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UPPER SEMICONTINUOUS QUANTUM STOCHASTIC DIFFERENTIAL INCLUSIONS VIA KAKUTANI-FAN FIXED POINT THEOREM

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ABSTRACT. The existence of solutions of Discontinuous Quantum Stochastic Differential Inclusions (QSDI) with upper semicontinuous coefficients is our concerned in this work. A non commutative generalization of Kakutani-Fan fixed point theorem is established in the work. By employing this result, the existence of solution of upper semicontinuous QSDI is established.

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1. INTRODUCTION

The problem of existence of solutions of Lipschitzian quantum stochastic differential inclusions of Hudson and Parthasarathy quantum stochastic calculus formulation was established in [7]. The properties of these solution sets were established in [3] and [4]. The quantum stochastic calculus is driven by quantum stochastic processes called annihilation, creation and gauge arising from quantum field operators. The multivalued generalization of this non commutative stochastic differential equation is essential in the applications of quantum control theory, quantum evolution inclusions[9] and differential equation with discontinuous coefficients.

For a classical differential equation with discontinuous coefficients the existence of solutions was established via a multivalued regularization procedure [2]. This multivalued regularization is upper semicontinuous. The existence of solutions of upper semicontinuous differential inclusions in the classical setting was established by using Kakutani fixed point approach [6] which is a multivalued generalization of Schauder fixed point theorem. The aim of this work is to establish this result in our non commutative setting. However, this result does not naturally transcends to our upper semicontinuous quantum stochastic differential inclusions. In this work we shall first establish a form of Kakutani-Fan fixed point theorem and then employ it to prove the existence of solution of our quantum stochastic differential inclusions. Hence we extend the existence of solution results in the literatures on quantum stochastic differential inclusions [7], [8] and [10] to discontinuous case.

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The work shall be arranged as follows; in section 2 we state the definitions and notations while section 3 shall be for results on the fixed point theorem and existence of solutions of upper semicontinuous quantum stochastic differential inclusions via this fixed point theorem.

2. PRELIMINARIES

2.1. **Notations and Definitions.** In what follows, if U is a topological space, we denote by $\operatorname{clos}(U)$, the collection of all non-empty closed subsets of U.

To each pair (D, H) consisting of a pre-Hilbert space D and its completion H, we associate the set $L^+_w(D, H)$ of all linear maps x from D into H, with the property that the domain of the operator adjoint contains D. The members of $L^+_w(D, H)$ are densely-defined linear operators on H which do not necessarily leave D invariant and $L^+_w(D, H)$ is a linear space when equipped with the usual notions of addition and scalar multiplication.

To H corresponds a Hilbert space $\Gamma(H)$ called the boson Fock space determined by H. A natural dense subset of $\Gamma(H)$ consists of linear space generated by the set of exponential vectors(Guichardet, [11]) in $\Gamma(H)$ of the form

$$e(f) = \bigoplus_{n=0}^{\infty}(n!)^{-\frac{1}{2}} \bigotimes^{n} f, \ f \in H$$

where $\bigotimes^0 f = 1$ and $\bigotimes^n f$ is the n-fold tensor product of f with itself for $n \geq 1$.

In what follows, $\mathbb{D}$ is some pre-Hilbert space whose completion is $\mathcal{R}$ and γ is a fixed Hilbert. $L^2_\gamma(\mathbb{R}_+)$ (resp. $L^2_\gamma([0, t))$, resp. $L^2_\gamma([t, \infty))$ $t \in \mathbb{R}_+$) is the space of square integrable γ-valued maps on $\mathbb{R}_+$ (resp. $[0, t)$, resp. $[t, \infty)$).

The inner product of the Hilbert space $\mathcal{R} \otimes \Gamma(L^2_\gamma(\mathbb{R}_+))$ will be denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ the norm induced by $\langle \cdot, \cdot \rangle$. Let $\mathbb{E}, \mathbb{E}_t$ and $\mathbb{E}^t$, $t > 0$ be linear spaces generated by the exponential vectors in Fock spaces $\Gamma(L^2_\gamma(\mathbb{R}_+)), \Gamma(L^2_\gamma([0, t)))$ and $\Gamma(L^2_\gamma([t, \infty)))$ respectively;

$$\mathcal{A} \equiv L^+_w(\mathbb{D}\underline{\otimes}\mathbb{E}, \mathcal{R} \otimes \Gamma(L^2_\gamma(\mathbb{R}_+)))$$
$$\mathcal{A}_t \equiv L^+_w(\mathbb{D}\underline{\otimes}\mathbb{E}_t, \mathcal{R} \otimes \Gamma(L^2_\gamma([0, t)))) \otimes \mathbb{I}^t$$
$$\mathcal{A}^t \equiv \mathbb{I}_t \otimes L^+_w(\mathbb{E}^t, \Gamma(L^2_\gamma([t, \infty)))), \ t > 0$$

where $\underline{\otimes}$ denotes algebraic tensor product and $\mathbb{I}_t$ (resp. $\mathbb{I}^t$) denotes the identity map on $\mathcal{R} \otimes \Gamma(L^2_\gamma([0, t)))$ (resp. $\Gamma(L^2_\gamma([t, \infty)))$), $t > 0$. For every $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ define

$$\| x \|_{\eta,\xi} = | \langle \eta, x\xi \rangle |, \ x \in \mathcal{A}$$

then the family of seminorms

$$\{\| \cdot \|_{\eta\xi} \colon \eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}\}$$

generates a topology τ_w, weak topology. The completion of the locally convex spaces $(\mathcal{A}, \tau_w), (\mathcal{A}_t, \tau_w)$ and $(\mathcal{A}^t, \tau_w)$ are respectively denoted by $\widetilde{\mathcal{A}}, \widetilde{\mathcal{A}}_t$ and $\widetilde{\mathcal{A}}^t$.

