1}x_1$. By Lemma 2.13, we have $x_1(t)\geq\frac{\beta}{\alpha+\beta}\rho(t)\|x_1\|=\frac{\beta}{\alpha+\beta}\rho(t)R,\forall\,t\in\mathbb{R}^+$. Then $\frac{x_1(t)}{1+t}\geq\frac{\beta}{\alpha+\beta}\widetilde{\rho}(t)R$. As a consequence, we derive the estimates:

$$\begin{aligned}
R &= \|x_1\|\\
&= \|\lambda_1A_{n_1}x_1\|\\
&\leq \|A_{n_1}x_1\|\\
&\leq M\|A_{n_1}x_1\|_2\\
&\leq M\sup_{t\geq0}\phi^{-1}\left(\int_t^{+\infty}q(\tau)f_{n_1}(\tau,x_1(\tau),x_1'(\tau))d\tau\right),\\
&\leq M\phi^{-1}\left(\int_0^{+\infty}q(\tau)F(\tau,\max\{\tfrac{1}{n_1},\tfrac{x_1(\tau)}{1+\tau}\},x_1'(\tau))d\tau\right),\\
&\leq M\phi^{-1}\left(\int_0^{+\infty}q(\tau)m(\tau)g(\max\{\tfrac{1}{n_1},\tfrac{x_1(\tau)}{1+\tau}\})\psi(x_1'(\tau))d\tau\right),\\
&\leq M\phi^{-1}(\int_0^{+\infty}q(\tau)m(\tau)h(\max\{\tfrac{1}{n_1},\tfrac{x_1(\tau)}{1+\tau}\})\frac{g(\max\{\frac{1}{n_1},\frac{x_1(\tau)}{1+\tau}\})}{h(\max\{\frac{1}{n_1},\frac{x_1(\tau)}{1+\tau}\})}\psi(x_1'(\tau))d\tau\\
&\leq M\phi^{-1}\left(\frac{g(R)}{h(R)}\psi(R)\int_0^{+\infty}q(\tau)m(\tau)h(\tfrac{\beta}{\alpha+\beta}\widetilde{\rho}(\tau)R)d\tau\right),
\end{aligned}$$

which is a contradiction to (4.5). Then by Lemma 2.3, we deduce that

$$i(A_n,\Omega_1\cap\mathbb{P},\mathbb{P})=1,\ \text{ for all }\ n\in\{n_0,n_0+1,\ldots\}. \tag{4.6}$$

Hence there exists an $x_n\in\Omega_1\cap\mathbb{P}$ such that $A_nx_n=x_n$, $\forall\,n\geq n_0$.

Step 2: a compactness argument.

(a) Since $\|x_n\|<R$, by $(\mathbf{H}_5)$ there exists $\psi_R\in C(\mathbb{R}^+,\mathbb{R}^+)$ such that

$$f_n(t,x_n(t),x_n'(t))\geq\psi_R(t),\quad\forall\,t\in I$$

with

$$\int_0^{+\infty}q(s)\psi_R(s)ds<+\infty.$$

Then

$$\begin{aligned} x_n(t) &= A_n x_n(t) \\ &\geq \int_0^t \phi^{-1}\left(\int_s^{+\infty} q(\tau)f_n(\tau, x_n(\tau), x_n'(\tau))d\tau\right) ds \\ &\geq \int_0^t \phi^{-1}\left(\int_s^{+\infty} q(\tau)\psi_R(\tau)d\tau\right) ds. \end{aligned}$$

Let

$$c^* = \phi^{-1}\left(\int_1^{+\infty} q(\tau)\psi_R(\tau)d\tau\right) > 0, \tag{4.7}$$

and distinguish between two cases:

(i): If $t \in [0, 1]$, then

$$x_n(t) \geq t\phi^{-1}\left(\int_t^{+\infty} q(\tau)\psi_R(\tau))d\tau\right) \geq t\phi^{-1}\left(\int_1^{+\infty} q(\tau)\psi_R(\tau)d\tau\right) = \rho(t)c^*.$$

(ii): If $t \in (1, +\infty)$, then

$$\begin{aligned} x_n(t) &\geq \int_0^1 \phi^{-1}\left(\int_s^{+\infty} q(\tau)\psi_R(\tau)d\tau\right) ds \\ &\geq \int_0^1 \phi^{-1}\left(\int_1^{+\infty} q(\tau)\psi_R(\tau)d\tau\right) ds \\ &\geq \phi^{-1}\left(\int_1^{+\infty} q(\tau)\psi_R(\tau)d\tau\right) \\ &\geq \tfrac{1}{t}\phi^{-1}\left(\int_1^{+\infty} q(\tau)\psi_R(\tau)d\tau\right) \\ &\geq \rho(t)c^*. \end{aligned}$$

Then, we deduce that $\frac{x_n(t)}{1+t} \geq c^*\widetilde{\rho}(t),\ \forall\, t \in \mathbb{R}^+,\ \forall\, n \geq n_0$.

(b) For any $T > 0$ and $t, t' \in [0, T]$ $(t > t')$, the following estimates hold:

$$\begin{aligned} &\left|\tfrac{x_n(t)}{1+t} - \tfrac{x_n(t')}{1+t'}\right| \\ \leq\ &\left|\tfrac{1}{1+t} - \tfrac{1}{1+t'}\right| \phi^{-1}\left(\int_0^{+\infty} q(\tau)f_n(\tau, x_n(\tau), x_n'(\tau))d\tau\right) \\ &+\left|\frac{\int_0^t \phi^{-1}\left(\int_s^{+\infty} q(\tau)f_n(\tau,x_n(\tau),x_n'(\tau))d\tau\right)ds}{1+t} - \frac{\int_0^{t'} \phi^{-1}\left(\int_s^{+\infty} q(\tau)f_n(\tau,x_n(\tau),x_n'(\tau))d\tau\right)ds}{1+t'}\right| \\ \leq\ &\left|\tfrac{1}{1+t} - \tfrac{1}{1+t'}\right| \phi^{-1}\left(\int_0^{+\infty} q(\tau)m(\tau)h(c^*\widetilde{\rho}(s))\tfrac{g(R)}{h(R)}\psi(R)d\tau\right) \\ &+2\left|\tfrac{1}{1+t} - \tfrac{1}{1+t'}\right| \int_0^{+\infty} \phi^{-1}\left(\int_s^{+\infty} q(\tau)m(\tau)h(c^*\widetilde{\rho}(\tau))\tfrac{g(R)}{h(R)}\psi(R)d\tau\right) ds \\ &+\tfrac{1}{1+t'}\int_{t'}^t \phi^{-1}(\int_s^{+\infty} q(\tau)m(\tau)h(c^*\widetilde{\rho}(\tau))\tfrac{p(R)}{h(R)}\psi(R)d\tau)ds \end{aligned}$$

and

$$\begin{aligned} |\phi(x_n'(t)) - \phi(x_n'(t'))| &= \left|\int_{t'}^t q(\tau)f_n(\tau, x_n(\tau), x_n'(\tau))d\tau\right| \\ &\leq \int_{t'}^t q(\tau)m(\tau)h(c^*\widetilde{\rho}(\tau))\frac{p(R)}{h(R)}\psi(R)d\tau. \end{aligned}$$

