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UPPER-SEMICONTINUITY OF ATTRACTORS FOR REACTION-DIFFUSION EQUATION AND DAMPED WAVE EQUATION IN $\mathbb{R}^n$ PERTURBED BY SMALL MULTIPLICATIVE NOISES

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ABSTRACT. In this paper, we establish the upper semicontinuity of random attractors for the stochastically perturbed reaction-diffusion equation and damped wave equation with multiplicative noises defined on the entire space $\mathbb{R}^n$ as the coefficient of the white noise term tends to zero.

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1. INTRODUCTION

It is known that the asymptotic behavior of an infinite dimensional random dynamical system (RDS) is determined by a random attractor. Many authors studied the existence of random attractors for the partial differential equations (PDEs) in bounded domains [6, 7, 8, 9, 10, 11, 12, 13, 14, 20, 23, 31, 33] and in unbounded domains [2, 25, 26, 27, 28, 29, 30], and the ordinary differential equations (ODEs) on infinite lattices [3, 32, 17]. Concerning the properties of attractors for parameters-depending dynamical systems, it is important to consider the dependence of attractors on the parameters. In the deterministic case, the upper semicontinuity of global attractors with respect to some parameters were investigated in [4, 15, 16, 18, 21, 24] and many others. For stochastic PDEs defined in bounded domains, this problem has been studied by the authors of [6, 7, 19, 22]. Recently, Wang in [28] gave some conditions for the upper semicontinuity of perturbed random attractors to a global attractor of the limiting autonomous dynamical system, and then applied it to establish the upper semicontinuity of random attractors for stochastic reaction-diffusion equation with small additive white noise defined in $\mathbb{R}^n$ as the coefficient of the white noise term tends to zero, where Wang overcome the obstacles caused by the non-compactness of embedding by using uniform estimates for far-field values of functions inside the perturbed random attractors and showed that the values of all functions in all perturbed random attractors are uniformly convergent to zero when spatial variables approach infinity.

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In this paper, we consider the following stochastic reaction-diffusion equation and damped wave equation with small multiplicative noise defined in the entire space $\mathbb{R}^n$:

$$\begin{cases} du + (\lambda u - \Delta u)dt = (f(x,u) + g(x))dt + \varepsilon u \circ dW, & t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

and

$$\begin{cases} d(u_t + \alpha u) + (\lambda u - \Delta u + f(u))dt = g(x)dt + \varepsilon u dW, & t > 0, \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.2}$$

where Δ is the Laplacian with respect to the variable $x \in \mathbb{R}^n$, $u = u(x,t)$ is a real-valued function of $x \in \mathbb{R}^n$ and $t \geq 0$; $\lambda > 0$, $\alpha > 0$, ε are constants, g is a given function defined on $\mathbb{R}^n$; f is a nonlinear function satisfying certain conditions; $W(t)$ is a two-sided real-valued Wiener process on probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is a subset of $\{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\}$, the Borel σ-algebra $\mathcal{F}$ on Ω is generated by the compact open topology, and $\mathbb{P}$ is the corresponding Wiener measure on $\mathcal{F}$; $\circ$ denotes the Stratonovich sense in the stochastic term. $W(t)$ acting at $\omega \in \Omega$ is identified with $\omega(t)$, i.e., $W(t)(\omega) = W(t,\omega) = \omega(t)$ for $t \in \mathbb{R}$ and $\omega \in \Omega$. The authors have proved the existence of random attractors for the random dynamical systems associated with (1.1)–(1.2) in [29, 30], respectively. Here we will study the limiting behavior of random attractors for the stochastic equations (1.1)–(1.2) as $\varepsilon \to 0$, and establish the upper semicontinuity of these perturbed random attractors.

The rest of this paper is organized as follows. In the next section, we recall some basic concepts related to random attractor for general random dynamical systems and present some conditions for the upper semicontinuity of perturbed random attractors to a global attractor. In section 3 and section 4, we show the upper semicontinuity of random attractors for stochastically perturbed equations (1.1) and (1.2) to the global attractor of corresponding limiting determined equations as $\varepsilon \to 0$, respectively.

2. PRELIMINARIES

Let $(X, \|\cdot\|_X)$ be a separable Banach space with the Borel σ-algebra $\mathcal{B}(X)$. Let $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ be an ergodic metric dynamical system. *A continuous random dynamical system (RDS)* on X over $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ is a $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$-measurable mapping:

$$\varphi : \mathbb{R}^+ \times \Omega \times X \to X, \quad (t, \omega, u) \mapsto \varphi(t, \omega, u)$$

such that the following properties hold: (i) $\varphi(0, \omega, \cdot)$ is the identity on X; (ii) $\varphi(t + s, \omega, \cdot) = \varphi(t, \theta_s\omega, \varphi(s, \omega, \cdot))$ for all $s, t \geq 0$; (iii) $\varphi(t, \omega, \cdot) : X \to X$ is continuous for all $t \geq 0$ (see [1, 8]). A set-valued mapping $\{D(\omega)\} : \Omega \to 2^X$, $\omega \to D(\omega)$, is said to be a *random set* if the mapping $\omega \mapsto d(u, D(\omega))$ is measurable for any $u \in X$.

If $D(\omega)$ is also closed (compact) for each $\omega \in \Omega$, $\{D(\omega)\}$ is called a *random closed (compact) set*. A random set $\{D(\omega)\}$ is said to be *bounded* if there exist $u_0 \in X$ and a random variable $R(\omega) > 0$ such that $D(\omega) \subset \{u \in X : \|u - u_0\|_X \leq R(\omega)\}$ for all $\omega \in \Omega$. A random set $\{D(\omega)\}$ is called *tempered* provided for $\mathbb{P}$-a.e. $\omega \in \Omega$, $\lim\limits_{t\to+\infty} e^{-\beta t} d(D(\theta_{-t}\omega)) = 0$ for all $\beta > 0$, where $d(D) = \sup\{\|b\|_X : b \in D\}$. Let $\mathcal{D}(X)$ denote the set of all tempered random sets of X. A random set $\{B(\omega)\}$ is said to be a *random absorbing set* if for any tempered random set $\{D(\omega)\}$, and $\mathbb{P}$-a.e. $\omega \in \Omega$, there exists $t_D(\omega) \geq 0$, such that $\varphi(t, \theta_{-t}\omega, D(\theta_{-t}\omega)) \subset B(\omega)$ for all $t \geq t_D(\omega)$. A random set $\{B_1(\omega)\}$ is said to be a *random attracting set* if for any tempered random set $\{D(\omega)\}$, and $\mathbb{P}$-a.e. $\omega \in \Omega$, we have $\lim_{t\to+\infty} \mathrm{d}_H(\varphi(t, \theta_{-t}\omega, D(\theta_{-t}\omega)), B_1(\omega)) = 0$, where d_H is the Hausdorff semi-distance given by $d_H(E, F) = \sup_{u\in E} \inf_{v\in F} \|u - v\|_X$ for any $E, F \subset X$. A random compact set $\{A(\omega)\}$ is said to be a *random attractor* if it is a random attracting set and $\varphi(t, \omega, A(\omega)) = A(\theta_t\omega)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$ and all $t \geq 0$.