We define the Hausdorff topology on $\text{clos}(\widetilde{\mathcal{A}})$ as follows: For $x \in \widetilde{\mathcal{A}}$, $\mathcal{M}, \mathcal{N} \in \text{clos}(\widetilde{\mathcal{A}})$ and $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, define

$$\rho_{\eta\xi}(\mathcal{M}, \mathcal{N}) \equiv \max(\delta_{\eta\xi}(\mathcal{M}, \mathcal{N}), \delta_{\eta\xi}(\mathcal{N}, \mathcal{M}))$$

where

$$\delta_{\eta\xi}(\mathcal{M}, \mathcal{N}) \equiv \sup_{x \in \mathcal{M}} \mathbf{d}_{\eta\xi}(x, \mathcal{N}) \text{ and}$$
$$\mathbf{d}_{\eta\xi}(x, \mathcal{N}) \equiv \inf_{y \in \mathcal{N}} \| x - y \|_{\eta\xi} .$$

The Hausdorff topology which shall be employed in what follows, denoted by, τ_H, is generated by the family of pseudometrics $\{\rho_{\eta\xi}(\cdot) : \eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}\}$. Moreover, if $\mathcal{M} \in clos(\widetilde{\mathcal{A}})$, then $\| \mathcal{M} \|_{\eta\xi}$ is defined by

$$\| \mathcal{M} \|_{\eta\xi} \equiv \rho_{\eta\xi}(\mathcal{M}, \{0\});$$

for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$. For $A, B \in \text{clos}(\mathbb{C})$ and $x \in \mathbb{C}$, a complex number, define

$$d(x, B) \equiv \inf_{y \in B} | x - y |$$
$$\delta(A, B) \equiv \sup_{x \in A} d(x, B)$$
$$\text{and } \rho(A, B) \equiv \max(\delta(A, B), \delta(B, A)).$$

Then ρ is a metric on $\text{clos}(\mathbb{C})$ and induces a metric topology on the space. Let $I \subseteq \mathbb{R}_+$. A *stochastic process* indexed by I is an $\widetilde{\mathcal{A}}$-valued measurable map on I. A stochastic process X is called *adapted* if $X(t) \in \widetilde{\mathcal{A}}_t$ for each $t \in I$. We write $\text{Ad}(\widetilde{\mathcal{A}})$ for the set of all adapted stochastic processes indexed by I.

Definition 2.1. A member X of $\text{Ad}(\widetilde{\mathcal{A}})$ is called

(i) weakly absolutely continuous if the map $t \mapsto \langle \eta, X(t)\xi \rangle$, $t \in I$ is absolutely continuous for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$

(ii) locally absolutely p-integrable if $\| X(\cdot) \|^p_{\eta\xi}$ is Lebesgue -measurable and integrable on $[0, t) \subseteq I$ for each $t \in I$ and arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.

We denote by $\text{Ad}(\widetilde{\mathcal{A}})_{wac}$ (resp. $L^p_{loc}(\widetilde{\mathcal{A}})$) the set of all weakly, absolutely continuous (resp. locally absolutely p-integrable) members of $\text{Ad}(\widetilde{\mathcal{A}})$.

Stochastic integrators: Let $L^\infty_{\gamma,loc}(\mathbb{R}_+)$ [resp. $L^\infty_{B(\gamma),loc}(\mathbb{R}_+)$] be the linear space of all measurable, locally bounded functions from $\mathbb{R}_+$ to γ [resp. to B(γ), the Banach space of bounded endomorphisms of γ]. If $f \in L^\infty_{\gamma,loc}(\mathbb{R}_+)$ and $\pi \in L^\infty_{B(\gamma),loc}(\mathbb{R}_+)$, then πf is the member of $L^\infty_{\gamma,loc}(\mathbb{R}_+)$ given by $(\pi f)(t) = \pi(t)f(t)$, $t \in \mathbb{R}_+$.

For $f \in L^2_\gamma(\mathbb{R})_+$ and $\pi \in L^\infty_{B(\gamma),loc}(\mathbb{R}_+)$; the annihilation, creation and gauge operators, $a(f), a^+(f)$ and $\lambda(\pi)$ in $L^+_w(\mathbb{D}, \Gamma(L^2_\gamma(\mathbb{R})_+))$ respectively, are defined as:

$$\begin{aligned} a(f)\mathbf{e}(g) &= \langle f, g\rangle_{L^2_\gamma(\mathbb{R}_+)}\mathbf{e}(g) \\ a^+(f)\mathbf{e}(g) &= \frac{d}{d\sigma}\mathbf{e}(g+\sigma f)\mid_{\sigma=0} \\ \lambda(\pi)\mathbf{e}(g) &= \frac{d}{d\sigma}\mathbf{e}(e^{\sigma\pi}f)\mid_{\sigma=0} \end{aligned}$$