Then, for any $\varepsilon > 0$ and $T > 0$, there exists $\delta > 0$ such that $|\frac{x_n(t)}{1+t} - \frac{x_n(t')}{1+t'}| < \varepsilon$ and $|x_n'(t) - x_n'(t')| < \varepsilon$ for all $t, t' \in [0, T]$ with $|t - t'| < \delta$.

(c) For any $n \geq 0$, we have, by $(\mathbf{H}_4)$, $\lim\limits_{t\to+\infty} \frac{x_n(t)}{1+t} = \lim\limits_{t\to+\infty} x'_n(t) = 0$. Therefore

$$\begin{aligned}
& \sup_{n\geq n_0} \left|\frac{x_n(t)}{1+t} - \lim_{t\to+\infty} \frac{x_n(t)}{1+t}\right| \\
= & \sup_{n\geq n_0} \frac{\frac{\beta}{\alpha}\phi^{-1}\left(\int_0^{+\infty} q(\tau)f_n(\tau,x_n(\tau),x'_n(\tau))d\tau\right)+\int_0^t \phi^{-1}\left(\int_s^{+\infty} q(\tau)f_n(\tau,x_n(\tau),x'_n(\tau))d\tau\right)ds}{1+t} \\
\leq & \frac{\frac{\beta}{\alpha}\phi^{-1}\left(\frac{g(R)}{h(R)}\psi(R)\int_0^{+\infty} q(\tau)m(\tau)h(c^*\widetilde{\rho}(\tau))d\tau\right)}{1+t} + \frac{\int_0^{+\infty} \phi^{-1}\left(\frac{g(R)}{h(R)}\psi(R)\int_s^{+\infty} q(\tau)m(\tau)h(c^*\widetilde{\rho}(\tau))d\tau\right)ds}{1+t}.
\end{aligned}$$

where the right-hand side tends to 0 as $t \to +\infty$. Also

$$\begin{aligned}
& \lim_{t\to+\infty} \sup_{n\geq n_0} |x'_n(t) - \lim_{t\to+\infty} x'_n(t)| \\
= & \lim_{t\to+\infty} \sup_{n\geq n_0} \phi^{-1}\left(\int_t^{+\infty} q(\tau)f_n(\tau,x_n(\tau),x'_n(\tau))d\tau\right) \\
\leq & \lim_{t\to+\infty} \phi^{-1}\left(\frac{g(R)}{h(R)}\psi(R)\int_t^{+\infty} q(\tau)m(\tau)h(c^*\widetilde{\rho}(\tau))d\tau\right) = 0.
\end{aligned}$$

Therefore $\{x_n\}_{n\geq n_0}$ is relatively compact in E by Lemma 2.6 and hence there exists a subsequence $\{x_{n_k}\}_{k\geq 1}$ converging to some limit x_0. Since $x_{n_k}(t) \geq \widetilde{\rho}(t)c^*, \forall\, k \geq 1$, we infer that $x_0(t) \geq \widetilde{\rho}(t)c^*, \forall\, t \in \mathbb{R}^+$. From (4.5), we have $\|x_0\| < R$. Consequently, the continuity of f implies that, for all $s \in \mathbb{R}^+$, we have

$$\begin{aligned}
\lim_{k\to+\infty} f_{n_k}(s,x_{n_k}(s),x'_{n_k}(s)) &= \lim_{k\to+\infty} f(s,\max\{(1+s)/n_k, x_{n_k}(s)\}, x'_{n_k}(s)) \\
&= f(s,\max\{0,x_0(s)\},x'_0(s)) = f(s,x_0(s),x'_0(s)).
\end{aligned}$$

By the Lebesgue dominated convergence theorem, we finally deduce that

$$\begin{aligned}
x_0(t) &= \lim_{k\to+\infty} x_{n_k}(t) \\
&= \lim_{k\to+\infty} \frac{\beta}{\alpha}\phi^{-1}\left(\int_0^{+\infty} q(\tau)f_{n_k}(\tau,x_{n_k}(\tau),x'_{n_k}(\tau))d\tau\right) \\
&\quad + \int_0^t \phi^{-1}\left(\int_s^{+\infty} q(\tau)f_{n_k}(\tau,x_{n_k}(\tau),x'_{n_k}(\tau))d\tau\right) ds \\
&= \frac{\beta}{\alpha}\phi^{-1}\left(\int_0^{+\infty} q(\tau)f(\tau,x_0(\tau),x'_0(\tau))d\tau\right) \\
&\quad + \int_0^t \phi^{-1}\left(\int_s^{+\infty} q(\tau)f(\tau,x_0(\tau),x'_0(\tau))d\tau\right) ds.
\end{aligned}$$

Therefore x_0 is a positive solution of problem (1.1). □

4.2. **Two positive solutions.**

Theorem 4.3. *Assume that* $(\mathbf{H}_4)-(\mathbf{H}_7)$ *hold and* ϕ^{-1} *is super-multiplicative. Then problem* (1.1) *has at least two positive solutions.*

The proof is identical to that of Theorem 3.3 and is omitted.