Theorem 2.1 (See [28])**.** *Let $(X, \|\cdot\|_X)$ be a separable Banach space with Borel σ-algebra $\mathcal{B}(X)$ and $\{\phi(t)\}_{t\geq 0}$ be an autonomous dynamical system defined on X. Suppose $\{\phi^\varepsilon(t, \omega)\}_{\varepsilon>0, t\geq 0, \omega\in\Omega}$ is a family of random dynamical systems on X over metric system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$. Suppose that (i) ϕ has a global attractor A_0 in X which is compact and invariant and attracts every bounded subset of X uniformly; (ii) for any $\varepsilon > 0$, ϕ^ε has a random attractor $A_\varepsilon = \{A_\varepsilon(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ and a random absorbing set $B_\varepsilon = \{B_\varepsilon(\omega)\}_{\omega\in\Omega} \in \mathcal{D}$ such that for some deterministic positive constant c_0 and for $\mathbb{P}$-a.e. $\omega \in \Omega$, $\lim_{n\to\infty} \sup \|B_\varepsilon(\omega)\|_X \leqslant c_0$, where $\|B_\varepsilon(\omega)\|_X = \sup_{x\in B_\varepsilon(\omega)} \|x\|_X$; (iii) there exists $\varepsilon_0 > 0$ such that for $\mathbb{P}$-a.e. $\omega \in \Omega$, $\bigcup_{0<\varepsilon\leqslant\varepsilon_0} A_\varepsilon(\omega)$ is precompact in X; (iv) for $\mathbb{P}$-a.e. $\omega \in \Omega$, $t \geq 0$, $\varepsilon_n \to 0$, and x_n, $x_0 \in X$ with $x_n \to x_0$, it holds: $\lim_{n\to\infty} \phi^{\varepsilon_n}(t, \omega, x_n) = \phi(t)x_0$. Then for $\mathbb{P}$-a.e. $\omega \in \Omega$, $d_H(A_\varepsilon(\omega), A_0) \to 0$ as $\varepsilon \to 0$.*

3. STOCHASTIC REACTION-DIFFUSION EQUATION ON $\mathbb{R}^n$ WITH MULTIPLICATIVE NOISE

In this section, we consider the upper semicontinuity of random attractors for the following initial value problem of stochastic reaction-diffusion equation (1.1) with multiplicative noise defined in the entire space $\mathbb{R}^n$ ($n \in \mathbb{N}$):

$$\begin{cases} du + (\lambda u - \Delta u)dt = (f(x, u) + g(x))dt + \varepsilon u \circ dW, \\ u(x, 0) = u_0(x), \ x \in \mathbb{R}^n, \end{cases} \tag{3.1}$$

where $u = u(x, t) \in \mathbb{R}$, $x \in \mathbb{R}^n$, $t \geq 0$; $\lambda > 0$, ε are constants, $g \in L^2(\mathbb{R}^n)$; $f \in C^1(\mathbb{R}^n \times \mathbb{R}, \mathbb{R})$ satisfies certain conditions; $W(t)$ is a two-sided real-valued Wiener process on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ defined in section 1; $\circ$ denotes the Stratonovich sense in the stochastic term. Define $(\theta_t)_{t\in\mathbb{R}}$ on Ω via $\theta_t\omega(\cdot) =$

$\omega(\cdot+t)-\omega(t)$, $t\in\mathbb{R}$, then $(\Omega,\mathcal{F},\mathbb{P},(\theta_t)_{t\in\mathbb{R}})$ is an ergodic metric dynamical system [1, 8].

Note that Eq.(3.1) is interpreted as an integral equation:

$$u(t)=u(0)-\int_0^t(\lambda u(s)-\Delta u(s))ds+\int_0^t(f(x,u(s))+g(x))ds+\varepsilon\int_0^t u(s)\circ dW, \tag{3.2}$$

where the stochastic integral is understood in the Stratonovich sense.

We make the following assumptions on the nonlinearity $f(x,u)$:

$$\begin{cases} f\in C^1(\mathbb{R}^n\times\mathbb{R},\mathbb{R}), \\ f(x,0)=0, \qquad f(x,u)u\leqslant 0, \qquad \text{for all } x\in\mathbb{R}^n,\ u\in\mathbb{R}, \\ \left|\frac{\partial f}{\partial u}(x,u)\right|\leqslant\kappa, \quad \left|\frac{\partial f}{\partial x}(x,u)\right|\leqslant\widetilde{f}(x), \end{cases} \tag{3.3}$$

where κ is a positive constant, $\widetilde{f}(x)\in L^2(\mathbb{R}^n)$, $|\cdot|$ denotes the absolute value of real number in $\mathbb{R}$.

It is convenient to convert the problem (3.1) into a deterministic system with a random parameter. Put $z(\theta_t\omega):=-\int_{-\infty}^0 e^s(\theta_t\omega)(s)ds$, $t\in\mathbb{R}$, which is an Ornstein-Uhlenbeck process and solves Itô equation $dz+zdt=dW(t)$. Moreover, the random variable $z(\omega)$ is tempered, and there is a θ_t-invariant set $\widetilde{\Omega}\subset\Omega$ of full $\mathbb{P}$ measure such that for every $\omega\in\widetilde{\Omega}$, $t\mapsto z(\theta_t\omega)$ is continuous in t; $\lim_{t\to\pm\infty}\frac{|z(\theta_t\omega)|}{|t|}=0$ and $\lim_{t\to\pm\infty}\frac{1}{t}\int_0^t z(\theta_s\omega)ds=0$ (see [1, 5, 8]).

Let $v(t)=e^{-\varepsilon z(\theta_t\omega)}u(t)$, where u is a solution of problem (3.1). Then (3.1) can be written as the following evolution equation with random coefficients but without white noise:

$$\begin{cases} \frac{dv(t)}{dt}=\Delta v(t)-\lambda v(t)+e^{-\varepsilon z(\theta_t\omega)}\left(f(x,e^{\varepsilon z(\theta_t\omega)}v(t))+g(x)\right)+\varepsilon z(\theta_t\omega)v(t), \\ v(0,x)=v_0(x)=e^{-\varepsilon z(\omega)}u_0(x),\ x\in\mathbb{R}^n, \quad t>0. \end{cases} \tag{3.4}$$

We will consider (3.4) for $\omega\in\widetilde{\Omega}$ and still write $\widetilde{\Omega}$ as Ω. Concerning the solutions of (3.4), from [29] we knew the following results.

Theorem 3.1 (See [29].). *Suppose conditions* (3.3), $g\in L^2(\mathbb{R}^n)$ *hold and* $\varepsilon\in\mathbb{R}$. *Then*

(1) *for any* $v_0\in L^2(\mathbb{R}^n)$, *the system* (3.4) *has a unique solution* $v(\cdot,\omega,v_0)\in C([0,+\infty),L^2(\mathbb{R}^n))\cap C([0,T),H^1(\mathbb{R}^n))$ $(T>0)$ *and* $v(t,\omega,v_0)$ *is continuous in* v_0 *in* $L^2(\mathbb{R}^n)$ *for all* $t>0$. *Furthermore, for* $u_0\in L^2(\mathbb{R}^n)$, $t\geq 0$, $\omega\in\Omega$, *the mapping*

$$\varphi^\varepsilon:\mathbb{R}^+\times\Omega\times L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n),\varphi^\varepsilon(t,\omega,v_0)=v(t,\omega,v_0) \tag{3.5}$$

forms a continuous RDS $\{\varphi^\varepsilon(t,\omega)\}_{t\geq0,\omega\in\Omega}$ *over* $(\Omega,\mathcal{F},\mathbb{P},(\theta_t)_{t\in\mathbb{R}})$ *and*

$$\phi^\varepsilon(t,\omega,u_0)=u(t,\omega,u_0)=e^{\varepsilon z(\theta_t\omega)}\varphi^\varepsilon(t,\omega,e^{-\varepsilon z(\omega)}u_0) \tag{3.6}$$

generates a continuous RDS $\{\phi^\varepsilon(t,\omega)\}_{t\geq0,\omega\in\Omega}$ *associated with the problem* (3.1) *on* $L^2(\mathbb{R}^n)$.

