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UTILITY INDIFFERENCE PRICING OF INSURANCE CONTRACTS FOR HOME REVERSION PLAN UNDER STOCHASTIC INTEREST RATE

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ABSTRACT. In this paper, we explore the indifference pricing of the insurance contract relevant to the home reversion plan involving a single insured and a pair of insureds. Under the assumption that the risk-free bonds accumulate with a stochastic interest rate driven by a diffusion process, we applied the principle of equivalent utility to derive the partial differential equation system that the indifferent annuity benefits satisfy under the exponential utility function. Interestingly, the partial differential equation systems under stochastic interest rate coincide in form with those under the constant interest rate. However, while some parallels exist, there are subtle differences between them. In case that the value of stochastic interest rate at the beginning of signing the insurance contract is the same with the constant interest rate, the indifference annuity benefits under the stochastic interest rate coincide with those under the constant interest rate. The indifference annuity rates under the stochastic interest rate relate only with the initial value of stochastic interest rate at the start of writing the insurance contract, and have nothing to do with the specific paths of the diffusion process that drives the dynamics of stochastic interest rate.

Keywords: Stochastic interest rate; Home reversion plan; Long-term care; Markov model; Indifference continuous annuity; HJB equation.

AMS Classification: Primary: 60H15, 60J65; Secondary: 91D20.

1. INTRODUCTION

Reverse mortgages are specialized loans available to elderly homeowners. This loan, as opposed to a simple sale, enables the qualifying seniors to convert some of their home equity into cash while allowing them to continue to live in their own homes. Also with a reverse mortgage, such qualifying seniors can access a significant

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amount of cash to pay for living expenses, in-home services and supports, insurance premiums for long-term care.

With the increase in the aging population, the pressure on social security and medicare entitlements is on the rise. So, it becomes increasingly important to find effective ways to improve the seniors' long-term care financing system, as well as to decrease state medicare/medicaid budgets. Thus the idea if utilizing the reverse mortgage to pay for long-term care services and insurance has come into being couple of decades ago, see [3], [4], [5], [9], [10].

However, the published research on pricing the contract that links the reverse mortgage to long-term care are still scanty. In a recent work, [11] first designs a special insurance contract linking the home reversion plan to the long-term care involving a single insured, and then prices the contract with the equivalent utility principle. Our earlier work [8] modified and followed the method in this work to design and price the continuous annuities of home reversion plan and the insurance contract linking home reversion plan to long-term care for a 'pair of insureds' (meaning, husband and wife). The work [8] assumes that the insurer can choose investment proportion dynamically between the risky assets and the riskless bonds, the instantaneous yield of the risky assets is governed by a geometric Brownian motion, and the riskless bonds accumulate with a constant interest rate. Then, [7] generalizes the dynamics of the risky assets (i.e. home price) to follow a Lévy process, while the riskless bonds still accumulate with the constant interest rate. In this paper, we extend the constant interest rate to the stochastic interest rate and the stochastic interest rate is modeled by a diffusion process; however, the instantaneous yield of the risky assets is still governed by a geometric Brownian motion.

The remainder of the paper is organized as follows: Section 2 presents the results of the optimal investment without the insurance risk under the stochastic interest rates. As in the case of the insurance contract linking Home Reversion Plan (HRP) to Long-Term Care (LTC) involving a pair of insureds, we derive, in Section 3, a system of partial differential equations satisfied by the indifferent annuities under the exponential utility function. In Section 4, as in the home reversion plan involving a couple (presented by [8]), we derive the partial differential equation system that the indifferent annuities satisfy. In section 5, we modify the work of [11] and derive the partial differential equation system that the indifferent annuities satisfy under the stochastic interest rate. Interestingly, the partial differential equation system that the indifference annuities satisfy under the stochastic interest rates are the same in form with that under the constant interest rate; however, they have different implications and meanings. The final Section 6 presents the conclusion and indicate our future direction in such problems.

2. OPTIMAL INVESTMENT WITHOUT THE INSURANCE RISK

We consider a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{t\le s\le T}, P)$ satisfying the usual conditions, where T denotes the term of the trading horizon. The filtration $\mathcal{F}_s$ consists of two subfiltrations, i.e., $\mathcal{F}_s = \mathcal{F}^H_s \vee \mathcal{F}^r_s$, where $\mathcal{F}^H = (\mathcal{F}^H_s)_{t\le s\le T}$ covers the information about the risky asset, and $\mathcal{F}^r = (\mathcal{F}^r_s)_{t\le s\le T}$ contains the information about the stochastic interest rates. We make some blanket assumptions here.

- The filtrations $\mathcal{F}^H$ and $\mathcal{F}^r$ are independent. The insurer invests in both a risk-free asset and a risky asset.
- The value process $(H_s)_{t<s\le T}$ of risky asset is modeled by a geometric Brownian motion

$$dH_s = H_s(\mu\ ds + \sigma\ dB^H_s),\ H_t = H > 0,\ t \le s \le T, \tag{1}$$

where B^H_s is a standard Brownian motion on the probability space $(\Omega, \mathcal{F}, P)$, adapted to the filtration $\mathcal{F}^H_s = \sigma(B^H_u : 0 \le u \le s)$. The constant drift parameter μ and the diffusion parameter σ denote the mean return rate and the volatility of the risky asset, respectively.

- The value of risk-free asset accumulates with the stochastic interest rate $r_s > 0$, for $t < s \le T$. The dynamics of the stochastic interest rate is governed by

$$dr_s = a(s, r_s)\ ds + b(s, r_s)\ dB^r_s,\ r_t = r > 0,\ t \le s \le T, \tag{2}$$

where B^r_s is a standard Brownian motion on the probability space $(\Omega, \mathcal{F}, P)$, adapted to the filtration $\mathcal{F}^r_s = \sigma(B^r_u : 0 \le u \le s)$. Note that the Brownian motions $\{B^H_s\}$ and $\{B^r_s\}$, $0 \le s \le T$, are independent.

- The stochastic interest rate given by (2) satisfies the assumptions:

(B1) $a(s, r_s) : [0, T] \times (0, \infty) \mapsto \mathcal{R}$ and $b(s, r_s) : [0, T] \times (0, \infty) \mapsto \mathcal{R}$ are continuous functions, uniformly in s, locally Lipschitz continuous in r;

(B2) For any $(t, r) \in [0, T] \times (0, \infty)$, we have

$$\sup_{u\in[t,T]} E[|r_u|^2 \mid r_t = r] < \infty,$$

$$P(r_u \in (0, \infty), \forall u \in [t, T] \mid r_t = r) = 1.$$

Under the Assumptions (B1)–(B2), there exists a unique strong solution to the SDE (2) such that the mapping $(t, r, s) \mapsto r^{t,r}(s)$ is $P - a.s.$ continuous, and for each starting point $(t, r) \in [0, T] \times (0, \infty)$, the stochastic interest rate process is non-explosive on $[t, T]$.