Example 4.4. Consider the singular boundary value problem

$$(4.8) \qquad \begin{cases} ((x'(t))^{\frac{1}{5}})' + \delta e^{-t}\frac{(1+t)^{\omega-2}m(t)(x^2(t)+(1+t)^2)(x'^3(t)+1)}{x^{\omega}(t)} = 0, \\ \alpha x(0) - \beta x'(0) = 0, \quad \lim\limits_{t\to+\infty} x'(t) = 0, \end{cases}$$

where

$$m(t)=\begin{cases}\frac{t^\omega}{(1+t)^\omega}, & t\in[0,1]\\ \frac{1}{t^\omega(1+t)^\omega}, & t\in(1,+\infty),\end{cases}$$

$f(t,x,y)=\frac{m(t)(1+t)^{\omega-2}(x^2+(1+t)^2)(y^3+1)}{x^\omega}$ $(\omega>0)$, $\phi(t)=t^{\frac{1}{5}}$, and $q(t)=\delta e^{-t}$. Then ϕ is continuous, increasing and $\phi(0)=0$. Moreover

$$F(t,x,y)=f(t,(1+t)x,y)=\frac{m(t)(x^2+1)(y^3+1)}{x^\omega}.$$

($\mathbf{H}_4$): Let $g(x)=\frac{x^2+1}{x^\omega}$, $\psi(y)=(y^3+1)$, and $h(x)=\frac{1}{x^\omega}$. Then h is a decreasing function, $\psi,\frac{g}{h}$ are increasing functions, and

$$F(t,x,y)\le m(t)g(x)\psi(y),\ \forall\, t,y\in\mathbb{R}^+,\ \forall\, x>0.$$

Moreover, for any $c,c'>0$, we have

$$\int_0^{+\infty}q(\tau)m(\tau)h(c\widetilde{\rho}(\tau))d\tau=\frac{\delta}{c^\omega}\int_0^{+\infty}e^{-\tau}d\tau=\frac{\delta}{c^\omega}<+\infty$$

and

$$\begin{aligned}\int_0^{+\infty}\phi^{-1}\left(\int_s^{+\infty}\frac{g(c')\psi(c')}{h(c')}q(\tau)m(\tau)h(c\widetilde{\rho}(\tau))d\tau\right)ds &= \int_0^{+\infty}\phi^{-1}\left(\delta\frac{g(c')\psi(c')}{ch(c')}e^{-s}\right)ds\\ &= (\delta\frac{g(c')\psi(c')}{ch(c')})^5\int_0^{+\infty}e^{-5s}ds\\ &= \frac{1}{5}(\delta\frac{g(c')\psi(c')}{ch(c')})^5<+\infty.\end{aligned}$$

($\mathbf{H}_5$): For any $c>0$, there exists $\psi_c(t)=\frac{m(t)}{c^\omega}$ such that

$$F(t,x,y)\ge\psi_c(t),\ \forall\, t,y\in\mathbb{R}^+,\ \forall\, x\in(0,c]$$

and $\int_0^{+\infty}q(\tau)\psi_c(\tau)d\tau<+\infty$.

($\mathbf{H}_6$):

$$\begin{aligned}\sup_{c>0}\frac{c}{M\phi^{-1}\left(\frac{g(c)\psi(c)}{h(c)}\int_0^{+\infty}q(\tau)m(\tau)h(c\frac{\beta}{\alpha+\beta}\widetilde{\rho}(\tau))d\tau\right)} &= \sup_{c>0}\frac{c}{M\left(\delta\frac{(\alpha+\beta)^\omega}{\beta^\omega}\frac{g(c)\psi(c)}{c^\omega h(c)}\right)^5}\\ &= \frac{\beta^{5\omega}}{M\delta^5(\alpha+\beta)^{5\omega}}\sup_{c>0}\frac{c^{5\omega+1}}{(c^2+1)^5(c^3+1)^5}.\end{aligned}$$

If we choose α,β,ω, and δ such that $\frac{M\delta^5(\alpha+\beta)^{5\omega}}{\beta^{5\omega}}<\sup_{c>0}\frac{c^{5\omega+1}}{(c^2+1)^5(c^3+1)^5}$, then all conditions of Theorem 4.2 are fulfilled, which implies that problem (4.8) has at least one positive solution.

Example 4.5. Consider the singular boundary value problem

$$\text{(4.9)}\qquad\begin{cases}(a(x'(t))^{\frac{3}{5}})'+e^{-t}\frac{m(t)(x^2(t)+(1+t)^2)(x'^3(t)+1)}{(1+t)x(t)}=0,\\ \alpha x(0)-\beta x'(0)=0,\quad \lim\limits_{t\to+\infty}x'(t)=0.\end{cases}$$

Here

$$m(t)=\begin{cases}\frac{t^2}{(1+t)^2}, & t\in[0,1]\\ \frac{1}{t^2(1+t)^2}, & t\in(1,+\infty)\end{cases}$$

and $\phi(t) = at^{\frac{3}{5}}$, where $a > \max\{1, (\sup_{c>0} \frac{c}{M\left(\frac{(c^2+1)(c^3+1)(\alpha+\beta)^2}{\sqrt{c}\beta^2}\right)^{\frac{5}{3}}})^{-1}\}$. Then ϕ is continuous, increasing, $\phi(0) = 0$ and, for all $x, y \geq 0$, satisfies

$$\phi^{-1}(xy) \geq \phi^{-1}(x)\phi^{-1}(y).$$

Moreover $F(t, x, y) = \frac{m(t)(x^2+1)(y^3+1)}{\sqrt{x}}$. Let $h(x) = \frac{1}{x^2}$, $g(x) = \frac{x^2+1}{\sqrt{x}}$, and $\psi(y) = (y^3+1)$; then it is easy to show $(\mathbf{H}_4)$ and $(\mathbf{H}_5)$.

$(\mathbf{H}_6)$:

$$\begin{aligned}\sup_{c>0} \frac{c}{M\phi^{-1}\left(\frac{g(c)\psi(c)}{h(c)}\int_0^{+\infty} q(\tau)m(\tau)h(c\frac{\beta}{\alpha+\beta}\widetilde{\rho}(\tau))d\tau\right)} &= \sup_{c>0} \frac{c}{M\phi^{-1}\left(\frac{g(c)\psi(c)(\alpha+\beta)^2}{\beta^2 c^2 h(c)}\right)} \\ &= \sup_{c>0} \frac{c}{M\left(\frac{g(c)\psi(c)(\alpha+\beta)^2}{\beta^2 a}\right)^{\frac{5}{3}}} \\ &= a^{\frac{5}{3}} \sup_{c>0} \frac{c}{M\left(\frac{g(c)\psi(c)(\alpha+\beta)^2}{\beta^2}\right)^{\frac{5}{3}}} > 1.\end{aligned}$$

$(\mathbf{H}_7)$: It is clear that, for any compact interval $[a, b] \subseteq (0, +\infty)$, we have

$$\lim_{x\to+\infty} \frac{F(t, x, y)}{\phi(x)} = \lim_{x\to+\infty} \frac{m(t)(x^2+1)(y^3+1)}{ax^{\frac{3}{5}}\sqrt{x}} = +\infty,\ \forall\, t \in [a, b], \forall\, y \in \mathbb{R}^+.$$

Then all conditions of Theorem 4.3 are met; consequently problem (4.9) has at least two positive solutions.