$g \in L^2_\gamma(\mathbb{R})_+$

For arbitrary $f \in L^\infty_{\gamma,loc}(\mathbb{R}_+)$ and $\pi \in L^\infty_{B(\gamma),loc}(\mathbb{R}_+)$, they give rise to the operator-valued maps A_f, A^+_f and Λ_π defined by:

$$\begin{aligned} A_f(t) &\equiv a(f\chi_{[0,t)}) \\ A^+_f(t) &\equiv a^+(f\chi_{[0,t)}) \\ \Lambda_\pi(t) &\equiv \lambda(\pi\chi_{[0,t)}) \end{aligned}$$

$t \in \mathbb{R}_+$, where χ_I denotes the indicator function of the Borel set $I \subseteq \mathbb{R}_+$. The maps A_f, A^+_f and Λ_π are stochastic processes, called annihilation, creation and gauge processes, respectively, when their values are identified with their amplifications on $\mathcal{R} \otimes \Gamma(L^2_\gamma(\mathbb{R}_+))$. These are the stochastic integrators in Hudson and Parthasarathy [12] formulation of boson quantum stochastic integration.

For processes $p, q, u, v \in L^2_{loc}(\widetilde{\mathcal{A}})$, the quantum stochastic integral:

$$\int_{t_0}^t \big(p(s)d\Lambda_\pi(s) + q(s)dA_f(s) + u(s)dA^+_g(s) + v(s)ds\big),\ t_0, t \in \mathbb{R}_+$$

is interpreted in the sense of Hudson-Parthasarathy[12]. The definition of Quantum stochastic differential Inclusions follows as in [7]. A relation of the form

$$\begin{aligned} dX(t) &\in E(t, X(t))d\Lambda_\pi(t) + F(t, X(t))dA_f(t) \\ &\quad + G(t, X(t))dA^+_g(t) + H(t, X(t))dt \text{ almost all } t \in I \\ X(t_0) &= x_0 \end{aligned} \tag{2.1}$$

is called Quantum stochastic differential inclusions(QSDI) with coefficients E, F, G, H and initial data (t_0, x_0). Equation(2.1) is understood in the integral form:

$$\begin{aligned} X(t) \in x_0 + \int_{t_0}^t &\big(E(s, X(s))d\Lambda_\pi(s) + F(s, X(s))dA_f(s) \\ &+ G(s, X(s))dA^+_g(s) + H(s, X(s))ds\big),\ t \in I \end{aligned}$$

called a stochastic integral inclusion with coefficients E, F, G, H and initial data (t_0, x_0). An equivalent form of (2.1) has been established in [7], Theorem 6.2 as:

$$\begin{aligned}
(\mu E)(t,x)(\eta,\xi) &= \{\langle \eta, \mu_{\alpha\beta}(t)p(t,x)\xi\rangle : p(t,x) \in E(t,x)\} \\
(\nu F)(t,x)(\eta,\xi) &= \{\langle \eta, \nu_{\beta}(t)q(t,x)\xi\rangle : q(t,x) \in F(t,x)\} \\
(\sigma G)(t,x)(\eta,\xi) &= \{\langle \eta, \sigma_{\alpha}(t)u(t,x)\xi\rangle : u(t,x) \in G(t,x)\} \\
\mathbb{P}(t,x)(\eta,\xi) &= (\mu E)(t,x)(\eta,\xi) + (\nu F)(t,x)(\eta,\xi) \\
&\quad + (\sigma G)(t,x)(\eta,\xi) + H(t,x)(\eta,\xi) \\
H(t,x)(\eta,\xi) &= \{v(t,x)(\eta,\xi) : v(\cdot, X(\cdot)) \\
&\quad \text{is a selection of } H(\cdot, X(\cdot)) \forall\, X \in L^2_{loc}(\widetilde{\mathcal{A}})\}
\end{aligned} \tag{2.2}$$

Then Problem (2.1) is equivalent to

$$\begin{aligned}
\frac{d}{dt}\langle \eta, X(t)\xi\rangle &\in \mathbb{P}(t, X(t))(\eta,\xi) \\
X(t_0) &= x_0
\end{aligned} \tag{2.3}$$

for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, almost all $t \in I$. Hence the existence of solution of (2.1) implies the existence of solution of (2.3) and vice-versa. As explained in [7], for the map $\mathbb{P}$,

$$\mathbb{P}(t,x)(\eta,\xi) \neq \widetilde{\mathbb{P}}(t, \langle \eta, x\xi\rangle)$$

for some complex-valued multifunction $\widetilde{\mathbb{P}}$ defined on $I \times \mathbb{C}$ for $t \in I$, $x \in \widetilde{\mathcal{A}}$, $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$.

Definition 2.2. Let $D \subset \widetilde{\mathcal{A}}$ be a non-empty bounded subset of $\widetilde{\mathcal{A}}$. For each $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, $\sup_{x\in D} \| x \|_{\eta\xi} < \infty$. We define the diameter of D with respect to $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ by,

$$diam.(D) = \sup_{x,y\in D} \| x - y \|_{\eta\xi} .$$

Definition 2.3. For arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, let

$$\mathcal{B}_{\eta\xi} = \{D \subset \widetilde{\mathcal{A}} : \sup_{x,y\in D} \| x - y \|_{\eta\xi} < \infty\}$$

Then the map: $\alpha_{\eta\xi} : \mathcal{B}_{\eta\xi} \to \mathbb{R}_+$, defined by

$$\alpha_{\eta\xi}(D) = \inf\{d > 0 : D \text{ admits a finite cover by sets of diameter} \leq d\}, D \in \mathcal{B}_{\eta\xi}$$

is called (Kuratowski-)measure of non compactness.