With the dynamics of the stochastic interest rate for the single-factor structure governed by the SDE (2), [1], [2] and [13] analyze the pricing of the insurance contract, where they assume that $a(r_t, t) \geq 0$ and $b(r_t, t) \geq 0$ so that $r_t \geq 0$, for all $t \geq 0$.

- Now, the dynamics of risk-free asset M_s $(t \leq s \leq T)$ is given by

$$dM_s = r_s M_s ds, \ t \leq s \leq T.$$

- Let W_s denote the wealth of insurer at time s, with initial wealth $W_t = w$.
- The insurer can adjust the dynamic proportion of risky to risk-free asset. Particularly, the insurer invests π_s into the risky asset (in our case, the real estate) at time s $(t \leq s \leq T)$, and the remainder of the asset $W_s - \pi_s$ into the riskless asset. Then, the wealth process of insurer W_s associated with π_s is a solution to the following SDE

$$\begin{aligned} dW_s &= \pi_s \frac{dH_s}{H_s} + (W_s - \pi_s) \frac{dM_s}{M_s} \\ &= [r_s W_s + (\mu - r_s)\pi_s] \, ds + \sigma \pi_s \, dB_s^H, \ (t \leq s \leq T), \end{aligned}$$

with the initial wealth $W_t = w$.

- Without the insurance risk, the value function of the insurer is defined by

(3) $$U^{(0)}(w, r, t) = \sup_{\{\pi_s \in \mathcal{A}\}} E[u(W_T) | W_t = w, r_t = r],$$

where the utility function $u : R \to R$ is assumed to be strictly increasing and concave.

- Let $\mathcal{A}$ denote the set of all admissible strategies π_s that are $\mathcal{F}_s$-adapted, self-financing and square integrable (i.e. $E\left(\int_t^T \pi_s^2 ds\right) < \infty$).
- Assume that the utility u is given by the exponential utility function

(4) $$u(w) = -\frac{1}{\alpha} e^{-\alpha w}, \quad (\alpha > 0),$$

where the parameter α measures the *absolute risk aversion of the insurer.*

Definition 1. For notational brevity, we introduce the following partial differential operators:

(5) $$\begin{aligned} {}_{\mathbf{2}}\mathcal{A}_b^\pi f(w, r, H, t) \ &:= \ \frac{\partial f}{\partial t} + (rw + (\mu - r)\pi - b)\frac{\partial f}{\partial w} + \frac{1}{2}\sigma^2\pi^2\frac{\partial^2 f}{\partial w^2} + a(t, r)\frac{\partial f}{\partial r} \\ &\quad + \frac{1}{2}b^2(t, r)\frac{\partial^2 f}{\partial r^2} + \mu H \frac{\partial f}{\partial H} + \frac{1}{2}\sigma^2 H^2 \frac{\partial^2 f}{\partial H^2} + \sigma^2 \pi H \frac{\partial^2 f}{\partial w \partial H}, \end{aligned}$$

(6)
$${}_{\mathbf{3}}\mathcal{A}_b^\pi f(w, r, t) := \frac{\partial f}{\partial t} + (rw + (\mu - r)\pi - b)\frac{\partial f}{\partial w} + \frac{1}{2}\sigma^2\pi^2\frac{\partial^2 f}{\partial w^2} + a(t, r)\frac{\partial f}{\partial r} + \frac{1}{2}b^2(t, r)\frac{\partial^2 f}{\partial r^2},$$

$$ {}_{\mathbf{0}}\mathcal{L}_b^{r,\sigma} f(H,t) := \frac{\partial f}{\partial t} + rH\frac{\partial f}{\partial H} + \frac{1}{2}\sigma^2 H^2 \frac{\partial^2 f}{\partial H^2} + b\alpha e^{r(T-t)}. \tag{7} $$

Here the partial derivatives in equation (5) and (6) are defined as functions of (w, r, H, t) and (w, r, t), respectively; for instance, $\frac{\partial f}{\partial w}$ in (5) means that $\frac{\partial f}{\partial w} \equiv \frac{\partial f}{\partial w}(w, r, H, t)$, and $\frac{\partial f}{\partial w}$ in (6) means that $\frac{\partial f}{\partial w} \equiv \frac{\partial f}{\partial w}(w, r, t)$; the parameter α is the same as in (4); $T,\ \mu,\ \sigma$ denote, respectively, trade horizon, the mean return rate, and volatility of risky asset given by the Equation (1); and r denotes the initial value of stochastic interest rate given by (2).

We will freely use the following standard results:

1. Applying Itô formula to $U^{(0)}(w, r, t)$ we get

$$ \begin{aligned} dU^{(0)}(w,r,t) \;=\; & \left[\frac{\partial U^{(0)}}{\partial t} + (r_t w_t + (\mu - r_t)\pi_t)\frac{\partial U^{(0)}}{\partial w} + \frac{1}{2}\sigma^2\pi_t^2\frac{\partial^2 U^{(0)}}{\partial w^2} + a(t,r_t)\frac{\partial U^{(0)}}{\partial r} \right. \\ & \left. + \frac{1}{2}b^2(t,r_t)\frac{\partial^2 U^{(0)}}{\partial r^2}\right] dt + \sigma\pi_t\frac{\partial U^{(0)}}{\partial w}dB_t^H + b(t,r_t)\frac{\partial U^{(0)}}{\partial r}dB_t^r. \end{aligned} $$

2. Using the standard stochastic control methods, we obtain that $U^{(0)}(w, r, t)$ solves the following HJB equation

$$ \begin{aligned} & \frac{\partial U^{(0)}}{\partial t} + rw\frac{\partial U^{(0)}}{\partial w} + a(t,r)\frac{\partial U^{(0)}}{\partial r} + \frac{1}{2}b^2(t,r)\frac{\partial^2 U^{(0)}}{\partial r^2} \\ & + \max_{\pi}\left\{\frac{1}{2}\sigma^2\pi^2\frac{\partial^2 U^{(0)}}{\partial w^2} + (\mu - r)\pi\frac{\partial U^{(0)}}{\partial w}\right\} = 0 \end{aligned} \tag{8} $$

3. Let $\bar{\alpha} := \alpha e^{r(T-t)}$. In order to simplify the above HJB equation for $U^{(0)}(w, r, t)$, we make an ansatz of the form

$$ U^{(0)}(w,r,t) := -\frac{1}{\alpha}\exp(-\alpha w e^{r(T-t)})g(r,t). \tag{9} $$

Then $g(r, t)$ solves the following equation with the boundary condition $g(r, T) = 0$

$$ \begin{aligned} & \left\{\bar{\alpha}w(T-t)\left[-a(t,r) - \frac{1}{2}b^2(t,r)(T-t) + \frac{1}{2}b^2(t,r)\bar{\alpha}w(T-t)\right] \right. \\ & \left. - \frac{(\mu - r)^2}{2\sigma^2}\right\}g(r,t) + \frac{\partial g}{\partial t} \\ & + \left[a(t,r) - \bar{\alpha}w(T-t)b^2(t,r)\right]\frac{\partial g}{\partial r} + \frac{1}{2}b^2(t,r)\frac{\partial^2 g}{\partial r^2} = 0. \end{aligned} \tag{10} $$