4.3. **A further result.** Theorem 4.2 still holds true if we keep $(\mathbf{H}_5)$ and replace conditions $(\mathbf{H}_4)$ and $(\mathbf{H}_6)$ by the following one:

$(\mathbf{H}_4)'$: there exist $m \in C(\mathbb{R}^+, \mathbb{R}^+)$ and a decreasing function $l \in C(I, I)$ such that

$$F(t, x, y) \leq m(t)l(x/y), \quad \forall\, t \in \mathbb{R}^+,\ \forall\, x, y \in I \tag{4.10}$$

and for any $c > 0$

$$\begin{cases} \int_0^{+\infty} q(\tau)m(\tau)l(c\widetilde{\rho}(\tau))d\tau < +\infty, \\ \int_0^{+\infty} \phi^{-1}\left(\int_s^{+\infty} q(\tau)m(\tau)l(c\widetilde{\rho}(\tau))d\tau\right) ds < +\infty. \end{cases} \tag{4.11}$$

Now, given $f \in C(\mathbb{R}^+ \times I \times \mathbb{R}^+, \mathbb{R}^+)$, define a sequence of approximating functions $\{f_n\}_{n\geq1}$ by

$$f_n(t, x, y) = f(t, \max\{(1 + t)/n, x\}, \max\{1/n, y\}), \quad n \in \{1, 2, \ldots\}.$$

Next for $x \in \mathbb{P}$, define a sequence of operators by

$$\begin{aligned} A_n x(t) = &\frac{\beta}{\alpha}\phi^{-1}\left(\int_0^{+\infty} q(\tau)f_n(\tau, x(\tau), x'(\tau))d\tau\right) \\ &+ \int_0^t \phi^{-1}\left(\int_s^{+\infty} q(\tau)f_n(\tau, x(\tau), x'(\tau))d\tau\right) ds. \end{aligned}$$

We have

Theorem 4.6. *Assume that Assumptions $(\mathbf{H}_4)'$ and $(\mathbf{H}_5)$ hold. Then problem (1.1) has at least one positive solution.*

Proof. Lemma 2.15 implies that $A_n\mathbb{P} \subseteq \mathbb{P}$. The proof that A_n is completely continuous is similar to that of the operator A in Theorem 3.2 and is omitted. Let R be such that

$$R > M\phi^{-1}\left(\int_0^{+\infty} q(\tau)m(\tau)l(\min\left(\frac{\beta}{\alpha},1\right)\widetilde{\rho}(\tau))d\tau\right)$$

and set

$$\Omega_1 = \{x \in E: \ \|x\| < R\}.$$

We claim that $x \neq \lambda A_n x$, for any $x \in \partial\Omega_1 \cap \mathbb{P}, \lambda \in (0,1]$, and $n \geq 1$. On the contrary, suppose that there exists $n_0 \geq 1$, $x_0 \in \partial\Omega_1 \cap \mathbb{P}$, and $\lambda_0 \in (0,1]$ such that $x_0 = \lambda_0 A_{n_0} x_0$. By Lemma 2.14, we have $\frac{x_0(t)}{x_0'(t)} \geq \frac{\beta}{\alpha}\rho(t)$ and, for each $n \geq 1$

$$\frac{\max\{1/n, \frac{x_0(t)}{1+t}\}}{\max\{\frac{1}{n}, x_0'(t)\}} \geq \min\{\frac{\beta}{\alpha},1\}\widetilde{\rho}(t),\ \forall t \in \mathbb{R}^+.$$

Therefore

$$\begin{aligned}
R &= \|x_0\| \\
&= \|\lambda_0 A_{n_0} x_0\| \\
&\leq \|A_{n_0} x_0\| \\
&\leq M\|A_{n_0} x_0\|_2 \\
&\leq M\sup_{t\geq 0}\phi^{-1}\left(\int_t^{+\infty} q(\tau)f_{n_0}(\tau, x_0(\tau), x_0'(\tau))d\tau\right), \\
&\leq M\phi^{-1}\left(\int_0^{+\infty} q(\tau)F(\tau, \max\{\tfrac{1}{n_0}, \tfrac{x_0(\tau)}{1+\tau}\}, \max\{\tfrac{1}{n_0}, x_0'(\tau)\}\}d\tau\right), \\
&\leq M\phi^{-1}\left(\int_0^{+\infty} q(\tau)m(\tau)l(\tfrac{\max\{\frac{1}{n_0}, \frac{x_0(\tau)}{1+\tau}\}}{\max\{\frac{1}{n_0}, x_0'(\tau)\}})d\tau\right), \\
&\leq M\phi^{-1}(\int_0^{+\infty} q(\tau)m(\tau)l(\min\{\tfrac{\beta}{\alpha},1\}\widetilde{\rho}(\tau)))d\tau,
\end{aligned}$$

which is a contradiction. By Lemma 2.3, we conclude that

$$i(A_n, \Omega_1 \cap \mathbb{P}, \mathbb{P}) = 1, \ \text{ for all } \ n \in \{1, 2, \ldots\}. \tag{4.12}$$

Hence there exists an $x_n \in \Omega_1 \cap \mathbb{P}$ such that $A_n x_n = x_n$, $\forall\, n \geq 1$. Arguing as in the proof of Theorem 4.2, Step 2 together with the condition $(\mathbf{H}_4)'$ and the fact that

$$q(\tau)f_n(\tau, x_n(\tau), x_n'(\tau)) \leq q(\tau)m(\tau)l\left(\min\{\frac{\beta}{\alpha},1\}\widetilde{\rho}(\tau)\right),$$

we can prove that $\{x_n\}_n$ is relatively compact. Hence there exists a subsequence $\{x_{n_k}\}_{k\geq 1}$ with limit $\lim_{k\to+\infty} x_{n_k} = \bar{x}$. Since $\|\bar{x}\| \leq R$, then from $(\mathbf{H}_5)$ we deduce that $\bar{x}(t) \geq c^*\rho(t), \forall\, t \in \mathbb{R}^+$, where c^* is defined by (4.7); thus $\bar{x}$ is a positive solution of problem (1.1). □