The following are properties of $\alpha_{\eta\xi}$, established in [5]

Proposition 2.4. *Suppose* $\alpha_{\eta\xi} : \mathcal{B}_{\eta\xi} \to \mathbb{R}_+$, *then*

(a) $\alpha_{\eta\xi}(D) = 0$ *if and only if* D *is compact*

(b) $\alpha_{\eta\xi}$ *is a seminorm, that is; for* $\lambda > 0$,

$$\alpha_{\eta\xi}(\lambda D) =\mid \lambda \mid \alpha_{\eta\xi}(D) \ \textit{and}\ \alpha_{\eta\xi}(D_1 + D_2) \leq \alpha_{\eta\xi}(D_1) + \alpha(D_2)$$

(c) $D_1 \subset D_2$ *implies*

$$\alpha_{\eta\xi}(D_1) \leq \alpha_{\eta\xi}(D_2),\ \ \alpha_{\eta\xi}(D_1 \cup D_2) = \max\{\alpha_{\eta\xi}(D_1), \alpha_{\eta\xi}(D_2)\}$$

(d) $\alpha_{\eta\xi}(coD) = \alpha_{\eta\xi}(D)$.

(e) $\alpha_{\eta\xi}$ *is continuous with respect to the Hausdorff distance; that is*

$$\mid \alpha_{\eta\xi}(D_1) - \alpha_{\eta\xi}(D_2) \mid \leq \rho_{\eta\xi}(D_1, D_2)$$

for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ *where*

$$\rho_{\eta\xi}(D_1, D_2) = \max\big\{ \sup_{x\in D_1} d_{\eta\xi}(x, D_2), \sup_{x\in D_2} d_{\eta\xi}(x, D_1)\big\},\ \ D_1, D_2 \subset \mathcal{B}_{\eta\xi}$$

Definition 2.5. (a) Let $v_0, v_1, \ldots, v_n$ be an affinely independent set of $n+1$ points in a vector space E. The convex hull

$$\{x \in E : x = \sum_{i=0}^{n} \lambda_i v_i, 0 \leq \lambda_i \leq 1, \sum_{i=0}^{n} \lambda_i = 1\}$$

is called (closed)n-simplex and is denoted by $v_0v_1 \ldots v_n$. The points $v_0, v_1, \ldots, v_n$ are called the vertices of the simplex. For $0 \leq k \leq n$ and $0 \leq i_0 < i_1 < \cdots < i_k \leq n$, the k-simplex $v_{i_0}v_{i_1} \ldots v_{i_k}$ is a subset of the n-simplex $v_0v_1 \ldots v_n$; it is called a k-dimensional face(or simply k-face) of $v_0v_1 \ldots v_n$. In addition , if $y = \sum_{i=0}^{n} \lambda_i v_i$ we let $\chi(y) = \{i : \lambda_i > 0\}$

(b) A real-valued function ϕ on $\widetilde{\mathcal{A}}$ is lower (resp. upper) semicontinuous if the set $\{x \in \widetilde{\mathcal{A}} : \phi(x) \leq \lambda\}$ (resp. $\{x \in \widetilde{\mathcal{A}} : \phi(x) \geq \lambda\}$) is closed in $\widetilde{\mathcal{A}}$ for each $\lambda \in \mathbb{R}$. If Q is a convex set in a vector space then a real-valued function ϕ on Q is said to be quasiconcave (resp. quasiconvex) if $\{x \in Q : \phi(x) > \lambda\}$ (resp. $\{x \in Q : \phi(x) < \lambda\}$) is convex for each $\lambda \in \mathbb{R}$

(c) Let K be a non empty set, and $\Phi : K \to 2^K$ a multifunction, an element $x \in K$ is said to be a fixed point of Φ if $x \in \Phi(x)$.

(d) Let Q be a convex set in a vector space X, A a non-empty subset of Q and $F : A \to 2^Q$, a multivalued map. The family $\{F(x) : x \in A\}$ is said to be a KKM covering for Q if

$$co\{x : x \in N\} \subseteq \bigcup_{x\in N} F(x)$$

for any finite set $N \subseteq A$

2.2. **Preliminary results.** In the Locally convex spaces, Schauder-Tychonoff fixed point theorem is the generalization of Schauder fixed point theorem on Banach spaces[13]. For the case of multifunctions, Kakutani fixed point theorem is the multi-valued analogue of Schauder fixed point theorem and Kakutani-Fan fixed point theorem is the generalization of Schauder-Tychonoff theorem[1]. The following theorems due to Knaster, Kuratowski and Mazurkiewicz (KKM) shall be employed.