(2) *For any* $\{D(\omega)\} \in \mathcal{D}(L^2(\mathbb{R}^n))$ *and* $\mathbb{P}$*-a.e.* $\omega \in \Omega$*, there is* $T_D(\omega) > 0$ *such that for any* $u_0 \in D(\theta_{-t}\omega) \cap L^2(\mathbb{R}^n)$*,*

$$\begin{aligned} \|\phi^\varepsilon(t, \theta_{-t}\omega, u_0)\|^2 &\leqslant e^{2\varepsilon z(\omega)} \left(1 + \frac{\|g\|^2}{\lambda} \int_{-\infty}^0 e^{-2\varepsilon z(\theta_s\omega) + 2\varepsilon \int_s^0 z(\theta_\tau\omega)d\tau + \lambda s} ds\right) \\ &= R(\varepsilon, \omega), \end{aligned} \tag{3.7}$$

$$\begin{aligned} \|\nabla\phi^\varepsilon(t, \theta_{-t}\omega, u_0)\|^2 \leqslant\ & e^{2\varepsilon z(\omega)}(C_1 + 2\varepsilon e\gamma(\omega))R_0(\varepsilon, \omega) \\ &+ e^{2\varepsilon z(\omega)} \int_{-1}^0 e^{-2\varepsilon z(\theta_\tau\omega)} (\|g\|^2 + C_2\|\widetilde{f}\|^2) d\tau, \end{aligned} \tag{3.8}$$

where C_1, C_2 *are positive constants independent of* ε *and*

$$R_0(\varepsilon, \omega) = 1 + \frac{\|g\|^2}{\lambda} \int_{-\infty}^0 e^{\varepsilon\left(-2z(\theta_s\omega) + 2e \max_{0 \leqslant \tau \leqslant 1} |z(\theta_\tau\omega)| + 2\int_s^0 z(\theta_\tau\omega)d\tau\right) + \lambda(s+1)} ds. \tag{3.9}$$

(3) *For any* $\eta > 0$*, any* $\{D(\omega)\} \in \mathcal{D}(L^2(\mathbb{R}^n))$ *and* $\mathbb{P}$*-a.e.* $\omega \in \Omega$*, there exist* $T_1 = T_1(\eta, \omega, B) > 0$ *and* $r_1 = r_1(\eta, \omega) > 0$ *(independent of* ε*), such that the solution* ϕ^ε *of* (3.1) *satisfies for* $\mathbb{P}$*-a.e.* $\omega \in \Omega$*,* $\forall\ t \geq T_1$*,*

$$\int_{|x| \geq r_1} |\phi^\varepsilon(t, \theta_{-t}\omega, u_0)(x)|^2 dx \leqslant \eta. \tag{3.10}$$

(4) *RDS* $\{\phi^\varepsilon(t, \omega)\}_{t \geq 0, \omega \in \Omega}$ *has a unique global random attractor* $\{A^\varepsilon(\omega)\}$ *in* $L^2(\mathbb{R}^n)$*.*

According to Theorem 2.1, we have the following upper semicontinuity of random attractors $\{A^\varepsilon(\omega)\}$ when $\varepsilon \to 0$.

Theorem 3.2. *Suppose conditions* (3.3) *and* $g \in L^2(\mathbb{R}^n)$ *hold. Then for* $\mathbb{P}$*-a.e.* $\omega \in \Omega$*,*

$$d_H(A^\varepsilon(\omega), A^0) = \sup_{x \in A^\varepsilon(\omega)} \inf_{y \in A^0} \|x - y\|_{L^2(\mathbb{R}^n)} \to 0 \quad as \quad \varepsilon \to 0, \tag{3.11}$$

where A^0 *is the global attractor of the autonomous dynamical system associated with the limiting deterministic equation*

$$\begin{cases} \frac{du}{dt} + \lambda u - \Delta u = f(x, u) + g(x), & t > 0, \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{3.12}$$

Proof. Let us check that RDS $\{\phi^\varepsilon(t, \omega)\}_{t \geq 0, \omega \in \Omega}$ satisfies the conditions in Theorem 2.1 one by one.

(i) It is clearly that under the conditions (3.3) and $g \in L^2(\mathbb{R}^n)$, the solutions of limiting deterministic equation (3.12) determines a continuous autonomous dynamical system $\{\phi^0(t)\}_{t \geq 0}$ and $\{\phi^0(t)\}_{t \geq 0}$ has a global attractor A^0 in $L^2(\mathbb{R}^n)$ (see [28, 29]).

(ii) Given $|\varepsilon| \leqslant 1$. By the properties of $z(\theta_t\omega)$, there exists $T_\lambda > 0$ (independent of ε) such that

$$\left| \frac{-2z(\theta_s\omega) + 2\int_s^0 z(\theta_\tau\omega)d\tau}{s} \right| < \frac{\lambda}{2}, \quad \forall\, |s| \geq T_\lambda. \tag{3.13}$$

Then

(3.14)
$$\begin{aligned} R(\varepsilon,\omega) &= e^{2\varepsilon z(\omega)} \left(1 + \frac{\|g\|^2}{\lambda} \int_{-T_\lambda}^0 e^{\varepsilon\left(-2z(\theta_s\omega)+2\varepsilon\int_s^0 z(\theta_\tau\omega)d\tau\right)+\lambda s}\right) \\ &\quad + \frac{\|g\|^2}{\lambda} e^{2\varepsilon z(\omega)} \int_{-\infty}^{-T_\lambda} e^{\varepsilon\left(-2z(\theta_s\omega)+2\int_s^0 z(\theta_\tau\omega)d\tau\right)+\lambda s} ds \\ &\leqslant e^{2|z(\omega)|} \left(1 + \frac{\|g\|^2}{\lambda} \int_{-T_\lambda}^0 e^{\left|2z(\theta_s\omega)+2\int_s^0 z(\theta_\tau\omega)d\tau\right|+\lambda s} ds + \frac{2\|g\|^2}{\lambda^2} e^{-\frac{\lambda}{2}T_\lambda}\right) = R_1(\omega), \end{aligned}$$

and similarly,

(3.15)
$$e^{2\varepsilon z(\omega)} \left((C_1 + 2\varepsilon e\gamma(\omega)) R_0(\varepsilon,\omega) + \int_{-1}^0 e^{-2\varepsilon z(\theta_\tau\omega)} (\|g\|^2 + C_2\|\widetilde{f}\|^2) d\tau \right) \leqslant R_2(\omega),$$

where $R_1(\omega)$, $R_2(\omega)$ are independent of ε. Let

$$\begin{aligned} B^\varepsilon(\omega) &= \left\{u \in L^2(\mathbb{R}^n) : \|u\|^2 \leqslant R(\varepsilon,\omega)\right\}, \\ B^0(\omega) &= \left\{u \in L^2(\mathbb{R}^n) : \|u\|^2 \leqslant R_1(\omega)\right\}. \end{aligned} \tag{3.16}$$

Then by (3.14), $B^\varepsilon = \{B^\varepsilon(\omega)\}_{\omega\in\Omega} \in \mathcal{D}(L^2(\mathbb{R}^n))$ is a closed tempered random absorbing set for $\{\phi^\varepsilon(t)\}_{t\geq 0}$ and

$$\bigcup_{|\varepsilon|\leqslant 1} B^\varepsilon(\omega) \subseteq B^0(\omega). \tag{3.17}$$

By (3.7),

$$\begin{aligned} &\limsup_{\varepsilon\to 0} R(\varepsilon,\omega) \\ &\quad = \limsup_{\varepsilon\to 0} e^{2\varepsilon z(\omega)} \left(1 + \frac{\|g\|^2}{\lambda} \int_{-\infty}^0 e^{\varepsilon\left(-2z(\theta_s\omega)+2\int_s^0 z(\theta_\tau\omega)d\tau\right)+\lambda s} ds\right) \\ &\quad = 1 + \frac{\|g\|^2}{\lambda^2}. \end{aligned} \tag{3.18}$$

So, for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\begin{aligned} \limsup_{\varepsilon\to 0} \|B^\varepsilon(\omega)\| &= \limsup_{\varepsilon\to 0} \sup_{u\in B^\varepsilon(\omega)} \|u\| \\ &\leqslant \limsup_{\varepsilon\to 0} \sqrt{R(\varepsilon,\omega)} \leqslant \left(1 + \frac{\|g\|^2}{\lambda^2}\right)^{\frac{1}{2}}. \end{aligned} \tag{3.19}$$

By Theorem 3.1 (4), the random attractor $A^\varepsilon = \{A^\varepsilon(\omega)\}_{\omega\in\Omega} \in \mathcal{D}(L^2(\mathbb{R}^n))$ for $\{\phi^\varepsilon(t,\omega)\}_{t\geq 0,\omega\in\Omega}$ satisfies that for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$A^\varepsilon(\omega) \subseteq B^\varepsilon(\omega), \quad \bigcup_{|\varepsilon|\leqslant 1} A^\varepsilon(\omega) \subseteq \bigcup_{|\varepsilon|\leqslant 1} B^\varepsilon(\omega) \subseteq B^0(\omega). \tag{3.20}$$