Now, consider the following assumptions:

$(\mathbf{A}_1)$: There exist $m \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $g, \psi, l \in C(I, I)$ such that l is a decreasing function and ψ, g are increasing functions with

$$F(t,x,y) \leq m(t)g(x)\psi(y)l(x/y), \quad \forall\, t \in \mathbb{R}^+,\ \forall\, x, y \in I. \tag{4.13}$$

For each $c, c' > 0$,

$$\int_0^{+\infty} q(\tau)m(\tau)l(c\widetilde{\rho}(\tau))d\tau < +\infty, \tag{4.14}$$

$$\int_0^{+\infty} \phi^{-1}\left(g(c')\psi(c')\int_s^{+\infty} q(\tau)m(\tau)l(c\widetilde{\rho}(\tau))d\tau\right) ds < +\infty. \tag{4.15}$$

($\mathbf{A}_2$):

$$\sup_{c>0} \frac{c}{M\phi^{-1}\left(g(c)\psi(c)\int_0^{+\infty} q(\tau)m(\tau)l(\min\{\frac{\beta}{\alpha},1\}\widetilde{\rho}(\tau))d\tau\right) ds} > 1.$$

($\mathbf{A}_3$): For any $c > 0$, there exists $\psi_c \in C(\mathbb{R}^+, \mathbb{R}^+)$ and there exists an interval $J \subset (1, +\infty)$ such that $\psi_c(t) > 0$ on J and

$$F(t, x, y) \geq \psi_c(t), \quad \forall\, t, y \in \mathbb{R}^+,\ \forall\, x \in (0, c]$$

with

$$\int_0^{+\infty} q(\tau)\psi_c(\tau)d\tau < +\infty. \tag{4.16}$$

($\mathbf{A}_4$): There exist positive numbers $a < b$ such that

$$\lim_{x\to+\infty} \frac{F(t, x, y)}{\phi(x)} = +\infty, \quad \text{uniformly in } t \in [a, b] \text{ and } y \geq 0.$$

We state without proof another existence result:

Theorem 4.7. *Assume that Assumptions* $(\mathbf{A}_1) - (\mathbf{A}_4)$ *hold. Then problem* (1.1) *has at least two positive solutions.*

5. SINGULARITIES AT $x = 0$ AND AT $x' = 0$

In this final section, we suppose that the nonlinearity f is positive, continuous on $\mathbb{R}^+ \times I \times I$, and ϕ is multiplicative, i.e.

$$\phi(xy) = \phi(x)\phi(y),\ \forall\, x, y \geq 0.$$

Next, we list some assumptions:

($\mathbf{H}_8$): there exist $m \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $g, h, \psi, l \in C(I, I)$ such that h, l are decreasing functions and $\frac{\psi}{l}$, $\frac{g}{h}$ are increasing functions with

$$F(t, x, y) \leq m(t)g(x)\psi(y),\ \forall\, t \in \mathbb{R}^+,\ \forall\, x, y \in I, \tag{5.1}$$

and for each $c > 0$,

$$\int_0^{+\infty} q(\tau)m(\tau)h(c\widetilde{\rho}(\tau))d\tau < +\infty. \tag{5.2}$$

($\mathbf{H}_9$): For any $c > 0$, there exists $\psi_c \in C(\mathbb{R}^+, \mathbb{R}^+)$ and there exists an interval $J \subset (1, +\infty)$ such that $\psi_c(t) > 0$, in J and

$$F(t,x,y) \geq \psi_c(t), \quad \forall\, t \in I,\ \forall\, x, y \in (0, c]$$

with

$$\gamma_c(t) = \int_t^{+\infty} q(\tau)\psi_c(\tau) d\tau \leq \int_0^{+\infty} q(\tau)\psi_c(\tau) d\tau < +\infty,$$

and for each $k > 0$,

$$\int_0^{+\infty} \phi^{-1}\left(\int_s^{+\infty} q(\tau)m(\tau)h(k\widetilde{\rho}(\tau))l(\phi^{-1}(\gamma_c(\tau))d\tau\right) ds < +\infty.$$

($\mathbf{H}_{10}$):

$$\sup_{c>0} \frac{c}{M\phi^{-1}\left(L^{-1}\left(\frac{g(c)\psi(c)}{h(c)l(c)}\int_0^{+\infty} q(\tau)m(\tau)h(c\frac{\beta}{\alpha+\beta}\widetilde{\rho}(\tau)), d\tau\right)\right)} > 1,$$

where L is defined by

$$L(u) = \int_0^u \frac{ds}{l(\phi^{-1}(s))},\ \forall\, u \in \mathbb{R}^+.$$

($\mathbf{H}_{11}$): There exist positive numbers $a < b$ such that

$$\lim_{x\to+\infty} \frac{F(t,x,y)}{\phi(x)} = +\infty, \quad \text{uniformly in } t \in [a,b] \text{ and } y > 0.$$

Now, given $f \in C(\mathbb{R}^+ \times I \times I, \mathbb{R}^+)$, define a sequence of approximating functions $\{f_n\}_{n\geq 1}$ by

$$f_n(t,x,y) = f(t, \max\{(1+t)/n, x\}, \max\{1/n, y\}), \quad n \in \{1, 2, \ldots\}$$

and for $x \in \mathbb{P}$, define a sequence of operators by

$$\begin{aligned} A_n x(t) = &\frac{\beta}{\alpha}\phi^{-1}\left(\int_0^{+\infty} q(\tau)f_n(\tau, x(\tau), x'(\tau))d\tau\right) \\ &+ \int_0^t \phi^{-1}\left(\int_s^{+\infty} q(\tau)f_n(\tau, x(\tau), x'(\tau))d\tau\right) ds. \end{aligned}$$

We have

Lemma 5.1. *Suppose that* ($\mathbf{H}_8$) *holds. Then, for each $n \geq 1$, the operator A_n sends $\mathbb{P}$ into $\mathbb{P}$ and is completely continuous.*

Proof. Lemma 2.15 yields that $A_n\mathbb{P} \subseteq \mathbb{P}$. The proof that A_n is completely continuous is similar to that of the operator A in Theorem 3.2; hence it is omitted. ∎

5.1. **Existence of a single positive solution.**

Theorem 5.2. *Assume that Assumptions* $(\mathbf{H}_8)-(\mathbf{H}_{10})$ *hold. Then problem* (1.1) *has at least one positive solution.*

Proof.