Theorem 2.6. [1] *Let* $\{F_0, \ldots, F_n\}$ *be a family of* $n+1$ *closed subsets of an* n*-simplex* $v_0v_1 \ldots v_n$*. Suppose that for each* $0 \leq k \leq n$ *and* $0 \leq i_0 < i_1 < \cdots < i_k \leq n$ *we have*

$$v_{i_0}v_{i_1} \ldots v_{i_k} \subseteq F_{i_0} \cup F_{i_1} \cup \cdots \cup F_{i_k}$$

Then

$$\bigcap_{i=0}^{n} F_i \neq \emptyset$$

The infinite dimensional version of the KKM theorem, Theorem 2.1, above is:

Theorem 2.7. [1] *Let* Q *be a convex set in* $\widetilde{\mathcal{A}}$*,* $\mathcal{N}$ *a non-empty subset of* Q*,* $F : \mathcal{N} \to 2^Q$ *a multivalued map and* $\{F(x) : x \in \mathcal{N}\}$ *a KKM covering for* Q*. If there exists an* $a \in \mathcal{N}$ *with* $\overline{F(a)}$ *compact, then*

$$\bigcap_{x \in \mathcal{N}} \overline{F(x)} \neq \emptyset$$

The following is a non commutative analogue of the Ky Fan's minimax theorem, as established in[1]

Theorem 2.8. *Let* $K \neq \emptyset$*, convex and compact subset in* $\widetilde{\mathcal{A}}$ *and* ϕ *a real-valued function on the product space* $K \times K$ *satisfying the following conditions;*

$$\text{for each fixed } x \in K, \phi(x, \cdot) \text{ is lower semicontinuous on } K \text{ and} \tag{2.4}$$

$$\text{for each fixed } y \in K, \phi(\cdot, y) \text{ is quasiconcave on } K \tag{2.5}$$

Then there exists $y^* \in K$ *with*

$$\phi(x, y^*) \leq \sup_{z \in K} \phi(z, z) \text{ for all } x \in K$$

(and therefore $\min_{y \in K} \sup_{x \in K} \phi(x, y) \leq \sup_{x \in K} \phi(x, x)$*)*

Proof. Let $\lambda = \sup_{x \in K} \phi(x, x)$. We may assume that $\lambda \neq \infty$. For each $x \in K$ let

$$F(x) = \{y \in K : \phi(x, y) \leq \lambda\}$$

condition 2.4 guarantees that each $F(x)$ is closed and hence compact in K (note that K is compact). We claim that $\{F(x) : x \in K\}$ is a KKM covering for K. If the claim is true then Theorem 2.2 guarantees that $\bigcap_{x \in K} F(x) \neq \emptyset$. Take $y^* \in \bigcap_{x \in K} F(x)$ and the proof is concluded.

To prove the claim . Suppose it is not true. Then there exists $\{x_1, \ldots, x_n\} \subset K$ and $\alpha_i > 0$ $(i = 0, 1, \ldots, n)$ with $\sum_{i=0}^{n} \alpha_i = 1$ such that

$$w = \sum_{i=0}^{n} \alpha_i x_i \in \big(\bigcup_{i=0}^{n} F(x_i)\big)'$$

This together with the definition of $F(x)$ yields

$$\phi(x_i, \sum_{i=0}^{n} \alpha_i x_i) = \phi(x_i, w) > \lambda, \text{ for } i = 0, 1, \ldots, n \tag{2.6}$$

Finally (2.4) together with the quasiconcavity of $\phi(\cdot, w)$ guarantees that $\phi(w, w) > \lambda$, a contradiction. □

In the following result, we shall employ the notation: $\langle x, g\rangle$ to denote the duality pairing for each $g \in \widetilde{\mathcal{A}}'$ and $x \in \widetilde{\mathcal{A}}$

Theorem 2.9. *Let $X : I \to \widetilde{\mathcal{A}}$, Q a non-empty subset of $\widetilde{\mathcal{A}}$ and $\Phi : Q \to 2^Q$ be upper semicontinuous with $\Phi(X(t))$ non-empty and bounded for each $X(t) \in Q$. Then for any $g \in \widetilde{\mathcal{A}}'$(dual), the map $\phi_g : Q \to \mathbb{R}$, defined by $\phi_g(Y(t)) = \sup_{X(t)\in\Phi(Y(t))} Re\langle X(t), g\rangle$ is upper semicontinuous in the sense of real-valued function.*

Proof. Fix $y_0 \in Q$. Let $\epsilon > 0$ be given and let

$$U_\epsilon = \{X(t) \in Q :\mid \langle X(t), g\rangle \mid< \frac{\epsilon}{2}\}$$

Notice that U_ϵ is an open neighbourhood of 0. Since $\Phi(y_0) + U_\epsilon$ is an open set containing $\Phi(y_0)$, it follows from the upper semicontinuity of Φ at y_0 that there exists a neighbourhood $N(y_0)$ of y_0 in Q with

$$\Phi(Y(t)) \subseteq \Phi(y_0) + U_\epsilon \text{ for all } Y(t) \in N(y_0)$$

Thus for each $Y(t) \in N(y_0)$ we have that

$$\begin{aligned}\phi_g(Y(t)) = \sup_{X(t)\in\Phi(Y(t))} Re\langle X(t), g\rangle &\le \sup_{X(t)\in\Phi(y_0)+U_\epsilon} Re\langle X(t), g\rangle \\ &\le \sup_{X(t)\in\Phi(y_0)} Re\langle X(t), g\rangle + \sup_{X(t)\in U_\epsilon} Re\langle X(t), g\rangle \\ &< \phi_g(y_0) + \epsilon\end{aligned}$$

therefore ϕ_g is upper semicontinuous. □

The following separation theorem shall be employed in what follows:

Theorem 2.10. [1] *Suppose that A and B are disjoint, non-empty, convex sets in $\widetilde{\mathcal{A}}$. If in addition A is compact and B is closed, then there exist $f :\in \widetilde{\mathcal{A}}'$ and $\gamma \in \mathbb{R}$ with*

$$\max Ref(A) < \gamma \le \inf Ref(B)$$

3. MAIN RESULTS

Theorem 3.1. *Suppose $K \neq \emptyset$, $K \subset \widetilde{\mathcal{A}}$ is a convex and compact subset of $\widetilde{\mathcal{A}}$, such that the following conditions hold:*

(i) *$X(t)$ is a stochastic process; $X : I \to \widetilde{\mathcal{A}}$ such that $X(t) \in K, \forall t \in I$*

(ii) *The map $\Phi : K \to 2^K$ is upper semicontinuous with respect to a pair $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, with $\Phi(X(t))$ a non-empty closed and convex subset of K for each $X(t) \in K$. Then there exists a $y(t) \in K$ with $y(t) \in \Phi(y(t))$.*

Proof. Suppose that the result is not true, that is suppose $y(t) \notin \Phi(y(t))$ for such $y(t) \in K$. Now for each $y(t) \in K$, Theorem 2.4 guarantees that there exists $f_{y(t)} \in \widetilde{\mathcal{A}}'$ with

$$Re\langle y(t), f_{y(t)}\rangle - \sup_{X(t)\in\Phi(y(t))} Re\langle X(t), f_{y(t)}\rangle > 0. \tag{3.1}$$

For each $g \in \widetilde{\mathcal{A}}'$, let

$$V(g) = \{y(t) \in K : Re\langle y(t), g\rangle - \sup_{X(t)\in\Phi(y(t))} \langle X(t), g\rangle > 0\}$$

We observe that (3.1) ensures that $K = \bigcup_{g\in\widetilde{\mathcal{A}}'} V(g)$. In addition Theorem 2.2 implies that $V(g)$ is open in K. The compactness of K guarantees the existence of $g_1, g_2, \ldots, g_n \in \widetilde{\mathcal{A}}'$ with $K = \bigcup_{i=1}^n V(g_i)$. Let $\{\lambda_1, \ldots, \lambda_n\}$ be a partition of unity on K subordinate to the covering $\{V(g_1), \ldots, V(g_n)\}$ (let $V_i = V(g_i)$ for $i - 1, \ldots, n$), that is $\lambda_1, \ldots, \lambda_n$ are continuous non negative real valued functions on K with λ_i vanishing on $K \setminus V_i$ for each $i = 1, \ldots, n$ and $\sum_{i=1}^n \lambda_i(X(t)) = 1$ for all $X(t) \in K$. Therefore K is a non-empty,convex and compact subset of $\widetilde{\mathcal{A}}$. Let $\phi : K \times K \to \mathbb{R}$ be given by

$$\phi(X(t), y(t)) = \sum_{i=1}^n \lambda_i(y(t)) Re\langle y(t) - X(t), g_i\rangle$$

For each $X(t) \in K$ $\phi(X(t), \cdot)$ is lower semicontinuous on K and for each $y(t) \in K, \lambda \in \mathbb{R}$ the set, $\{X(t) \in K : \phi(X(t), y(t)) > \lambda\}$ is convex, then by Ky Fan's minimax theorem (Theorem 2.5), there exists $y_0 \in K$ with

$$\phi(X(t), y_0) \leq 0, \text{ for all } X(t) \in K$$

that is,

$$\sum_{i=1}^n \lambda_i(y_0) Re\langle y_0 - X(t), g_i\rangle \leq 0 \text{ for all } X(t) \in K \tag{3.2}$$

Suppose that $i \in \{1, 2, \ldots, n\}$ is such that $\lambda_i(y_0) > 0$. Then $y_i \in V(g_i)$ (since λ_i vanishes on $K \setminus V_i$) and consequently,

$$Re\langle y_0, g_i\rangle > \sup_{X(t)\in\Phi(y_0)} Re\langle X(t), g_i\rangle \geq Re\langle x_0, g_i\rangle$$

for all $x_0 \in \Phi(y_0)$ (that is, $Re\langle y_0 - x_0, g_i\rangle > 0$ for all $x_0 \in \Phi(y_0)$). Thus $\lambda_i(y_0)Re\langle y_0 - x_0, g_i\rangle > 0$ whenever $\lambda_i(y_0) > 0$ (for $i = 1, \ldots, n$) for all $x_0 \in \Phi(y_0)$. Since $\lambda_i(y_0) > 0$ for at least one $i \in \{1, 2, \ldots, n\}$, it follows that

$$\sum_{i=1}^{n} \lambda_i(y_0)Re\langle y_0 - x_0, g_i\rangle > 0$$

for all $x_0 \in \Phi(y_0)$. This contradicts (3.2). Therefore the conclusion of the theorem is true. $\square$

Theorem 3.2. *Assume that the maps $E, F, G, H \in L^2_{loc}(I \times \widetilde{\mathcal{A}})_{mvs}$ and $\mathbb{P} : I \times \widetilde{\mathcal{A}} \to 2^{sesq(\mathbb{D}\underline{\otimes}\mathbb{E})^2}$, a sesquilinear form valued map with closed and convex values such that*