(iii) Given $|\varepsilon| \leq 1$. By (3.7), (3.8), (3.14) and (3.15), for $\mathbb{P}$-a.e. $\omega \in \Omega$, there is $T_{B^0}(\omega) > 0$ (independent of ε) such that

$$\begin{aligned} &\|\phi^\varepsilon(t,\theta_{-t}\omega,B^0(\theta_{-t}\omega))\|^2_{H^1(\mathbb{R}^n)} \\ &\quad= \|\phi^\varepsilon(t,\theta_{-t}\omega,B^0(\theta_{-t}\omega))\|^2 + \|\nabla\phi^\varepsilon(t,\theta_{-t}\omega,B^0(\theta_{-t}\omega))\|^2 \\ &\quad\leqslant R_1(\omega)+R_2(\omega), \quad t \geq T_{B^0}(\omega). \end{aligned} \tag{3.21}$$

Thus, by (3.20), we have that $\bigcup_{|\varepsilon|\leqslant 1} A^\varepsilon(\theta_{-t}\omega) \subseteq B^0(\omega)$ and for $|\varepsilon| \leq 1$,

$$\|\phi^\varepsilon(t,\theta_{-t}\omega,A^\varepsilon(\theta_{-t}\omega))\|^2_{H^1(\mathbb{R}^n)} \leqslant R_1(\omega)+R_2(\omega), \quad t \geq T_{B^0}(\omega). \tag{3.22}$$

By the invariance of the random attractor $A^\varepsilon(\omega)$, it follows that for $\mathbb{P}$-a.e. $\omega \in \Omega$ and $t \geq 0$,

$$\phi^\varepsilon(t,\theta_{-t}\omega,A^\varepsilon(\theta_{-t}\omega)) = A^\varepsilon(\omega). \tag{3.23}$$

By (3.22),

$$\sup_{u\in\bigcup_{|\varepsilon|\leqslant 1} A^\varepsilon(\omega)} \|u\|^2_{H^1(\mathbb{R}^n)} \leqslant R_1(\omega)+R_2(\omega). \tag{3.24}$$

Now let us show that the precompactness of $\bigcup_{|\varepsilon|\leqslant 1} A^\varepsilon(\omega)$ in $L^2(\mathbb{R}^n)$, i.e., given any $\eta > 0$, find a finite covering of balls of radius less than η for $\bigcup_{|\varepsilon|\leqslant 1} A^\varepsilon(\omega)$ in $L^2(\mathbb{R}^n)$. Let $r > 0$, and write

$$Q^r = \{x \in \mathbb{R}^n : |x| < r\}, \quad \bar{Q}^r = \{x \in \mathbb{R}^n : |x| \geq r\} = \mathbb{R}^n \backslash Q^r. \tag{3.25}$$

By Theorem 3.1 (3), for $\mathbb{P}$-a.e. $\omega \in \Omega$, there exist $T_2 = T_2(\eta,\omega,B^0) > 0$ and $r_2 = r_2(\eta,\omega) > 0$ (independent of ε), such that for $\forall t \geq T_2$ and $u_0 \in A^\varepsilon(\theta_{-t}\omega)) \cap L^2(\mathbb{R}^n) \subseteq B^0(\theta_{-t}\omega)$,

$$\int_{|x|\geq r_2} |\phi^\varepsilon(t,\theta_{-t}\omega,u_0)(x)|^2 dx \leqslant \frac{\eta^2}{16}. \tag{3.26}$$

From the invariance of $A^\varepsilon(\omega)$, for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\sup_{u\in\bigcup_{|\varepsilon|\leqslant 1} A^\varepsilon(\omega)} \|u\|_{L^2(\bar{Q}^{r_2})} = \sup_{u\in\bigcup_{|\varepsilon|\leqslant 1} A^\varepsilon(\omega)} \left(\int_{|x|\geq r_2} |u(x)|^2 dx\right)^{\frac{1}{2}} \leqslant \frac{\eta}{4}. \tag{3.27}$$

By (3.24), for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\sup_{u\in\bigcup_{|\varepsilon|\leqslant 1} A^\varepsilon(\omega)} \|u\|^2_{H^1(Q^{r_2})} \leqslant R_1(\omega)+R_2(\omega), \tag{3.28}$$

implying that $\bigcup_{|\varepsilon|\leqslant 1} A^\varepsilon(\omega)$ is bounded in $H^1(Q^{r_2})$. Since the embedding $H^1(Q^{r_2}) \hookrightarrow L^2(Q^{r_2})$ is compact, $\bigcup_{|\varepsilon|\leqslant 1} A^\varepsilon(\omega)$ is a compact subset of $L^2(Q^{r_2})$, hence, $\bigcup_{|\varepsilon|\leqslant 1} A^\varepsilon(\omega)$

has a finite covering of balls of radii less than $\frac{\eta}{4}$ in $L^2(Q^{r_2})$. Combining with (3.27), $\bigcup_{|\varepsilon|\leqslant 1} A^\varepsilon(\omega)$ can be covered by finite balls with radii less than η in $L^2(\mathbb{R}^n)$. Therefore, for $\mathbb{P}$-a.e. $\omega\in\Omega$, $\bigcup_{|\varepsilon|\leqslant 1} A_\varepsilon(\omega)$ is precompact in $L^2(\mathbb{R}^n)$.

(iv) Let v^ε and u be the solutions of (3.4) and (3.12) with initial data v_0^ε and u_0, respectively. Set $w=v^\varepsilon-u$, then

$$\begin{cases}\frac{dw}{dt}+\lambda w-\Delta w=e^{-\varepsilon z(\theta_t\omega)}f(x,e^{\varepsilon z(\theta_t\omega)}v^\varepsilon(t))-f(x,u)\\ \quad +(e^{-\varepsilon z(\theta_t\omega)}-1)g(x)+\varepsilon z(\theta_t\omega)v^\varepsilon, & t>0,\\ w(0,x)=v_0^\varepsilon(x)-u_0(x), & x\in\mathbb{R}^n.\end{cases}\tag{3.29}$$

Taking the inner product of (3.29) with w in $L^2(\mathbb{R}^n)$, we find that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|w\|^2+\|\nabla w\|^2+\lambda\|w\|^2\\ &\quad=\int_{\mathbb{R}^n}\left(e^{-\varepsilon z(\theta_t\omega)}f(x,e^{\varepsilon z(\theta_t\omega)}v^\varepsilon)-f(x,u)\right)wdx\\ &\qquad+((e^{-\varepsilon z(\theta_t\omega)}-1)g,w)+(\varepsilon z(\theta_t\omega)v^\varepsilon,w).\end{aligned}\tag{3.30}$$

We find by condition (3.3) that

$$\begin{aligned}&\int_{\mathbb{R}^n}\left(e^{-\varepsilon z(\theta_t\omega)}f(x,e^{\varepsilon z(\theta_t\omega)}v^\varepsilon)-f(x,u)\right)wdx\\ &\quad=e^{-\varepsilon z(\theta_t\omega)}\int_{\mathbb{R}^n}\left(f(x,e^{\varepsilon z(\theta_t\omega)}v^\varepsilon)-f(x,e^{\varepsilon z(\theta_t\omega)}u)\right)wdx\\ &\quad+e^{-\varepsilon z(\theta_t\omega)}\int_{\mathbb{R}^n}\left(f(x,e^{\varepsilon z(\theta_t\omega)}u)-f(x,u)\right)wdx\\ &\quad+\left(e^{-\varepsilon z(\theta_t\omega)}-1\right)\int_{\mathbb{R}^n}f(x,u)wdx\\ &=\int_{\mathbb{R}^n}\frac{\partial f}{\partial u}(x,\xi_1)w^2dx+\left(1-e^{-\varepsilon z(\theta_t\omega)}\right)\int_{\mathbb{R}^n}\frac{\partial f}{\partial u}(x,\xi_2)uwdx\\ &\quad+\left(e^{-\varepsilon z(\theta_t\omega)}-1\right)\int_{\mathbb{R}^n}\frac{\partial f}{\partial u}(x,\xi_3)uwdx\\ &\leqslant\kappa\|w\|^2+2\kappa\left|1-e^{-\varepsilon z(\theta_t\omega)}\right|\int_{\mathbb{R}^n}|u|\cdot|w|dx\\ &\leqslant\kappa\|w\|^2+\kappa\left|1-e^{-\varepsilon z(\theta_t\omega)}\right|(2\|u\|^2+\|v^\varepsilon\|^2),\end{aligned}\tag{3.31}$$

$$((e^{-\varepsilon z(\theta_t\omega)}-1)g,w)\leqslant\frac{\|g\|^2}{\lambda}(e^{-\varepsilon z(\theta_t\omega)}-1)^2+\frac{\lambda}{4}\|w\|^2,\tag{3.32}$$

$$(\varepsilon z(\theta_t\omega)v^\varepsilon,w)\leqslant\frac{\varepsilon^2}{\lambda}|z(\theta_t\omega)|^2\|v^\varepsilon\|^2+\frac{\lambda}{4}\|w\|^2.\tag{3.33}$$