Step 1: An approximating solution. From condition $(\mathbf{H}_{10})$, there exists $R>0$ such that

$$\frac{R}{M\phi^{-1}\left(L^{-1}\left(\frac{g(R)\psi(R)}{h(R)l(R)}\int_0^{+\infty}q(\tau)m(\tau)h(\frac{\beta}{\alpha+\beta}R\widetilde{\rho}(\tau))d\tau\right)\right)}>1. \tag{5.3}$$

Define the open ball

$$\Omega_1=\{x\in E:\ \|x\|<R\}.$$

We claim that $x\neq\lambda A_nx$ for any $x\in\partial\Omega_1\cap\mathbb{P},\lambda\in(0,1]$ and $n\geq n_0>1/R$. On the contrary, suppose that there exists $n_1\geq n_0$, $x_1\in\partial\Omega_1\cap\mathbb{P}$, and $\lambda_1\in(0,1]$ such that $x_1=\lambda_1A_{n_1}x_1$. By Lemma 2.13, we have

$$x_1(t)\geq\frac{\beta}{\alpha+\beta}\rho(t)\|x_1\|=\frac{\beta}{\alpha+\beta}\rho(t)R,\ \forall\,t\in\mathbb{R}^+.$$

Then $\frac{x_1(t)}{1+t}\geq\frac{\beta}{\alpha+\beta}\widetilde{\rho}(t)R$. As a consequence, the following estimates hold:

$$\begin{aligned}
&-(\phi(x_1'(t)))'=\phi(\lambda_1)q(t)f_n(t,x_1(t),x_1'(t))\\
&\leq q(t)F(t,\max\{1/n_1,\tfrac{x_1(t)}{1+t}\},\max\{1/n_1,x_1'(t)\})\\
&\leq q(t)m(t)g(\max\{1/n_1,\tfrac{x_1(t)}{1+t}\})\psi(\max\{1/n_1,x_1'(t)\})\\
&\leq q(t)m(t)h(\max\{\tfrac{1}{n_1},\tfrac{x_1(t)}{1+t}\})l(\max\{1/n_1,x_1'(t)\})\tfrac{g(\max\{1/n_1,\frac{x_1(t)}{1+t}\})}{h(\max\{1/n_1,\frac{x_1(t)}{1+t}\})}\tfrac{\psi(\max\{1/n_1,x_1'(t)\})}{l(\max\{1/n_1,x_1'(t)\})}\\
&\leq\tfrac{g(R)\psi(R)}{h(R)l(R)}q(t)m(t)h(\tfrac{\beta}{\alpha+\beta}R\widetilde{\rho}(t))l(x_1'(t)).
\end{aligned}$$

Hence

$$\frac{-(\phi(x_1'(t)))'}{l(x_1'(t))}\leq\frac{g(R)\psi(R)}{h(R)l(R)}q(t)m(t)h\left(\frac{\beta}{\alpha+\beta}R\widetilde{\rho}(t)\right).$$

An integration from t to $+\infty$ yields

$$\int_t^{+\infty}\frac{-(\phi(x_1'(\tau)))'}{l(x_1'(\tau))}d\tau\leq\frac{g(R)\psi(R)}{h(R)l(R)}\int_t^{+\infty}q(\tau)m(\tau)h\left(\frac{\beta}{\alpha+\beta}R\widetilde{\rho}(\tau)\right)d\tau.$$

Therefore

$$L\left(\phi(x_1'(t))\right)\leq\frac{g(R)\psi(R)}{h(R)l(R)}\int_t^{+\infty}q(\tau)m(\tau)h\left(\frac{\beta}{\alpha+\beta}R\widetilde{\rho}(\tau)\right)d\tau.$$

Then, for all $t\in\mathbb{R}^+$

$$x_1'(t)\leq\phi^{-1}\left(L^{-1}\left(\frac{g(R)\psi(R)}{h(R)l(R)}\int_1^{+\infty}q(\tau)m(\tau)h\left(\frac{\beta}{\alpha+\beta}R\widetilde{\rho}(\tau)\right)d\tau\right)\right).$$

By Lemma 2.12, we deduce that

$$\begin{aligned} R &= \|x_1\| \leq M\|x_1\|_2 \\ &\leq M \sup_{t\in\mathbb{R}^+} x_1'(t) \\ &\leq M\phi^{-1}\left(L^{-1}\left(\frac{g(R)\psi(R)}{h(R)l(R)}\int_1^{+\infty} q(\tau)m(\tau)h(\tfrac{\beta}{\alpha+\beta}R\widetilde{\rho}(\tau))d\tau\right)\right), \end{aligned}$$

which is a contradiction to (5.3). Finally, Lemma 2.3 yields that

$$i(A_n, \Omega_1 \cap \mathbb{P}, \mathbb{P}) = 1, \text{ for all } n \in \{n_0, n_0+1, \ldots\}. \tag{5.4}$$

Hence there exists an $x_n \in \Omega_1 \cap \mathbb{P}$ such that $A_n x_n = x_n$, $\forall\, n \geq n_0$.

Step 2: a compactness argument. Since $\|x_n\| < R$, from $(\mathbf{H}_9)$, there exists $\psi_R \in C(\mathbb{R}^+, \mathbb{R}^+)$ such that

$$f_n(t, x_n(t), x_n'(t)) \geq \psi_R(t), \quad \forall\, t \in \mathbb{R}^+,$$

with

$$\int_0^{+\infty} q(s)\psi_R(s)ds < +\infty.$$

Then

$$\begin{aligned} x_n(t) = A_n x_n(t) &\geq \int_0^t \phi^{-1}\left(\int_s^{+\infty} q(\tau)f_n(\tau, x_n(\tau), x_n'(\tau))d\tau\right) ds \\ &\geq \int_0^t \phi^{-1}\left(\int_s^{+\infty} q(\tau)\psi_R(\tau)d\tau\right) ds. \end{aligned}$$