(a) *$t \mapsto \mathbb{P}(t, X(t))(\eta, \xi)$ has a measurable selection,*

(b) *$X \mapsto \mathbb{P}(t, X(t))(\eta, \xi)$ is upper semicontinuous,*

(c) *$\rho(\mathbb{P}(t, X(t))(\eta, \xi), \{0\}) \leq c(t)(1+ \| X \|_{\eta\xi})$ on $I \times \widetilde{\mathcal{A}}$ with $c \in L^1_{loc}(I)$,*

(d) *$\lim_{\tau\to 0^+} \alpha_{\eta\xi}\Big(\mathbb{P}(I_{t,\tau} \times B)(\eta, \xi)\Big) \leq k(t)\alpha_{\eta\xi}(B)$ on I, where $\mathbb{P}(I_{t,\tau} \times B)(\eta, \xi) = \{\mathbb{P}(t, X(t))(\eta, \xi) : (t, X) \in I_{t,\tau} \times B\}$, $I_{t,\tau} = [t - \tau, t + \tau] \cap I$ for $B \in \mathcal{B}_{\eta\xi}$, $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$ and $k \in L^1_{loc}(I)$. Then the quantum stochastic differential inclusion*

$$\frac{d}{dt}\langle \eta, X(t)\xi\rangle \in \mathbb{P}(t, X(t))(\eta, \xi) \;\; X(t_0) = x_0 \text{ a.e. on } I$$

has a solution on I.

Proof. If $v \in Ad(\widetilde{\mathcal{A}})_{wac} \cap L^2_{loc}(\widetilde{\mathcal{A}})$, by (a), for an arbitrary pair of $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$, $\mathbb{P}(\cdot, v(\cdot))(\eta, \xi)$ has a measurable selection. That is there exists $\omega_{\eta\xi}(\cdot) \in \mathbb{P}(\cdot, v(\cdot))(\eta, \xi)$, such that $t \to \omega_{\eta\xi}(t)$ is measurable. By (c) we find that there exists $\psi_1(t)$ and $\psi_2(t) = c(t)(1 + \psi_1(t))$. Now we define K as:

$$K = \{v \in Ad(\widetilde{\mathcal{A}})_{wac} \cap L^2_{loc}(\widetilde{\mathcal{A}}) : v(t_0) = x_0, \| v(t) \|_{\eta\xi} \leq \psi_1(t) \text{ and}$$
$$\| v(t) - v(s) \|_{\eta\xi} \leq | \int_s^t \psi_2(\tau)d\tau \mid \forall \eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}\}$$

Also, since $\omega_{\eta\xi}(\cdot) \in \mathbb{P}(\cdot, v(\cdot))(\eta, \xi)$, there exists $\omega : I \to \widetilde{\mathcal{A}}$ such that $\omega_{\eta\xi}(\cdot) = \langle \eta, \omega(\cdot)\xi\rangle$, for arbitrary $\eta, \xi \in \mathbb{D}\underline{\otimes}\mathbb{E}$. Let $\widehat{K} \subset K$ be defined as

$$\widehat{K} = \{u \in K : \text{ there exist } v(\cdot) \in Ad(\widetilde{\mathcal{A}})_{wac} \cap L^2_{loc}(\widetilde{\mathcal{A}}), \omega_{\eta\xi}(\cdot)$$
$$= \langle \eta, \omega(\cdot)\xi\rangle \in \mathbb{P}(\cdot, v(\cdot))(\eta, \xi),$$
$$\text{with } \langle \eta, u(t)\xi\rangle = \langle \eta, x_0\xi\rangle + \int_0^t \omega_{\eta\xi}(s)ds\} \tag{3.3}$$

and a multivalued map $G : \widehat{K} \to \widehat{K}$ defined by

$$G(v) = \{u \in Ad(\widetilde{\mathcal{A}})_{wac} \cap L^2_{loc}(\widetilde{\mathcal{A}}) : \langle \eta, u(t)\xi\rangle = \langle \eta, x_0\xi\rangle + \int_0^t \omega_{\eta\xi}(s)ds,$$
$$\omega_{\eta\xi}(\cdot) = \langle \eta, \omega(\cdot)\xi\rangle \in \mathbb{P}(\cdot, v(\cdot))(\eta, \xi)\forall \eta\xi \in \mathbb{D}\underline{\otimes}\mathbb{E}\}.$$

G maps $\widehat{K}$ into itself, since for any $u \in G(v)$; $\langle \eta, u(t)\xi \rangle = \langle \eta, x_0\xi \rangle + \int_0^t \omega_{\eta\xi}(s)ds$, $u(0) = x_0$, $\| u(t) \|_{\eta\xi} \leq \psi_1(t)$ and

$$\begin{aligned} | \langle \eta, (u(t) - u(s))\xi \rangle | &= | \int_0^t \omega_{\eta\xi}(s)ds - \int_0^s \omega_{\eta\xi}(s)ds | \\ &= | \int_s^t \omega_{\eta\xi}(s)ds | \\ &\leq | \int_s^t c(\tau)(1+ | \psi_1(\tau) |)d\tau | \\ &\leq | \int_s^t \psi_2(\tau)d\tau | \end{aligned}$$

$\widehat{K}$ is bounded and weakly-equicontinuous, since for any $v \in \widehat{K}$; $v \in Ad(\widetilde{\mathcal{A}})_{wac} \cap L^2_{loc}(\widetilde{\mathcal{A}}$, $t, s \in I$, given $\epsilon > 0$, there exists $\delta > 0$ such that $\| v(t) - v(s) \|_{\eta\xi} < \epsilon$ whenever $| t - s | < \delta$. The weak equicontinuity follows by setting $\delta = \frac{\epsilon}{\lambda}$ where $\lambda = \max_{\tau \in [s,t]} | \psi_2(\tau) |$.