Since

$$\|u(t,u_0)\|^2\leqslant e^{-\lambda t}\|u_0\|^2+\frac{\|g\|^2}{\lambda}\int_0^t e^{-\lambda(t-s)}ds=P_1(t).\tag{3.34}$$

and for $|\varepsilon| \leqslant 1$,

$$\begin{aligned}\|v^\varepsilon(t,\omega,v_0^\varepsilon(x))\|^2 &\leqslant e^{2\int_0^t \varepsilon z(\theta_s\omega)ds-\lambda t}\|v_0^\varepsilon(x)\|^2 \\ &\quad + \frac{\|g\|^2}{\lambda}e^{2\varepsilon\int_0^t z(\theta_s\omega)ds-\lambda t}\int_0^t e^{-2\varepsilon z(\theta_s\omega)-2\varepsilon\int_0^s z(\theta_\tau\omega)d\tau+\lambda s}ds \\ &\leqslant P_2(t,\omega),\end{aligned} \tag{3.35}$$

where $P_1(t)$, $P_2(t,\omega)$ are positive-valued and continuous in t but independent of ε.

By (3.30)–(3.35), we get

$$\begin{aligned}\frac{d}{dt}\|w\|^2 &\leqslant (2\kappa-\lambda)\|w\|^2 + \frac{2\|g\|^2}{\lambda}\left|1-e^{-\varepsilon z(\theta_t\omega)}\right|^2 \\ &\quad + 2\kappa\left|1-e^{-\varepsilon z(\theta_t\omega)}\right|(2P_1(t)+P_2(t,\omega)) + \frac{2\varepsilon^2}{\lambda}|z(\theta_t\omega)|^2 P_2(t,\omega).\end{aligned} \tag{3.36}$$

By the Gronwall inequality to (3.36),

$$\begin{aligned}\|v^\varepsilon(t,\omega,v_0^\varepsilon)-u(t,u_0)\|^2 &= \|w(t,\omega,w(0))\|^2 \\ &\leqslant e^{(2\kappa-\lambda)t}\|v_0^\varepsilon-u_0\|^2 + \int_0^t e^{(\lambda-2\kappa)(t-s)}\frac{2\varepsilon^2}{\lambda}|z(\theta_s\omega)|^2P_2(s,\omega)ds \\ &\quad + \int_0^t e^{(\lambda-2\kappa)(t-s)}\Big(\frac{\|g\|^2}{\lambda}\left|1-e^{-\varepsilon z(\theta_s\omega)}\right|^2 \\ &\quad + 2\kappa\left|1-e^{-\varepsilon z(\theta_s\omega)}\right|\big(2P_1(s)+P_2(s,\omega)\big)\Big)ds.\end{aligned} \tag{3.37}$$

From (3.37), we see that for $\mathbb{P}$-a.e. $\omega\in\Omega$, $t\geq 0$, $\varepsilon_n\to 0$, and $v_0^{\varepsilon_n}$, $u_0\in L^2(\mathbb{R}^n)$ with $v_0^{\varepsilon_n}\to u_0$, it holds: $\lim_{n\to\infty} v^{\varepsilon_n}(t,\omega,v_0^{\varepsilon_n}) = u(t,u_0)$. Thus, for $\mathbb{P}$-a.e. $\omega\in\Omega$, $t\geq 0$, $\varepsilon_n\to 0$, and $u_0^{\varepsilon_n}$, $u_0\in L^2(\mathbb{R}^n)$ with $u_0^{\varepsilon_n} = e^{\varepsilon_n z(\omega)}v_0^{\varepsilon_n}\to u_0$,

$$\lim_{n\to\infty}\phi^{\varepsilon_n}(t,\omega,u_0^{\varepsilon_n}) = \lim_{n\to\infty} v^{\varepsilon_n}(t,\omega,v_0^{\varepsilon_n}) = u(t,u_0) = \phi^0(t)u_0. \tag{3.38}$$

By Theorem 2.1, the proof is completed.

□

4. STOCHASTIC DAMPED WAVE EQUATION ON $\mathbb{R}^n$ WITH MULTIPLICATIVE NOISE

Consider the following stochastic damped wave equation (1.2) with multiplicative noise defined in the entire space $\mathbb{R}^n$:

$$\begin{cases} d(u_t+\alpha u)+(\lambda u-\Delta u+f(u))dt = g(x)dt+\varepsilon u dW, & t>0, \\ u(x,0)=u_0(x),\ \ u_t(x,0)=u_1(x), & x\in\mathbb{R}^n,\end{cases} \tag{4.1}$$

where $u = u(x,t) \in \mathbb{R}$, $x \in \mathbb{R}^n$ and $t \geq 0$; $\alpha > 0$, $\lambda > 0$, ε are constants, g is a given function defined on $\mathbb{R}^n$; f is a nonlinear function satisfying the following conditions:

$$f \in C^1(\mathbb{R},\mathbb{R}), \quad |f'(u)| \leqslant c_0, \quad |f(u)| \leqslant c_1, \quad \forall\ u \in \mathbb{R}, \tag{4.2}$$

where c_0, c_1 are non-negative constants, $W(t)$ is as in section 3.

Let $\xi = u_t + \sigma u$, then (4.1) can be rewritten as the following equivalent system

$$\begin{cases} du = (\xi - \sigma u)dt, \\ d\xi = ((\sigma-\alpha)\xi + (\alpha\sigma - \lambda - \sigma^2)u + \Delta u - f(u) + g(x))\,dt + \varepsilon\xi dW, \\ u(x,0) = u_0(x),\ \xi(x,0) = \xi_0(x) = u_1(x) + \sigma u_0(x), \quad x \in \mathbb{R}^n. \end{cases} \tag{4.3}$$

Let the metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ be defined as in section 3. Put the Ornstein-Uhlenbeck process as

$$\tilde{z}(\theta_t\omega) := -\alpha \int_{-\infty}^{0} e^{\alpha s}(\theta_t\omega)(s)ds, \quad t \in \mathbb{R},$$

which solves the Itô equation $d\tilde{z} + \alpha\tilde{z}dt = dW(t)$ [5]. From [2, 11, 13], it is known that the random variable $|\tilde{z}(\omega)|$ is tempered, and there is a θ_t-invariant set $\widetilde{\Omega}_1 \subset \Omega$ of full $\mathbb{P}$ measure such that for every $\omega \in \widetilde{\Omega}_1$, $t \mapsto \tilde{z}(\theta_t\omega)$ is continuous in t and

$$\lim_{t\to\pm\infty} \frac{|\tilde{z}(\theta_t\omega)|}{|t|} = \lim_{t\to\pm\infty} \frac{1}{t}\int_0^t \tilde{z}(\theta_s\omega)ds = 0, \tag{4.4}$$

$$\lim_{t\to\pm\infty} \frac{1}{t}\int_0^t |\tilde{z}(\theta_s\omega)|ds = \frac{1}{\sqrt{\pi\alpha}}, \lim_{t\to\pm\infty} \frac{1}{t}\int_0^t |\tilde{z}(\theta_s\omega)|^2 ds = \frac{1}{2\alpha}. \tag{4.5}$$