Let

$$c^* = \phi^{-1}\left(\int_1^{+\infty} q(\tau)\psi_R(\tau)d\tau\right) > 0.$$

Arguing as in the proof of Theorem 4.2, we get $\frac{x_n(t)}{1+t} \geq c^*\widetilde{\rho}(t)$ and $x_n'(t) \geq \phi^{-1}(\gamma_R(t))$, $\forall\, t \in \mathbb{R}^+$, $\forall\, n \geq n_0$. Condition $(\mathbf{H}_8)$ implies that

$$q(\tau)f_n(\tau, x_n(\tau), x_n'(\tau)) \leq \frac{g(R)\psi(R)}{h(R)l(R)} q(\tau)m(\tau)h(c^*\widetilde{\rho}(\tau))l(\phi^{-1}(\gamma_R(\tau))).$$

Finally, as in proof of Theorem 4.2, we can show that $\{x_n\}_{n\geq n_0}$ has a convergent subsequence $\{x_{n_j}\}_{j\geq 1}$ with limit $\lim_{j\to+\infty} x_{n_j} = \bar{x}$ and $\bar{x}(t) \geq c^*\widetilde{\rho}(t), \forall\, t \in \mathbb{R}^+$. Then $\bar{x}$ is a positive solution of problem (1.1). □

5.2. **Two positive solutions.** Similarly to Theorem 4.3, we also obtain the following result the proof of which is omitted.

Theorem 5.3. *Assume that* $(\mathbf{H}_8) - (\mathbf{H}_{11})$ *hold. Then problem* (1.1) *has at least two positive solutions.*

We end the paper with two examples of applications illustrating Theorem 5.2 and Theorem 5.3 respectively.

Example 5.4. Consider the singular boundary value problem

$$\begin{cases} ((x'(t))^3)' + e^{-t}\frac{m(t)(x^2(t)+(1+t)^2)(x'(t)+1)}{(1+t)x(t)x'(t)} = 0, \\ \alpha x(0) - \beta x'(0) = 0, \quad \lim_{t\to+\infty} x'(t) = 0, \end{cases} \tag{5.5}$$

where

$$m(t) = \begin{cases} \frac{t}{1+t}, & t \in [0,1] \\ \frac{1}{t(1+t)}, & t \in (1,+\infty). \end{cases}$$

Here $f(t,x,y) = \frac{m(t)(x^2+(1+t)^2)(y+1)}{(1+t)xy}$, $\phi(t) = t^3$, and $q(t) = e^{-t}$. Then ϕ is continuous, increasing, and $\phi(0) = 0$. Moreover $F(t,x,y) = f(t,(1+t)x,y) = \frac{m(t)(x^2+1)(y+1)}{xy}$. Set $g(x) = \frac{x^2+1}{x}$, $\psi(y) = \frac{(y+1)}{y}$, $h(x) = \frac{1}{x}$, and $l(y) = \frac{1}{y}$. Then, for any $u \geq 0$, we have

$$L(u) = \int_0^u \frac{ds}{l(\phi^{-1}(s))} = \int_0^u \frac{ds}{l(s^{\frac{1}{3}})} = \int_0^u s^{\frac{1}{3}} ds = \frac{3}{4}u^{\frac{4}{3}}.$$

Hence

$$L^{-1}(u) = \left(\frac{4u}{3}\right)^{\frac{3}{4}}.$$

(H$_8$): It is clear that h, l are decreasing functions, $\frac{g}{h}, \frac{\psi}{l}$ are increasing functions, and $F(t,x,y) \leq m(t)g(x)\psi(y)$, $\forall\, t \in \mathbb{R}^+$, $\forall\, x, y > 0$. Moreover, for any $c > 0$, we have

$$\int_0^{+\infty} q(\tau)m(\tau)h(c\widetilde{\rho}(\tau))d\tau = \frac{1}{c}\int_0^{+\infty} e^{-\tau} d\tau = \frac{1}{c} < +\infty.$$

(H$_9$): For any $c > 0$, there exists $\psi_c(t) = \frac{m(t)}{c^2}$ such that

$$F(t,x,y) \geq \psi_c(t),\ \forall\, t \in \mathbb{R}^+, \forall\, x, y \in (0,c]$$

and $\int_0^{+\infty} q(\tau)\psi_c(\tau)d\tau < +\infty$. In addition, for any $t \geq 0$, we have

$$\gamma_c(t) = \int_t^{+\infty} q(\tau)\psi_c(\tau)d\tau \geq \frac{1}{c^2}\int_t^{t+1} q(\tau)m(\tau)d\tau \geq \frac{1}{c^2}\frac{e^{-(t+1)}}{(t+1)(t+2)}.$$

Then, for each $k > 0$, we have

$$\begin{aligned} &\int_0^{+\infty} \phi^{-1}\left(\int_s^{+\infty} q(\tau)m(\tau)h(k\widetilde{\rho}(\tau))l(\phi^{-1}(\gamma_c(\tau)))d\tau\right) ds \\ \leq\ &\int_0^{+\infty} \phi^{-1}\left(\frac{1}{k}\int_s^{+\infty} \frac{e^{-\tau}d\tau}{(\gamma_c(\tau))^{\frac{1}{3}}} ds\right) \\ \leq\ &\int_0^{+\infty} \phi^{-1}\left(\frac{1}{k}c^{\frac{2}{3}}\int_s^{+\infty} e^{-\tau}e^{\frac{\tau+1}{3}}(\tau+1)^{\frac{1}{3}}(\tau+2)^{\frac{1}{3}}d\tau\right) ds \\ \leq\ &\phi^{-1}\left(\frac{1}{k}c^{\frac{2}{3}}e^{\frac{1}{3}}\right)\int_0^{+\infty} \phi^{-1}\left(\int_s^{+\infty} e^{\frac{-2\tau}{3}}(\tau+1)^{\frac{1}{3}}(\tau+2)^{\frac{1}{3}}d\tau\right) ds < +\infty. \end{aligned}$$

(H$_{10}$):

$$\begin{aligned}
&\sup_{c>0} \frac{c}{M\phi^{-1}\Big(L^{-1}\big(\frac{g(c)\psi(c)}{h(c)l(c)}\int_0^{+\infty} q(\tau)m(\tau)h(c\frac{\beta}{\alpha+\beta}\widetilde{\rho}(\tau))d\tau\big)\Big)ds}\\
=\ &\sup_{c>0} \frac{c}{M\phi^{-1}\Big(L^{-1}\big(\frac{\alpha+\beta}{\beta}\frac{g(c)\psi(c)}{cl(c)h(c)}\big)\Big)}\\
=\ &\sup_{c>0} \frac{c}{M\phi^{-1}\Big(\big(\frac{(4(\alpha+\beta)g(c)\psi(c)}{3\beta cl(c)h(c)}\big)^{\frac{3}{4}}\Big)}\\
=\ &\sup_{c>0} \frac{c}{M\Big(\big(\frac{(4(\alpha+\beta)g(c)\psi(c)}{3\beta cl(c)h(c)}\big)^{\frac{1}{4}}\Big)}\\
=\ &\sup_{c>0} \frac{c}{M\Big(\big(\frac{(4(\alpha+\beta)(c^2+1)(c+1)}{3\beta c}\big)^{\frac{1}{4}}\Big)} > 1.
\end{aligned}$$

Therefore all conditions of Theorem 5.2 hold; this implies that problem (5.5) has at least one positive solution.