3. INSURANCE CONTRACT LINKING HRP TO LTC: PAIR OF INSUREDS

Under the financial market described in Section 2, we first adopt Markov models to describing the actuarial construct of the contract linking Home Reversion Plan to

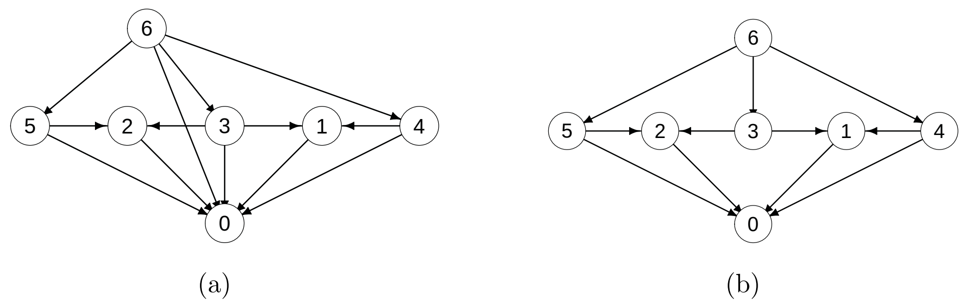

FIGURE 1. The Markov model for a pair of insureds

Long Term Care, and then show the indifference annuity rates are controlled by a system of nonlinear partial differential equations.

3.1. **Markov Model Distinguishing the Gender of Insureds.** The study of this section is mainly based on a seven-state continuous time Markov model illustrated in Figure 1(a), which describes the states and transitions for the contract linking home reversion plan to long term care for a pair of jointly insureds. In the following, the pair of insureds are denoted by (x) and (y), with the interpretation $(x) = x$-year old husband and $(y) = y$-year old wife. The corresponding states in the Markovian model are as follows:

(1) State **6** represents that the insured pair (x) and (y) lives at home;
(2) State **5** represents that (x) is dead and (y) lives at home;
(3) State **4** represents that (y) is dead and (x) lives at home;
(4) State **3** represents that (x) and (y) both live at nursing home;
(5) State **2** represents that the only survivor (y) lives at nursing home;
(6) State **1** represents that the only survivor (x) lives at nursing home;
(7) State **0** represents that both (x) and (y) are dead.

The model is assumed to work under the following implicit hypotheses:

(A1) The two events *One insured dies* and *The other insured moves into a nursing home* cannot happen simultaneously.
(A2) For the convenience of analysis, we forbid the event that *One of the insureds lives at home, while the other lives at nursing home.*

NOTE: (1) By assuming that the couple (x) and (y) cannot die at the same time, we can employ Figure 1(b) to illustrate the states and transitions. In the absence of this assumption, we can use Figure 1(a) to describe the actuarial structure.
(2) Comparing Figure 1(a) with Figure 1(b), we observe that the transitions $6 \to 0$ and $3 \to 0$ appear in Figure 1(a) but not in Figure 1(b).
(3) Under the assumption (A1), both the transitions $6 \to 1$ and $6 \to 2$ cannot happen.

Assume that Z_s is a continuous-time Markov chain representing the policy state at the time $s \in [t, T)$. Let $P_{ij}(s,t) = P(Z_t = j | Z_s = i)$, $(i, j \in S,\ s \le t)$ denote the transition probabilities, and $\lambda_{ij}(t) = \lim_{h \to 0} \frac{P_{ij}(t,t+h)}{h}$, $i \ne j$, and $\lambda_{ii}(t) = -\sum_{j \ne i} \lambda_{ij}(t)$, denote the corresponding transition intensity. Let τ_i be the stopping time of entering the state i $(i = 0, 1, \dots, 5)$, *i.e.*, $\tau_i := \inf\{t;\ Z_t = i\}$, $i = 0, 1, \dots, 5$.

Cash Flow SDEs: When the policy states and transitions are illustrated by Figure 1(a), the insurance contract linking home reversion plan to long term care, for a pair of insureds, is designed as follows:

(I) *When the insureds are at state i $(i = 1, 2, \dots, 6)$, the insurer pay the insureds the continuous annuity with rate b_i $(i = 1, 2, \dots, 6)$.*

(II) *The insureds employ all cash from the sale of the house to repay the insurer at the time $\tau = \min\{\tau_0, \tau_1, \tau_2, \tau_3\}$.*

Then, when the insurer underwrites the above contract linking home reversion plan to long term care, the cash flows of the insurer are governed by the following system of SDEs:

$$
(11) \qquad \begin{cases}
dW_s = \mu_6\ ds + \sigma \pi_s\ dB_s^H, & t < s < \min(\tau_0, \tau_3, \tau_4, \tau_5), \\
dW_s = \mu_5\ ds + \sigma \pi_s\ dB_s^H, & \tau_3 = \tau_4 = \infty,\ t < \tau_5 < s < \min(\tau_0, \tau_2), \\
dW_s = \mu_4\ ds + \sigma \pi_s\ dB_s^H, & \tau_3 = \tau_5 = \infty,\ t < \tau_4 < s < \min(\tau_0, \tau_1), \\
dW_s = \mu_3\ ds + \sigma \pi_s\ dB_s^H, & \tau_4 = \tau_5 = \infty,\ t < \tau_3 < s < \min(\tau_0, \tau_1, \tau_2), \\
dW_s = \mu_2\ ds + \sigma \pi_s\ dB_s^H, & \tau_4 = \infty,\ t < \tau_2 < s < \tau_0 \le T, \\
dW_s = \mu_1\ ds + \sigma \pi_s\ dB_s^H, & \tau_5 = \infty,\ t < \tau_1 < s < \tau_0 \le T, \\
dW_s = \mu_0\ ds + \sigma \pi_s\ dB_s^H, & t < \tau_0 < s \le T.
\end{cases}
$$

where $b_0 \equiv 0$ and $\mu_i = r_s W_s + (\mu - r_s)\pi_s - b_i$ $(i = 0, 1, \dots, 6)$, with the boundary conditions:

$$
(12) \qquad \begin{cases}
W_t = w, & \\
W_{\tau_0^+} = W_{\tau_0^-} + H_{\tau_0}, & \tau_3 = \tau_4 = \tau_5 = \infty, \\
W_{\tau_1^+} = W_{\tau_1^-} + H_{\tau_1}, & \tau_3 = \tau_5 = \infty,\ t < \tau_1 < \tau_0 \le T, \\
W_{\tau_2^+} = W_{\tau_2^-} + H_{\tau_2}, & \tau_3 = \tau_4 = \infty,\ t < \tau_2 < \tau_0 \le T, \\
W_{\tau_3^+} = W_{\tau_3^-} + H_{\tau_3}, & \tau_4 = \tau_5 = \infty,\ t < \tau_3 < \tau_0 \le T.
\end{cases}
$$

3.2. **The HJB Equations for the Indifference Annuity Rates.**

Definition 2. A strategy $\{\pi_s : (t, T] \times \Omega \mapsto \mathcal{R}, t < s \le T\}$ is called *admissible*, if π_s is progressively measurable with respect to the filtration $\mathcal{F}_s$, the SDE system (11) has a unique strong solution, and $E[\int_t^T \pi_s^2 ds] < \infty$.