Let

$$v(x,t) = \xi(x,t) - \varepsilon u(x,t)\tilde{z}(\theta_t\omega) = u_t(x,t) + \sigma u(x,t) - \varepsilon u(x,t)\tilde{z}(\theta_t\omega), \tag{4.6}$$

then (4.3) can be rewritten as the following equivalent random system with random coefficients but without white noise

$$\begin{cases} \frac{du}{dt} = v - \sigma u + \varepsilon u \tilde{z}(\theta_t\omega), \\ \frac{dv}{dt} = (\sigma(\alpha-\sigma) - \lambda - A)u + (\alpha - \sigma)v - \varepsilon(v - 2\sigma u + \varepsilon u\tilde{z}(\theta_t\omega))\tilde{z}(\theta_t\omega) \\ \qquad\qquad +g(x) - f(u), \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x) = u_1(x) + \sigma u_0(x) - \varepsilon u_0(x)\tilde{z}(\omega),\ x \in \mathbb{R}^n, \end{cases} \tag{4.7}$$

where $A = -\Delta$. In the following, we still write $\widetilde{\Omega}_1$ as Ω.

Let $E = H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ be the phase space endowed with the usual norm $\|Y\|_{H^1\times L^2} = (\|\nabla u\|^2 + \|u\|^2 + \|v\|^2)^{\frac{1}{2}}$ for $Y = (u,v)^\top \in E$, where $\|\cdot\|$ denotes the usual norm in $L^2(\mathbb{R}^n)$. It is known from [30] that the following statements hold.

Theorem 4.1 (See [30]). (1) *Under condition* (4.2) *and* $g \in L^2(\mathbb{R}^n)$, *for any* $\psi_0 = (u_0, v_0)^\top \in E$, (4.7) *has a unique solution* $\psi_\varepsilon(t,\omega,\psi_0) = (u(t,\omega,u_0), v(t,\omega,v_0))^\top$ *with* $\psi_\varepsilon(0,\omega,\psi_0) = \psi_0$ *and* $\psi_\varepsilon(t,\omega,\psi_0)$ *is continuous with respect to* ψ_0*in* E *for all* $t > 0$, *furthermore, the solution mapping*

$$\psi_\varepsilon(t,\omega) : \psi_0 \mapsto \psi_\varepsilon(t,\omega,\psi_0) = (u(t,\omega,u_0), v(t,\omega,v_0))^\top, \quad E \to E \tag{4.8}$$

generates a continuous RDS to (4.7), *and* $\Psi_\varepsilon(t,\omega) = P(\theta_t\omega)\psi_\varepsilon(t,\omega)P^{-1}(\theta_t\omega)$ *generates a continuous RDS associated with* (4.3), *where* $P(\theta_t\omega) : (u,\xi) \to (u,\ \xi+\varepsilon u\tilde{z}(\theta_t\omega))$, $(u,\xi)^\top \in E$ *is a homeomorphism on* E *and* $P^{-1}(\theta_t\omega) : (u,\xi) \to (u,\ \xi - cu\tilde{z}(\theta_t\omega))$.

(2) *Take a sufficient small number* $\sigma > 0$ *such that* $\lambda + \sigma^2 - \alpha\sigma > 0$, $\alpha - 3\sigma > 0$. *Let*

$$\varepsilon_0 = \frac{4\sigma}{\sqrt{\pi}\alpha\left(\frac{1+\sigma\gamma}{\alpha} + \sqrt{(\frac{1+\sigma\gamma}{\alpha})^2 + \frac{8\sigma\gamma}{\sqrt{\pi}\alpha}}\right)}, \quad \gamma = 1 + \frac{1}{\lambda + \sigma^2 - \alpha\sigma}. \tag{4.9}$$

Define a new norm $\|\cdot\|_E$, *which is equivalent to the usual norm* $\|\cdot\|_{H^1\times L^2}$, *by*

$$\|\psi\|_E = (\|v\|^2 + (\lambda + \sigma^2 - \alpha\sigma)\|u\|^2 + \|\nabla u\|^2)^{\frac{1}{2}}, \quad \psi = (u,v)^\top \in E. \tag{4.10}$$

Then for $|\varepsilon| \le \varepsilon_0$, *there exists a random ball* $\{B_\varepsilon(\omega)\} \in \mathcal{D}(E)$:

$$B_\varepsilon(\omega) = \{\varphi \in E : \|\psi\|_E \leqslant \varrho(\varepsilon,\omega)\} \tag{4.11}$$

centered at 0 *with random radius* $\varrho(\varepsilon,\omega) > 0$, *where*

$$\varrho^2(\varepsilon,\omega) = \frac{4}{\alpha - \sigma}(c_1^2 + \|g\|^2) \int_{-\infty}^0 e^{2\int_s^0(-\sigma+|\varepsilon|\cdot|z(\theta_\tau\omega)|+\gamma_2(\sigma|\varepsilon|\cdot|z(\theta_\tau\omega)|+\frac{1}{2}\varepsilon^2|z(\theta_\tau\omega)|^2))d\tau} ds, \tag{4.12}$$

such that $\{B_\varepsilon(\omega)\}$ *is a closed random absorbing set for* $\psi_\varepsilon(t,\omega)$ *in* $\mathcal{D}(E)$, *that is, for any* $\{D(\omega)\} \in \mathcal{D}(E)$ *and* $\mathbb{P}$-*a.e.* $\omega \in \Omega$, *there is* $T_D(\omega) > 0$ *such that*

$$\psi_\varepsilon(t,\theta_{-t}\omega, B(\theta_{-t}\omega)) \subseteq B_\varepsilon(\omega) \quad \text{for all } t > T_D(\omega). \tag{4.13}$$

(3) *Given* $|\varepsilon| \le \varepsilon_0$. *Let* $\psi_0(\omega) \in B_\varepsilon(\omega)$. *Then, for every* $\eta > 0$, *there exist* $T_3 = T_3(B,\eta,\omega) > 0$ *and* $r_3 = r_3(\eta,\omega) \ge 1$ *(independent of* ε*) such that the solution* ψ *of* (4.7) *satisfies that for* $\mathbb{P}$-*a.e.* $\omega \in \Omega$, $\forall\ t \ge \widetilde{T}$, $r \ge \widetilde{R}$,

$$\|\psi_\varepsilon(t,\theta_{-t}\omega,\psi_0(\theta_{-t}\omega))\|^2_{E(\mathbb{R}^n\setminus Q^r)} \leqslant \eta. \tag{4.14}$$

(4) *Define a smooth decreasing function* $\rho \in C^1(\mathbb{R}_+,[0,1])$ *satisfying*

$$\begin{cases} \rho(s) = 1, & 0 \leqslant s \leqslant 1, \\ 0 < \rho(s) < 1, & 1 < s < 2, \\ \rho(s) = 0, & s \ge 2. \end{cases} \tag{4.15}$$

Fix $r \geq 1$ and set

$$\widehat{\psi} = \rho(\frac{|x|^2}{r^2})\psi = (\widehat{u}, \widehat{v}) = \rho(\frac{|x|^2}{r^2})(u, v). \tag{4.16}$$

It is known that the eigenvalue problem "$A\widehat{u} = \mu\widehat{u}$ in Q^{2r} with $\widehat{u} = 0$ on ∂Q^{2r}" has a family of eigenfunctions $\{e_i\}_{i\in\mathbb{N}}$ with the eigenvalues $\{\lambda_i\}_{i\in\mathbb{N}} : \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_i \leqslant \cdots$, $\lambda_i \to +\infty$ $(i \to +\infty)$, such that $\{e_i\}_{i\in\mathbb{N}}$ is an orthonormal basis of $L^2(Q^{2r})$. Given n, let $X_n = span\{e_1, \cdots, e_n\}$ and $P_n : L^2(Q^{2r}) \to X_n$ be the projection operator. Let $\psi_0(\omega) \in B_\varepsilon(\omega)$. Then, for given $|\varepsilon| \leq \varepsilon_0$ and any $\eta > 0$, there exist $T_4 = T_4(B, \eta, \omega) > 0$, $r_4 = r_4(\eta, \omega) \geq 1$ and $N = N(\eta, \omega) > 0$ (independent of ε) such that the solution ψ of (4.7) satisfies that for $\mathbb{P}$-a.e. $\omega \in \Omega$, $\forall\ t \geq T_4$, $r \geq r_4$ and $n \geq N$,

$$\|(I - P_n)\widehat{\psi}(t, \theta_{-t}\omega, \psi_0(\theta_{-t}\omega))\|^2_{E(Q^{2r})} \leqslant \eta. \tag{4.17}$$