Moreover let $\alpha_{\eta\xi,0}(\cdot) = \alpha_{\eta\xi}(\cdot)$ for $Ad(\widetilde{\mathcal{A}})_{wac} \cap L^2_{loc}(\widetilde{\mathcal{A}}$ and $B(t) = \{v(t) : v \in B\}$, then $\alpha_{\eta\xi,0}(B) = \max_I \alpha_{\eta\xi}(B(t))$ for $B \subseteq\subseteq \widehat{K}$. Let $K_0 = \widehat{K}$, $\widehat{K}_{n+1} = convG(\widehat{K}_n)$ for $n \geq 0$ and $\widehat{K}_\infty = \bigcap_{n\geq 0} \widehat{K}_n$. Then $\left(\widehat{K}_n\right)$ is a decreasing sequence of closed convex sets. To show that $\widehat{K}_\infty$ is compact. Let $\rho_{\eta\xi,n}(t) = \alpha_{\eta\xi}(\widehat{K}_n(t))$ and $\gamma_{\eta\xi,n}(t) = \alpha_{\eta\xi}(G(\widehat{K}_n)(t))$. $\gamma_{\eta\xi,n}$ is absolutely continuous with $\gamma_{\eta\xi,n}(0) = 0$ and for $0 < t - \tau < t \leq T$, we have

$$\gamma_{\eta\xi,n}(t) - \gamma_{\eta\xi,n}(t-\tau) \leq \alpha_{\eta\xi}\left(\{\int_{t-\tau}^t \omega_{\eta\xi}(s)ds; \langle \eta, \omega(\cdot)\xi \rangle \in \mathbb{P}(\cdot, v(\cdot))(\eta, \xi), v \in \widehat{K}_n\right).$$

Using

$$\int_{t-\tau}^t \omega_{\eta\xi}(s)ds \in \tau\overline{conv}\mathbb{P}(I_{t,\tau} \times \cup_{I_{t,\tau_0}} K_n(s))(\eta, \xi) \text{ for } \tau \leq \tau_0,$$

we obtain

$$\frac{d}{dt}\gamma_{\eta\xi,n}(t) \leq K(t)\alpha_{\eta\xi}(\bigcup_{I_{t,\tau_0}} \widehat{K}_n(s))$$

almost everywhere, from condition (c) and therefore

$$\frac{d}{dt}\gamma_{\eta\xi}(t) \leq K(t)\rho_n \ a.e.$$

by letting $\tau_0 \to 0+$, since $\widehat{K}_n$ is equicontinuous. But $(\overline{conv}A)(t) = \overline{conv}A(t)$, then

$$\rho_{n+1}(t) \leq \int_0^t K(s)\rho_n(s)ds$$

hence $\rho_n(t) \to 0$ uniformly, since (ρ_n) is decreasing. Consequently, $\alpha_{\eta\xi,0}(K_\infty) = \max_I \alpha_{\eta\xi}(K_\infty(t)) = 0$ that is $\widehat{K}_\infty$ is compact with respect to τ^{wac} and convex. We also have $\widehat{K}_\infty \neq \emptyset$, since we may pick $v_n \in \widehat{K}_n$ and proceed in the same way to get $v_m \to v_0$ for some subsequence, hence $v_0 \in \widehat{K}_\infty$. Now, $G : \widehat{K}_\infty \to 2^{\widehat{K}_\infty} \setminus \emptyset$ and has convex values. If $(u_n) \subset G(v)$ then the corresponding (ω_n) has a weakly convergent subsequence. Hence $G(v)$ is also compact, moreover $G |_{K_\infty}$ has closed graph, hence $G|_{K_\infty}$ is Upper semicontinuous and therefore G has a fixed point in K_∞ by Kakutani-Fan fixed point theorem (Theorem 3.1).

Let $\varphi \in \widehat{K}_\infty$ be a fixed point of G. Then $\varphi \in Ad(\widetilde{\mathcal{A}})_{wac} \cap L^2_{loc}(\widetilde{\mathcal{A}})$ and

$$\langle \eta, \varphi(t)\xi \rangle = \langle \eta, x_0\xi \rangle + \int_0^t \omega_{\eta\xi}(s)ds$$

But, $\omega_{\eta\xi}(\cdot) \in \mathbb{P}(\cdot, \varphi(\cdot))(\eta, \xi)$. Therefore,

$$\frac{d}{dt}\langle \eta, \varphi(t)\xi \rangle = \langle \eta, \omega(t)\xi \rangle \in \mathbb{P}(t, \varphi(t))(\eta, \xi)$$

and $\varphi(t_0) = x_0$, a.e. $t \in I$. Hence the fixed point of G is a solution of the problem $\frac{d}{dt}\langle \eta, X(t)\xi \rangle \in \mathbb{P}(t, X(t))(\eta, \xi)$ $X(t_0) = x_0$ a.e. on I. $\square$

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