Value Function: Let $\mathcal{A}$ denote the set of all admissible strategies. For the above insurance linking HRP to LTC in the above subsection, the *value function* of the insurer at state i $(i = 4, 5, 6)$ is defined by

$$U^{(i)}(w, r, H, t) = \sup_{\{\pi_s \in \mathcal{A}\}} E[u(W_T)|W_t = w, r_t = r, H_t = H, Z_t = i], \quad i = 4, 5, 6.$$

When the insureds are at state i $(i = 1, 2, 3)$ at time t $(0 \leq t < T)$, the insurer pays the insureds a continuous annuity at the constant rate b_i $(i = 1, 2, 3)$. This still leaves the insurer at risk of payment of the annuity. From article (II) of the insurance treaty described above, the insured repays the insurer with the cash of selling the house at time $\tau = \min\{\tau_0, \tau_1, \tau_2, \tau_3\}$. In other words, at the time $t > \tau = \min\{\tau_0, \tau_1, \tau_2, \tau_3\}$, the insured has repaid the insurer. Then, the maximum expected utility of terminal wealth for the insurer corresponding to the states i, $i = 1, 2, 3$, which are derived by the optimal strategy, are not dependent on the house price. So, the value function of the insurer at state i $(i = 1, 2, 3)$ are defined as follows

$$U^{(i)}(w, r, t) = \sup_{\{\pi_s \in \mathcal{A}\}} E[u(W_T)|W_t = w, r_t = r, Z_t = i].$$

The following Lemma 1 gives the nonlinear partial differential equation system that the value functions $U^{(i)}(w, r, H, t)$ $(i = 4, 5, 6)$ and $U^{(i)}(w, r, t)$ $(i = 1, 2, 3)$ solve. The PDE operators ${}_{\mathbf{2}}\mathcal{A}^{\pi}_{b_i} U^{(i)}(w, r, H, t)$ and ${}_{\mathbf{3}}\mathcal{A}^{\pi}_{b_i} U^{(i)}(w, r, t)$ are defined in (5) and (6), respectively. The definition of $U^{(0)}(w, r, t)$ are given by (3) and it satisfies HJB Equation (8).

Lemma 1. $U^{(6)}(w, r, H, t)$ *solve the following HJB equation*

$$\begin{aligned} &\max_{\pi}[{}_{\mathbf{2}}\mathcal{A}^{\pi}_{b_6} U^{(6)}(w, r, H, t)] + \sum_{j=4,5} \lambda_{6j}(t)[U^{(j)}(w, r, H, t) - U^{(6)}(w, r, H, t)] \\ &+ \sum_{j=0,3} \lambda_{6j}(t)[U^{(j)}(w + H, r, t) - U^{(6)}(w, r, H, t)] = 0, \end{aligned} \tag{13}$$

where $U^{(i)}(w, r, H, t)$ $(i = 4, 5)$ *and* $U^{(3)}(w, r, t)$*, respectively, satisfy the following HJB Equations*

$$\max_{\pi} \left[{}_{\mathbf{2}}\mathcal{A}^{\pi}_{b_5} U^{(5)}(w, r, H, t)\right] + \sum_{j=0,2} \lambda_{5j}(t)[U^{(j)}(w + H, r, t) - U^{(5)}(w, r, H, t)] = 0, \tag{14}$$

$$\max_{\pi} \left[{}_{\mathbf{2}}\mathcal{A}^{\pi}_{b_4} U^{(4)}(w, r, H, t)\right] + \sum_{j=0,1} \lambda_{4j}(t)[U^{(j)}(w + H, r, t) - U^{(4)}(w, r, H, t)] = 0, \tag{15}$$

$$\max_{\pi} \left[{}_{\mathbf{3}}\mathcal{A}^{\pi}_{b_3} U^{(3)}(w, r, t)\right] + \sum_{j=0,1,2} \lambda_{3j}(t)[U^{(j)}(w, r, t) - U^{(3)}(w, r, t)] = 0. \tag{16}$$

Furthermore, $U^{(i)}(w,r,t)$ $(i=1,2)$ *solve the following HJB Equation*

$$\max_{\pi}\left[{}_{\mathbf{3}}\mathcal{A}^{\pi}_{b_i}U^{(i)}(w,r,t)\right]+\lambda_{i0}(t)[U^{(0)}(w,r,t)-U^{(i)}(w,r,t)]=0. \tag{17}$$

The HJB Equations (13)–(17) *are subject to the following terminal conditions*

$$U^{(i)}(w,r,H,T)=u(w)\ (i=4,5,6),\ U^{(i)}(w,r,T)=u(w)\quad(i=1,2,3).$$

Proof. Assume the insurer fixes the strategy $\{\pi_s\}$ as $\{\pi\}$ from the time t to $t+h$, which may not be the optimal strategy. From the time $t+h$ to the end of horizon, the insurer invests with the optimal investment strategies. We consider the next possible state from State 6 and the optimal investment. From the definition of $U^{(6)}(w,r,H,t)$, we obtain

$$\begin{aligned}U^{(6)}(w,r,H,t) \;\geq\; & \sum_{j=0,3}P_{6j}(t,t+h)E^{w,r,H,t}[U^{(j)}(W_{t+h}+H_{t+h},r_{t+h},t+h)]\\ &+\sum_{j=4,5}P_{6j}(t,t+h)E^{w,r,H,t}[U^{(j)}(W_{t+h},r_{t+h},H_{t+h},t+h)]\\ &+P_{66}(t,t+h)E^{w,r,H,t}[U^{(6)}(W_{t+h},r_{t+h},H_{t+h},t+h)],\end{aligned} \tag{18}$$

where the notation $E^{w,r,H,t}$ denotes the conditional expectation with respect to $\{W_t = w, r_t=r, H_t=H\}$.