(5) *$\{\psi_\varepsilon(t, \omega)\}_{t\geq 0,\omega\in\Omega}$ has a unique global random attractor $\{A_\varepsilon(\omega)\} \in \mathcal{D}(E)$ in E.*

Theorem 4.2. *If conditions (4.2) and $g \in L^2(\mathbb{R}^n)$ hold, then for $\mathbb{P}$-a.e. $\omega \in \Omega$,*

$$d_H(A_\varepsilon(\omega), A_0) = \sup_{x\in A_\varepsilon(\omega)} \inf_{y\in A_0} \|x - y\|_E \to 0 \quad as \quad \varepsilon \to 0, \tag{4.18}$$

where A_0 is the global attractor for the autonomous dynamical system associated with the limiting deterministic equation

$$\begin{cases} \frac{du}{dt} = v - \sigma u, \\ \frac{dv}{dt} = (\sigma(\alpha - \sigma) - \lambda - A)u + (\alpha - \sigma)v + g(x) - f(u), \\ u(x, 0) = u_0(x),\ v(x, 0) = v_0(x) = u_1(x) + \sigma u_0(x), \quad x \in \mathbb{R}^n. \end{cases} \tag{4.19}$$

Proof. The proof is based on Theorem 2.1, we then check that $\{\psi_\varepsilon(t, \omega)\}_{t\geq 0,\omega\in\Omega}$ satisfies the conditions of Theorem 2.1.

(i) Let (4.2) and $g \in L^2(\mathbb{R}^n)$, then the solutions of limiting deterministic equation (4.19) generates a continuous autonomous dynamical system $\{\psi(t)\}_{t\geq 0}$ and $\{\psi(t)\}_{t\geq 0}$ has a global attractor A_0 in $L^2(\mathbb{R}^n)$ (see [27, 30, 24]).

(ii) Given $|\varepsilon| \leq \varepsilon_0$. By the properties (4.4)-(4.5) of $z(\theta_t\omega)$,

$$\begin{aligned} \varrho^2(\varepsilon, \omega) &\leqslant \frac{4}{\alpha - \sigma}(c_1^2 + \|g\|^2) \int_{-\infty}^0 e^{2\int_s^0 (-\sigma + \varepsilon_0|\tilde{z}(\theta_\tau\omega)| + \gamma_2(\sigma\varepsilon_0|\tilde{z}(\theta_\tau\omega)| + \frac{1}{2}\varepsilon_0^2|\tilde{z}(\theta_\tau\omega)|^2))d\tau} ds \\ &= R_4(\varepsilon_0, \omega). \end{aligned} \tag{4.20}$$

Let

$$B_0(\omega) = \left\{\varphi \in E : \|\varphi\|^2_E \leqslant R_4(\varepsilon_0, \omega)\right\}. \tag{4.21}$$

Then by (4.11), (4.20)-(4.21),

$$\bigcup_{|\varepsilon|\leqslant\varepsilon_0} A_\varepsilon(\omega) \subseteq \bigcup_{|\varepsilon|\leqslant\varepsilon_0} B_\varepsilon(\omega) \subseteq B_0(\omega) \subset E, \quad \mathbb{P}\text{-a.e. } \omega \in \Omega. \tag{4.22}$$

By (4.12),

$$\begin{aligned} \limsup_{\varepsilon\to 0} \varrho^2(\varepsilon,\omega) &= \limsup_{\varepsilon\to 0} \frac{4}{\alpha-\sigma}(c_1^2 + \|g\|^2)\int_{-\infty}^0 e^{2\sigma s} ds \\ &= \frac{2}{\sigma(\alpha-\sigma)}(c_1^2 + \|g\|^2), \end{aligned} \tag{4.23}$$

implying that for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\begin{aligned} \limsup_{\varepsilon\to 0} \|A_\varepsilon(\omega)\| &= \limsup_{\varepsilon\to 0} \sup_{u\in A_\varepsilon(\omega)} \|u\| \\ &\leqslant \left(\frac{2}{\sigma(\alpha-\sigma)}(c_1^2 + \|g\|^2)\right)^{\frac{1}{2}}. \end{aligned} \tag{4.24}$$

(iii) Let $\{\psi_n(\omega)\}_{n=1}^\infty \subset \bigcup_{|\varepsilon|\leqslant\varepsilon_0} A_\varepsilon(\omega)$ be a given sequence of $\bigcup_{|\varepsilon|\leqslant\varepsilon_0} A_\varepsilon(\omega)$ in E. Then $\{\psi_n(\omega)\}_{n=1}^\infty \subset B_0(\omega)$ is bounded and there exist $|\varepsilon_n| \leqslant \varepsilon_0$, such that $\psi_n(\omega) \in A_{\varepsilon_n}(\omega) = \psi_{\varepsilon_n}(t, \theta_{-t}\omega, A_{\varepsilon_n}(\theta_{-t}\omega))$, $n = 1, 2, \ldots$, thus there exist $\{t_n\}_{n=1}^\infty \subset \mathbb{R}^+$ and $\psi_{n,0}(\theta_{-t_n}\omega) \in A_{\varepsilon_n}(\theta_{-t_n}\omega) \subset B_0(\omega)$, $n = 1, 2, \ldots$, such that $t_n \to \infty$ and $\psi_n(\omega) = \psi_{\varepsilon_n}(t_n, \theta_{-t_n}\omega, \psi_{n,0}(\theta_{-t_n}\omega))$, and

$$\begin{aligned} \|\psi_n(\omega)\|_E &= \|\psi_{\varepsilon_n}(t_n, \theta_{-t_n}\omega, \psi_{n,0}(\theta_{-t_n}\omega))\|_E \\ &\leqslant R_4(\varepsilon_0, \omega), \quad n = 1, 2, \ldots. \end{aligned} \tag{4.25}$$

By the proof of Lemma 4.2 and Lemma 4.3 of [30], for $\mathbb{P}$-a.e. $\omega \in \Omega$ and any $\eta > 0$, there exist $N_1 = N_1(\eta,\omega) > 0$, $r_5 = r_5(\eta,\omega) > 0$ and $M_1 = M_1(\eta,\omega) > 0$ (independent of ε) such that for every $n \geq M_1$,

$$\|\psi_{\varepsilon_n}(t_n, \theta_{-t_n}\omega, \psi_{n,0}(\theta_{-t_n}\omega))\|^2_{E(\mathbb{R}^n\setminus Q^{r_5})} \leqslant \eta. \tag{4.26}$$

and

$$\|(I - P_N)\widehat{\psi_{\varepsilon_n}}(t_n, \theta_{-t_n}\omega, \psi_{n,0}(\theta_{-t_n}\omega))\|^2_{E(Q^{2r_5})} \leqslant \eta. \tag{4.27}$$

It follows from (4.16) and (4.26) that $\{P_N\widehat{\psi_{\varepsilon_n}}(t_n, \theta_{-t_n}\omega, \psi_{n,0}(\theta_{-t_n}\omega))\}$ is a bounded in $P_N E(Q^{2r_5})$, and $\{\widehat{\psi_{\varepsilon_n}}(t_n, \theta_{-t_n}\omega, \psi_{n,0}(\theta_{-t_n}\omega))\}$ is precompact in $H^1(Q^{2r_5}) \times L^2(Q^{2r_5})$. Note that $\rho(\frac{|x|^2}{r_5^2}) = 1$ for $|x| \leqslant r_5$, this implies that $\{\psi_{\varepsilon_n}(t_n, \theta_{-t_n}\omega, \psi_{n,0}(\theta_{-t_n}\omega))\}$ is precompact in $E(Q^{r_5})$, along with (4.14) shows that the precompactness of $\{\varphi_n(\omega)\}_{n=1}^\infty$ in E. Therefore $\bigcup_{|\varepsilon|\leqslant\varepsilon_0} A_\varepsilon(\omega)$ is precompact in E.