Example 5.5. Consider the singular boundary value problem

$$(5.6)\qquad \begin{cases} ((x'(t))^3)' + ae^{-t}\frac{m(t)(x^5(t)+(1+t)^5)(x'(t)+1)}{(1+t)^4x(t)x'(t)} = 0,\\ \alpha x(0) - \beta x'(0) = 0, \quad \lim\limits_{t\to+\infty} x'(t) = 0,\end{cases}$$

where

$$m(t) = \begin{cases} \frac{t}{1+t}, & t \in [0,1]\\ \frac{1}{t(1+t)}, & t \in (1,+\infty).\end{cases}$$

Here $f(t,x,y) = \frac{m(t)(x^5+(1+t)^2)(y+1)}{(1+t)xy}$, $\phi(t) = t^3$, and $q(t) = ae^{-t}$ where

$$0 < a < \frac{\beta}{M^4(\alpha+\beta)}\left(\sup_{c>0}\frac{c}{(c^5+1)((c+1)}\right)^4.$$

Then ϕ is continuous, increasing, and $\phi(0) = 0$, $F(t,x,y) = f(t,(1+t)x,y) = \frac{m(t)(x^5+1)(y+1)}{xy}$, $g(x) = \frac{x^5+1}{x}$, $\psi(y) = \frac{(y+1)}{y}$, $h(x) = \frac{1}{x}$, and $l(y) = \frac{1}{y}$. Then all conditions of Theorem 5.3 are fulfilled, which implies that problem (5.6) has at least two positive solutions.

Remark 5.6. We can prove a similar result when the nonlinearity presents a singularity at $x' = 0$ but not at $x = 0$. This case is omitted.

Acknowledgment. The authors are grateful to the referee for his careful reading of the manuscript which led to an improvement of the first version.

REFERENCES

[1] R. P. Agarwal, M. Meehan, and D. O'Regan, *Fixed Point Theory and Applications,* Cambridge Tracts in Mathematics, **141**, Cambridge University Press, 2001.

[2] R. P. Agarwal and D. O'Regan, *Infinite Interval Problems for Differential, Difference and Integral Equations,* Kluwer Academic Publisher, Dordrecht, 2001.

[3] C. Corduneanu, *Integral Equations and Stability of Feedback Systems,* Academic Press, New York, 1973.

[4] K. Deimling, *Nonlinear Functional Analysis,* Springer-Verlag, Berlin, Heidelberg, 1985.

[5] S. Djebali and K. Mebarki, *Multiple positive solutions for singular BVPs on the positive half-line,* Comput. Math. Appli. **55(12)** (2008) 2940–2952.

[6] S. Djebali and K. Mebarki, *On the singular generalized Fisher-like equation with derivative depending nonlinearity,* Appl. Math. Comput. **205** (2008) 336–351.

[7] S. Djebali and K. Mebarki, *Multiple unbounded positive solutions for three-point bvps with sign-changing nonlinearities on the positive half-line,* Acta Appl. Math., **109** (2010) 361–388

[8] S. Djebali and O. Saifi, *Positive solutions for singular BVPs on the positive half-line with sign changing and derivative depending nonlinearity,* Acta Appl. Math. **110** (2010) 639–665.

[9] S. Djebali and O. Saifi, *Positive solutions for singular $\phi-$Laplacian BVPs on the positive half-line,* E.J.Q.T.D.E., **56** (2009) 1–24.

[10] S. Djebali and O. Saifi, *Upper and lower solution method for singular $\phi-$Laplacian BVPs with derivative depending nonlinearity on* $[0, +\infty)$*,* Comm. Appl. Anal., **14(4)** (2010), 463–480

[11] D. Guo, V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, San Diego, 1988.

[12] Y. Guo, C. Yu, and J. Wang, *Existence of three positive solutions for $m-$point boundary value problems on infinite intervals,* Nonlin. Anal., T.M.A., **71** (2009) 717–722.

[13] H. Lian, H. Pang, W. Ge, *Triple positive solutions for boundary value problems on infinite intrvals,* Nonlin. Anal., T.M.A., **67** (2007) 2199-2207.

[14] S. Liang and J. Zhang, *The existence of countably many positive solutions for nonlinear singular $m-$point boundary value problems on the half line,* J. Comput. Appl. Math. **222** (2008) 229–243.

[15] S. Liang and J. Zhang, *The existence of countably many positive solutions for one-dimensional $p-$Laplacian with infinitely many singularities on the half-line,* Appl. Math. Comput. **201** (2008) 210–220.

[16] S. Liang, J. Zhang, and Z. Wang, *The existence of multiple positive solutions for multi-point boundary value problems on the half line,* J. Comput. Appl. Math., **228(1)** (2009) 10–19.

[17] Y. Tian, W. Ge, and W. Shan *Positive solutions for three-point boundary value problem on the half-line,* Comput. Math. Appl. **53** (2007) 1029–1039.

[18] B. Yan, D. O'Regan, and R.P. Agarwal, *Positive solutions for second order singular boundary value problems with derivative dependence on infinite intervals,* Acta Appl. Math., **103(1)** (2008) 19–57

[19] B. Yan and Y. Liu, *Unbounded solutions of the singular boundary value problems for second order differential equations on the half-line,* Appl. Math. Comput. **147** (2006) 629–644.

[20] X. Zhang, *Existence of positive solutions for multi-point boundary value problems on infinite intervals in Banach spaces,* Appl. Math. Comput. **206** (2008) 932–941.