Apply Itô formula to $U^{(i)}(w,r,H,t)$ $(i=4,5,6)$ to obtain

$$\begin{aligned}&U^{(i)}(W_{t+h},r_{t+h},H_{t+h},t+h)\\ &=U^{(i)}(w,r,H,t)+\int_t^{t+h}{}_{\mathbf{2}}\mathcal{A}^{\pi}_{b_2}U^{(i)}(W_s,r_s,H_s,s)ds\\ &\quad+\int_t^{t+h}b(s,r_s)\frac{\partial U^{(i)}}{\partial r}(W_s,r_s,H_s,s)dB^r_s\\ &\quad+\int_t^{t+h}\left[\sigma H_s\frac{\partial U^{(i)}}{\partial H}(W_s,r_s,H_s,s)+\sigma\pi\frac{\partial U^{(i)}}{\partial w}(W_s,r_s,H_s,s)\right]dB^H_s,\end{aligned} \tag{19}$$

Again, Itô formula applied to $U^{(i)}(w,r,t)$ yields

$$\begin{aligned}&U^{(i)}(W_{t+h},r_{t+h},t+h)\\ &=U^{(i)}(w,r,t)+\int_t^{t+h}{}_{\mathbf{3}}\mathcal{A}^{\pi}_{b_i}U^{(i)}(W_s,r_s,s)ds\\ &\quad+\int_t^{t+h}\sigma\pi\frac{\partial U^{(i)}}{\partial w}(W_s,r_s,s)dB^H_s+\int_t^{t+h}b(s,r_s)\frac{\partial U^{(i)}}{\partial r}(W_s,r_s,s)dB^r_s.\end{aligned} \tag{20}$$

where $b_0\equiv 0$. Note that, as $h\to 0$, we have

$$\begin{aligned}P_{66}(t,t+h) &\to 1,\\ P_{6j}(t,t+h) &\to 0\ (j=0,3,4,5),\end{aligned}$$

$$\begin{aligned}\frac{P_{6j}(t,t+h)}{h} &\to \lambda_{6j}(t)\ (j=0,3,4,5),\\ \sum_{j=0,3,4,5,6} P_{6j}(t,t+h) &= 1.\end{aligned}$$

Inserting (19) and (20) into (18), reorganizing terms appropriately, dividing the equation by h, and letting $h \to 0$ we obtain

$$\begin{aligned}0 \geq\ & {}_{\mathbf{2}}\mathcal{A}^{\pi}_{b_6} U^{(6)}(w,r,H,t) + \sum_{j=4,5} \lambda_{6j}(t)[U^{(j)}(w,r,H,t) - U^{(6)}(w,r,H,t)]\\ &+ \sum_{j=0,3} \lambda_{6j}(t)[U^{(j)}(w+H,r,t) - U^{(6)}(w,r,H,t)] = 0,\end{aligned} \tag{21}$$

Finally, we follow the optimal strategy $\pi = \pi^*$ in the time interval $[t, t+h]$ to observe that the Equality in (18) holds true. We obtain (13).

In the same way, we can show that (14)–(17) hold. We thus conclude the proof. □

While the insurer pays the insureds the agreed upon continuous annuity, the insureds repay the insurer with the cash on selling the house at a random time of the insurance period as compensation. When the insurer pays out the annuity so that the optimal investment with the insurance risk and paying the continuous annuity coincides with the optimal investment without insurance risk and not paying the annuity, the insurer is indifferent with and without underwriting the insurance risk. In this case, the annuity rates are known as the indifference annuity rates. Thus, when the annuity rates are the same with the indifference annuity rates, we have

$$U^{(0)}(w,r,t) = U^{(6)}(w,r,H,t; b_i, i=1,\ldots,6).$$

The following theorem presents the nonlinear PDE system that the indifference annuity rates solve.

Theorem 2. *Assume that the utility function is exponential, viz,* $u(w) = -\frac{1}{\alpha}e^{-\alpha w}$. *Let* $\phi_i(t)$ $(i=1,2)$ *be the solution of the following terminal value problem*

$$\frac{d\phi_i(t)}{dt} + b_i\bar{\alpha}\phi_i(t) + \lambda_{i0}(t)[1-\phi_i(t)] = 0, \quad \phi_i(T) = 1. \tag{22}$$

Then the indifference continuous annuity rates b_i $(i=1,2,\ldots,6)$ *satisfy the following equations*

$$\phi_6(H,t; b_1, b_2, \ldots, b_6) = 0, \tag{23}$$

where $\phi_6(H,t)$ *solves the following HJB Equation*

$${}_{\mathbf{0}}\mathcal{L}^{r,\sigma}_{b_6}\phi_6(H,t) + \sum_{j=4,5} \lambda_{6j}(t)[e^{\phi_j(H,t)-\phi_6(H,t)} - 1]$$

$$+\sum_{j=0,3}\lambda_{6j}(t)[e^{-\bar{\alpha}H-\phi_6(H,t)}(\phi_3(t)\mathbf{1}_{\{j=3\}}+\mathbf{1}_{\{j=0\}})-1]=0, \tag{24}$$

with the terminal condition $\phi_6(H,T)=0$. *The functions* $\phi_j(H,t)$ $(j=4,5)$ *and* $\phi_3(t)$ *in* (24) *solve the following HJB equations*

$${}_{\mathbf{0}}\mathcal{L}_{b_5}^{r,\sigma}\phi_5(H,t)+[\lambda_{52}(t)\phi_2(t)+\lambda_{50}(t)]e^{-(\bar{\alpha}H+\phi_5(H,t))}-[\lambda_{52}(t)+\lambda_{50}(t)]=0, \tag{25}$$

$${}_{\mathbf{0}}\mathcal{L}_{b_4}^{r,\sigma}\phi_4(H,t)+[\lambda_{41}(t)\phi_1(t)+\lambda_{40}(t)]e^{-(\bar{\alpha}H+\phi_4(H,t))}-[\lambda_{41}(t)+\lambda_{40}(t)]=0, \tag{26}$$

$$\frac{d\phi_3(t)}{dt}+b_3\bar{\alpha}\phi_3(t)+\sum_{j=1,2}\lambda_{3j}(t)[\phi_j(t)-\phi_3(t)]+\lambda_{30}(t)[1-\phi_3(t)]=0, \tag{27}$$

subject to the terminal conditions $\phi_i(H,T)=0$ $(i=4,5)$ *and* $\phi_3(T)=1$, *respectively.*

Proof. To reduce dimensions of the equations in Lemma 1, we make the transformation

$$U^{(i)}(w,r,H,t)=U^{(0)}(w,r,t)e^{\phi_i(H,t)},\quad i=4,5,6, \tag{28}$$

$$U^{(i)}(w,r,t)=U^{(0)}(w,r,t)\phi_i(t),\quad i=1,2,3. \tag{29}$$

We obtain the derivatives of $U^{(i)}(w,r,H,t)$ $(i=4,5,6)$ from Equation (28)

$$\begin{aligned}
\frac{\partial U^{(i)}}{\partial r} &= e^{\phi_i(H,t)}\frac{\partial U^{(0)}}{\partial r}, & \frac{\partial U^{(i)}}{\partial H} &= e^{\phi_i(H,t)}U^{(0)}\frac{\partial \phi_i}{\partial H},\\
\frac{\partial^2 U^{(i)}}{\partial r^2} &= e^{\phi_i(H,t)}\frac{\partial^2 U^{(0)}}{\partial r^2}, & \frac{\partial U^{(i)}}{\partial w\partial H} &= e^{\phi_i(H,t)}\frac{\partial U^{(0)}}{\partial w}\frac{\partial \phi_i}{\partial H},\\
\frac{\partial U^{(i)}}{\partial w} &= e^{\phi_i(H,t)}\frac{\partial U^{(0)}}{\partial w}, & \frac{\partial U^{(i)}}{\partial t} &= e^{\phi_i(H,t)}\left[\frac{\partial U^{(0)}}{\partial t}+U^{(0)}\frac{\partial \phi_i}{\partial t}\right],\\
\frac{\partial^2 U^{(i)}}{\partial w^2} &= e^{\phi_i(H,t)}\frac{\partial^2 U^{(0)}}{\partial w^2}, & \frac{\partial^2 U^{(i)}}{\partial H^2} &= e^{\phi_i(H,t)}U^{(0)}\left[\frac{\partial^2 \phi_i}{\partial H^2}+(\frac{\partial \phi_i}{\partial H})^2\right].
\end{aligned}$$

Substituting the partial derivatives of $U^{(6)}(w,r,H,t)$ into $\max_\pi\{{}_{\mathbf{2}}\mathcal{A}_{b_6}^{\pi}U^{(6)}(w,r,H,t)\}$, and noting (8) and (9), we can obtain that

$$\max_{\pi}\left[{}_{\mathbf{2}}\mathcal{A}_{b_6}^{\pi}U^{(6)}(w,r,H,t)\right]=U^{(0)}(w,r,t)e^{\phi_6(H,t)}\ {}_{\mathbf{0}}\mathcal{L}_{b_6}^{r,\sigma}\phi_6(H,t). \tag{30}$$

Inserting (28) and (29), and noting $\bar{\alpha}=\alpha e^{r(T-t)}$, we get

$$\begin{aligned}
&\sum_{j=4,5}\lambda_{6j}(t)[U^{(j)}(w,r,H,t)-U^{(6)}(w,r,H,t)]\\
&=U^{(0)}(w,r,t)e^{\phi_6(H,t)}\sum_{j=4,5}\lambda_{6j}(t)\left(e^{\phi_j(H,t)-\phi_6(H,t)}-1\right),
\end{aligned} \tag{31}$$

$$U^{(3)}(w+H,r,t)-U^{(6)}(w,r,H,t)=U^{(0)}(w,r,t)e^{\phi_6(H,t)}\left(e^{-\bar{\alpha}H-\phi_6(H,t)}\phi_3(t)-1\right), \tag{32}$$

$$U^{(0)}(w+H,r,t)-U^{(6)}(w,r,H,t)=U^{(0)}(w,r,t)e^{\phi_6(H,t)}\left(e^{-\bar{\alpha}H-\phi_6(H,t)}-1\right). \tag{33}$$

Substitute (30), (31), (32) and (33) into (13) to obtain the Equation (24). With similar steps, we get the Equations (25) and (26) from (14) and (15), respectively. Noting the operator ${}_{\mathbf{3}}\mathcal{A}^{\pi}_{b_3}(w,r,t)$ defined by (6) and Equation (8) and (9), we obtain

$$\max_{\pi}\left[{}_{\mathbf{3}}\mathcal{A}^{\pi}_{b_3}U^{(3)}(w,r,t)\right] = U^{(0)}(w,r,t)\left[\frac{d\phi_3(t)}{dt} + b_3\bar{\alpha}\phi_3(t)\right]. \tag{34}$$

Substitute $U^{(i)}(w,r,t) = U^{(0)}(w,r,t)\phi_i(t)$, $(i = 1,2,3)$, to obtain

$$\begin{aligned}&\sum_{j=0,1,2}\lambda_{3j}(t)[U^{(j)}(w,r,t) - U^{(3)}(w,r,t)]\\ &= U^{(0)}(w,r,t)\left[\sum_{j=1,2}\lambda_{3j}(t)\left(\phi_j(t) - \phi_3(t)\right) + \lambda_{30}(t)\left(1-\phi_3(t)\right)\right]\end{aligned} \tag{35}$$

Substituting (34) and (35) into (16) we obtain (27). Equation (22) follows similarly from the HJB Equation (17). This completes the proof. □

Remark 1. Theorem 2 presents the indifference pricing equations for the continuous annuity rate b_i $(i = 1,2,\ldots,6)$. Apparently, the b_i, $(i = 1,2,\ldots,6)$, do not appear in those equations. In fact, the b_i's are hidden in the appropriate differential operators. For example, noting the definition of ${}_{\mathbf{0}}\mathcal{L}^{r,\sigma}_{b}f(X,t)$ given by (7), the specific form of ${}_{\mathbf{0}}\mathcal{L}^{r,\sigma}_{b_i}\phi_i(H,t)$ $(i = 4,5,6)$ is expressed as

$${}_{\mathbf{0}}\mathcal{L}^{r,\sigma}_{b_i}\phi_i(H,t) = \frac{\partial\phi_i}{\partial t} + rH\frac{\partial\phi_i}{\partial H} + \frac{1}{2}\sigma^2H^2\frac{\partial^2\phi_i}{\partial H^2} + b_i\alpha e^{r(T-t)}, \tag{36}$$

where r denotes the value of stochastic interest rate r_t at time t.

Remark 2. For the contracts linking home reversion plan to the long term care based on Markov models in [8], we just need to make the corresponding modifications in the derivative details of Lemma 1 and Theorem 2. Then the results paralleling them under the stochastic interest rates can be derived without much effort.

Employing the methods of Theorem 3.4. and Corollary 3.5 of [8], we can obtain from Theorem 2 the Feynman-Kac formula for the indifference continuous annuity rates b_i, $i = 1,2,\ldots,6$.

4. HOME REVERSION PLAN FOR A PAIR OF INSUREDS

The contents of Sections 4 and 5 parallel that of Section 3. Therefore, we continue to follow similar notations as in Section 3. With constant interest rate, [8] prices the home reversion plan for a pair of jointly insureds with the principle of equivalent utility. In this Section, we continue the study of this problem under the assumption of stochastic interest rate. Allowing that a pair of jointly insureds die simultaneously, we can adopt Figure 2(b) to describe the policy states and transitions. Otherwise the corresponding policy states and the transitions are given in Figure 2(a). The

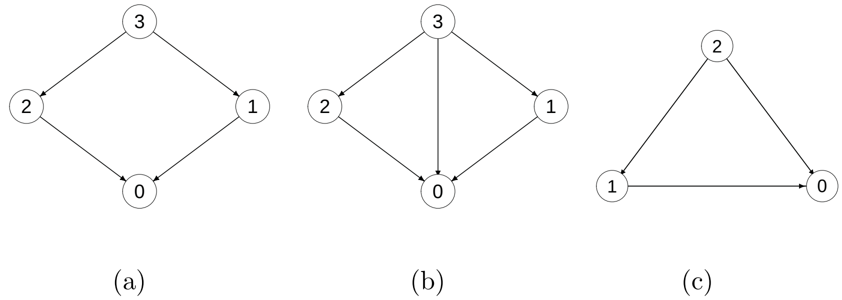

FIGURE 2. Markov model for home reversion plan for a pair of insureds

following are based on Figure 2(a). The associated policy states for the couple are as follows:

1. State 3 represents that both (x) and (y) are alive;
2. State 2 represents that (x) is dead and (y) is alive;
3. State 1 represents that (x) is alive and (y) is dead;
4. State 0 represents that both (x) and (y) are dead.

Here, as before, (x) and (y) denote the x-year old husband and y-year old wife, respectively. Let $\tau_i = \inf\{t;\ Z_t = i\}$ denote the stopping time of entrance into state i $(i = 0, 1, 2)$. The home reversion plan applied jointly by a couple is designed as follows

- As for benefits, we assume that a continuous annuity benefit is paid at an instantaneous constant rate b_i when the insureds are in state i, $i = 1, 2, 3$.
- In return, the insurer will be repaid with $\mathbf{g}(X_{\tau_0}, \tau_0)$ at the time of entering into state 0, where $0 \leq \mathbf{g}(X_{\tau_0}, \tau_0) \leq X_{\tau_0}$. In other words, the insureds agree as per the contract, to repay with whole or part of the cash generated from the sale of the house at time τ_0.

Dynamics of Wealth: Based on the financial market in Section 2, when the insurer underwrites home reversion plan for a couple, the dynamics of wealth is as follows

$$
\begin{cases}
W_t = w, & \\
W_{\tau_0^+} = W_{\tau_0^-} + \mathbf{g}(H_{\tau_0}, \tau_0), & t < \tau_0 < T, \\
dW_s = [r_s W_s + (\mu - r_s)\pi_s - b_3]\ ds + \sigma \pi_s\ dB_s^H, & t < s < \min(\tau_1, \tau_2) < T, \\
dW_s = [r_s W_s + (\mu - r_s)\pi_s - b_2]\ ds + \sigma \pi_s\ dB_s^H, & t < \tau_2 < s < \tau_0 < T,\ \tau_1 = \infty, \\
dW_s = [r_s W_s + (\mu - r_s)\pi_s - b_1]\ ds + \sigma \pi_s\ dB_s^H, & t < \tau_1 < s < \tau_0 < T,\ \tau_2 = \infty, \\
dW_s = [r_s W_s + (\mu - r_s)\pi_s]\ ds + \sigma \pi_s\ dB_s^H, & t < \tau_0 < s < T.
\end{cases}
$$

Value Functions: For the above home reversion plan, the corresponding value functions $U^{(i)}(w,r,H,t)$ $(i=1,2,3)$ are defined as follows

$$U^{(i)}(w,r,H,t) = \sup_{\{\pi_s \in \mathcal{A}\}} E[u(W_T)|W_t = w, r_t = r, H_t = H, Z_t = i]. \tag{37}$$

Here $\mathcal{A}$ represents the set of all admissible strategies that can be strictly defined in the same way as in Definition 2.

Computing the corresponding derivatives as in the proof of Lemma 1, we can show that the value functions defined by (37) solve the following HJB system.

Lemma 3. *$U^{(3)}(w,r,H,t)$ solves the HJB Equation*

$$\max_{\{\pi\in\mathcal{A}\}}[{}_{\mathbf{2}}\mathcal{A}^{\pi}_{b_3}U^{(3)}(w,r,H,t)] + \sum_{i=1,2}\lambda_{3i}(t)[U^{(i)}(w,r,H,t) - U^{(3)}(w,r,H,t)] = 0, \tag{38}$$

where $U^{(i)}(w,r,H,t)$ $(i=1,2)$ solve the HJB Equations

$$\max_{\{\pi\in\mathcal{A}\}}\left[{}_{\mathbf{2}}\mathcal{A}^{\pi}_{b_i}U^{(i)}(w,r,H,t)\right] + \lambda_{i0}(t)[U^{(0)}(w+\mathbf{g}(H,t),r,t) - U^{(i)}(w,r,H,t)] = 0, \tag{39}$$

subject to the terminal conditions

$$U^{(i)}(w,r,H,T) = u(w), \;\; i = 1,2,3,$$

respectively.

With the continuous indifference annuity rate b_1, b_2, b_3, the optimal investment with insurance risk is the same with the optimal investment without the insurance risk, i.e.

$$U^{(0)}(w,r,t) = U^{(3)}(w,r,H,t;b_1,b_2,b_3).$$

The following Theorem 4 gives the pricing equation system that the indifference annuity rates b_1, b_2, b_3 satisfy.

Theorem 4. *Assume that the utility function is exponential, viz $u(w) = -\frac{1}{\alpha}e^{-\alpha w}$. Then, the indifference continuous annuity rates b_i $(i=1,2,3)$ solve the following equation*

$$\phi_3(H,t;b_1,b_2,b_3) = 0, \tag{40}$$

where $\phi_3(H,t)$ solves the HJB Equation

$${}_{\mathbf{0}}\mathcal{L}^{r,\sigma}_{b_3}\phi_3(H,t) + \sum_{i=1,2}\lambda_{3i}(t)\left(e^{\phi_i(H,t)-\phi_3(H,t)} - 1\right) = 0, \tag{41}$$

and $\phi_i(H,t)$ $(i=1,2)$ satisfies the HJB Equation

$${}_{\mathbf{0}}\mathcal{L}^{r,\sigma}_{b_i}\phi_i(H,t) + \lambda_{i0}(t)\left(e^{-(\bar{\alpha}\mathbf{g}(H,t)+\phi_i(H,t))} - 1\right) = 0, \tag{42}$$

subject to $\phi_i(H,T) = 0$, $i = 1,2,3$.

Remark 3. The reader will notice that Theorem 4 coincides with Theorem 4.3. in [8] in form, just with different notations. However, there exist both differences and similarities between them.

- DIFFERENCE: The main differences are due to the different meanings of r in these two theorems. The rate r in Theorem 4 represents the initial value of stochastic interest rate whose dynamics are modeled by the diffusion process (2). Thus, Theorem 4 indicates that the indifference annuity rates only relate with the initial value of stochastic interest rate at the beginning of underwriting the insurance, and have nothing with how the diffusion process evolves. Note here that the r in Theorem 4.3 of [8] denotes a fixed deterministic interest rate during the whole insurance period.
- SIMILARITY: There exist mutual connections between them. Suppose here that the initial value of stochastic interest rate in Theorem 4 coincides with the constant interest rate in Theorem 4.3 of [8], the indifference annuity rates under stochastic interest rate are the same as for that under the constant interest rate.

5. INSURANCE CONTRACT LINKING HRP TO LTC: A SINGLE INSURED

Under the hypothesis that the interest rate is constant, the work [11] prices the insurance contract linking home reversion plan and long-term care insurance with the principle of equivalent utility. In this section, we will continue to price the linked contract, but now under the assumption of stochastic interest rate rather than a fixed rate. Since this section parallels Sections 3 and 4, we adopt similar notations and omit the proofs.

The article [11] adopts a three-state Markov model to illustrate the policy states and transitions of the linked contract. The corresponding policy states are:
- State (2): *the Insured is healthy and living at home*,
- State (1): *the Insured is in the nursing home*, and
- State (0): *the Insured is dead.*

Assume that there is no chance to recover from State (1) to health (State (2)), i.e. the transition $1 \to 2$ cannot appear in Figure 2(c).

The stopping time $\tau_i = \inf\{t;\ Z_t = i\}$ represent the time of entering the State i, $i = 0, 1$. The linked insurance contract designed by Xiao (2010) is characterized by the following clauses:

(I) *When the insured stays at state i ($i = 1, 2$), the insurer pay the continuous annuities with the constant rate b_i, $i = 1, 2$, and $b_2 < b_1$.*

(II) *At the stopping time $\tau = \min\{\tau_0, \tau_1\}$, the insureds repay the insurer with the cash of selling house.*

By the clause (II), if the insureds directly enters State 0 from State 2, then at time τ_0 the insurer will sell the insured's house and own all the cashes coming out of the sale of the house as compensation for annuity. If the insureds first enter State 1 from State 2 before entering state 0, then at time τ_1 the insurer will sell the house of insureds and own all cash proceeds.

Wealth Equations: In the current context, when the insurer signs the insurance contract linking home reversion plan and long term care, the dynamics of its wealth are as follows:

$$\begin{cases} W_t = w, & \\ W_{\tau_1^+} = W_{\tau_1^-} + H_{\tau_1}, & \tau_1 < \tau_0 < T, \\ W_{\tau_0^+} = W_{\tau_0^-} + H_{\tau_0}, & \tau_1 = \infty,\ \tau_0 < T, \\ dW_s = [r_s W_s + (\mu - r_s)\pi_s - b_2]\, ds + \sigma\pi_s\, dB_s^H, & t < s < \min(\tau_0, \tau_1), \\ dW_s = [r_s W_s + (\mu - r_s)\pi_s - b_1]\, ds + \sigma\pi_s\, dB_s^H, & \tau_1 < s < \tau_0, \\ dW_s = [r_s W_s + (\mu - r_s)\pi_s]\, ds + \sigma\pi_s\, dB_s^H, & \tau_0 < s < T. \end{cases}$$

Value Functions: The value functions describe the goal of the insurer which is to maximize the expected utility of terminal wealth. In the present case, the value functions at the State 2 and 1 are:

$$U^{(2)}(w, r, H, t; b_1, b_2) = \sup_{\{\pi_s \in \mathcal{A}\}} E[u(W_T) | W_t = w, r_t = r, H_t = H, Z_t = 2], \tag{43}$$

$$U^{(1)}(w, r, t; b_1) = \sup_{\{\pi_s \in \mathcal{A}\}} E[u(W_T) | W_t = w, r_t = r, Z_t = 1], \tag{44}$$

where $\mathcal{A}$ represents the set of all admissible strategies which can be defined in the same way as in Definition 2.

Modifying slightly the derivations in Lemma 1, we can show that the value function defined by (43) and (44) solve the following HJB equation system

Lemma 5. *The value function $U^{(2)}(w, r, H, t)$ solves the HJB equation*

$$\max_\pi [{}_{\mathbf{2}}\mathcal{A}^\pi_{b_2} U^{(2)}(w, r, H, t)] + \sum_{i=0,1} \lambda_{2i}(t)[U^{(i)}(w + H, r, t) - U^{(2)}(w, r, H, t)] = 0, \tag{45}$$

where the value function $U^{(1)}(w, r, t)$ satisfies

$$\max_\pi \left[{}_{\mathbf{3}}\mathcal{A}^\pi_{b_1} U^{(1)}(w, r, t)\right] + \lambda_{10}(t)[U^{(0)}(w, r, t) - U^{(1)}(w, r, t)] = 0, \tag{46}$$

subject to the respective terminal conditions

$$U^{(2)}(w, r, H, T) = u(w),\ U^{(1)}(w, r, T) = u(w).$$

If the insurer pays the insureds with the indifference continuous annuity rates b_1, b_2, then the maximum expectation utility of the insurer subject to the insurance risk coincides with the one without the insurance risk, i.e.

$$U^{(0)}(w,r,t) = U^{(2)}(w,r,H,t;b_1,b_2).$$

Pricing Equation: With the notation $\bar{\alpha} = \alpha e^{r(T-t)}$, the following Theorem 6 presents the pricing equations of indifference annuity rates b_i $(i = 1, 2)$ for a single insured.

Theorem 6. *Assume that the utility function is exponential, viz* $u(w) = -\frac{1}{\alpha}e^{-\alpha w}$. *The indifference continuous annuity rates* b_i $(i = 1, 2)$ *satisfy the following equation*

$$\phi_2(H,t;b_1,b_2) = 0, \tag{47}$$

where $\phi_2(H,t)$ *solves the HJB equation*

$${}_{\mathbf{0}}\mathcal{L}^{r,\sigma}_{b_2}\phi_2(H,t) + e^{-\bar{\alpha}H-\phi_2(H,t)}[\phi_1(t)\lambda_{21}(t) + \lambda_{20}(t)] - [\lambda_{21}(t) + \lambda_{20}(t)] = 0, \tag{48}$$

in which $\phi_1(t)$ *satisfies the following ordinary differential equation*

$$\frac{d\phi_1(t)}{dt} + [b_1\bar{\alpha} - \lambda_{10}(t)]\,\phi_1(t) + \lambda_{10}(t) = 0, \tag{49}$$

where (48) *and* (49) *are subject to the boundary conditions* $\phi_2(H,T) = 0$ *and* $\phi_1(T) = 1$, *respectively.*

Remark 4. Although Theorem 6 coincides in form with Theorem 2 of [11], these two theorems have some connections as well as differences. The interested readers can imitate Remark 3 to conclude their connections and differences. In particular, the notation r in Theorem 6 represents the value of the stochastic interest rate at the initial moment of signing the insurance contract, while the notation r of Theorem 2 in [11] is the fixed constant interest rate during the whole insurance period.

6. **CONCLUSION**

With a multi-state Markov modeling, this article explores the indifference pricing of continuous annuities of the insurance contract relevant to the home reversion plan involving a single insured and an insured couple. We assume that the risky asset (i.e., home value) follows a geometric Brownian motion, and the risk-free bonds accumulate with a stochastic interest rate driven by a diffusion process. Under such assumptions, we applied the principle of equivalent utility to derive the partial differential equation system that the indifferent annuity benefits satisfy under the exponential utility function. Interestingly, the partial differential equation system under stochastic interest rate coincides in form with those under the constant interest rate. However, there exist connections as well as differences between them. In case that the value of stochastic interest rate at the beginning of signing the insurance contract is the same as the constant interest rate in [11] and [8], then the indifference annuity benefits under the stochastic interest rate coincide with those under the constant interest rate. The indifference annuity rates under the stochastic interest rate relate only with the initial value of stochastic interest rate at the start of writing the insurance contract, and have nothing to do with the specific paths of the diffusion process that drives the dynamics of stochastic interest rate.

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