(iv) Let ψ_ε and ψ be the solutions of (4.7) and (4.19) with initial data $\psi_{\varepsilon,0}$ and ψ_0, respectively. Set $y=\psi_\varepsilon-\psi=(\zeta,\varsigma)=(u_\varepsilon-u,v_\varepsilon-v)$, then

$$\begin{cases} \frac{d\zeta}{dt}=\varsigma-\sigma\zeta+\varepsilon u_\varepsilon\tilde{z}(\theta_t\omega),\\ \frac{d\varsigma}{dt}=(\sigma(\alpha-\sigma)-\lambda-A)\zeta+(\alpha-\sigma)\varsigma-\varepsilon(v_\varepsilon-2\sigma u_\varepsilon+\varepsilon u_\varepsilon\tilde{z}(\theta_t\omega))\tilde{z}(\theta_t\omega)\\ \qquad\qquad -f(u_\varepsilon)+f(u),\\ \zeta(x,0)=v_{\varepsilon,0}(x)-u_0(x),\\ \varsigma(x,0)=u_{\varepsilon,1}(x)+\sigma u_{\varepsilon,0}(x)-\varepsilon u_{\varepsilon,0}(x)\tilde{z}(\omega)-u_1(x)-\sigma u_0(x),\quad x\in\mathbb{R}^n. \end{cases} \tag{4.28}$$

Taking the inner product of the second equation of (4.28) with ς in $L^2(\mathbb{R}^n)$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\varsigma\|^2&=(\sigma-\alpha)\|\varsigma\|^2-(\lambda+\sigma^2-\alpha\sigma)(\zeta,\varsigma)-(A\zeta,\varsigma)\\ &\quad-(\varepsilon(v_\varepsilon-2\sigma u_\varepsilon+\varepsilon u_\varepsilon\tilde{z}(\theta_t\omega))\tilde{z}(\theta_t\omega),\varsigma)-(f(u_\varepsilon)-f(u),\varsigma).\end{aligned} \tag{4.29}$$

For the terms in the right side of (4.29), we find that

$$(\zeta,\varsigma)=\left(\zeta,\frac{d\zeta}{dt}+\sigma\zeta-\varepsilon u_\varepsilon\tilde{z}(\theta_t\omega)\right)\geq\frac{1}{2}\frac{d}{dt}\|\zeta\|^2+\left(\sigma-\frac{1}{2}\right)\|\zeta\|^2-\frac{1}{2}|\varepsilon|^2|\tilde{z}(\theta_t\omega)|^2\|u_\varepsilon\|^2,$$

$$\begin{aligned}&-(\varepsilon(v_\varepsilon-2\sigma u_\varepsilon+\varepsilon u_\varepsilon\tilde{z}(\theta_t\omega))\tilde{z}(\theta_t\omega),\varsigma)\\ &\quad\leq|\varepsilon|\cdot|\tilde{z}(\theta_t\omega)|\cdot\|v_\varepsilon-u_\varepsilon(2\sigma-\varepsilon\tilde{z}(\theta_t\omega))\|\cdot\|\varsigma\|\\ &\quad\leqslant|\varepsilon|^2\cdot|\tilde{z}(\theta_t\omega)|^2[\|v_\varepsilon\|^2+\|u_\varepsilon\|^2(2\sigma-\varepsilon\tilde{z}(\theta_t\omega))^2]+\frac{1}{2}\|\varsigma\|^2,\end{aligned} \tag{4.30}$$

$$-(f(u_\varepsilon)-f(u),\varsigma)\leqslant c_0\|\zeta\|\cdot\|\varsigma\|\leqslant\frac{c_0}{2}(\|\zeta\|^2+\|\varsigma\|^2). \tag{4.31}$$

By (4.29)–(4.31), it follows that

$$\begin{aligned}&\frac{d}{dt}(\|\varsigma\|^2+(\lambda+\sigma^2-\alpha\sigma)\|\zeta\|^2+\|\nabla\zeta\|^2)\\ &\quad\leqslant c_0\|\varsigma\|^2+2\left((\lambda+\sigma^2-\alpha\sigma)(\frac{1}{2}-\sigma)+\frac{1}{2}+\frac{c_0}{2}\right)\|\zeta\|^2+(1-2\sigma)\|\nabla\zeta\|^2\\ &\qquad+2|\varepsilon|^2\cdot|\tilde{z}(\theta_t\omega)|^2\left[\|v_\varepsilon\|^2+\|u_\varepsilon\|^2\left((2\sigma-\varepsilon\tilde{z}(\theta_t\omega))^2+1\right)\right].\end{aligned} \tag{4.32}$$

By the inequality (4.13) in [30], for $|\varepsilon|\leqslant\varepsilon_0$,

$$\begin{aligned}&\|v_\varepsilon\|^2+(\lambda+\sigma^2-\alpha\sigma)\|u_\varepsilon\|^2\\ &\quad\leqslant e^{2\int_0^t(-\sigma+|\varepsilon|\cdot|\tilde{z}(\theta_\tau\omega)|+\gamma_2(\sigma|\varepsilon|\cdot|\tilde{z}(\theta_\tau\omega)|+\frac{1}{2}\varepsilon^2|\tilde{z}(\theta_\tau\omega)|^2))d\tau}\|\psi_0(\omega)\|_E^2\\ &\qquad+\frac{2}{\alpha-\sigma}(c_1^2+\|g\|^2)\int_0^t e^{2\int_s^t(-\sigma+|\varepsilon|\cdot|\tilde{z}(\theta_\tau\omega)|+\gamma_2(\sigma|\varepsilon|\cdot|\tilde{z}(\theta_\tau\omega)|+\frac{1}{2}\varepsilon^2|\tilde{z}(\theta_\tau\omega)|^2))d\tau}ds\\ &\quad\leqslant P_3(t,\omega),\qquad\text{(independent of }\varepsilon).\end{aligned} \tag{4.33}$$

Thus,

$$|\tilde{z}(\theta_t\omega)|^2\left[\|v_\varepsilon\|^2+\|u_\varepsilon\|^2\left((2\sigma-\varepsilon\tilde{z}(\theta_t\omega))^2+1\right)\right]\leqslant P_4(t,\omega), \tag{4.34}$$

where $P_4(t,\omega)$ is continuous in t but independent of ε. By (4.32) and (4.34),

$$\frac{d}{dt}\|y\|_E^2 \leqslant C_1\|y\|_E^2 + 2|\varepsilon|^2 \cdot P_4(t,\omega), \quad t \geq 0, \tag{4.35}$$

where C_1 depends on c_0, λ, σ, α but independent of ε. By the Gronwall inequality,

$$\begin{aligned} &\|\psi_\varepsilon(t,\omega,\psi_{\varepsilon,0}) - \psi(t,\psi_0)\|_E^2 \\ &= \|y(t,\omega,w(0))\|_E^2 \\ &\leqslant e^{C_1 t}\|\psi_{\varepsilon,0} - \psi_0\|_E^2 + 2|\varepsilon|^2 \int_0^t e^{C_1(t-s)} P_4(s,\omega)ds. \end{aligned} \tag{4.36}$$

From (4.36), we see that for $\mathbb{P}$-a.e. $\omega \in \Omega$, $t \geq 0$, $\varepsilon_n \to 0$, and $\psi_{\varepsilon_n,0}, \psi_0 \in E$ with $\psi_{\varepsilon_n,0} \to \psi_0$, it holds:

$$\lim_{n\to\infty} \psi_{\varepsilon_n}(t,\omega,\psi_{\varepsilon_n,0}) = \psi(t,\psi_0) \text{ and } \lim_{n\to\infty} \Psi_{\varepsilon_n}(t,\omega,\Psi_{\varepsilon_n,0}) = \psi(t)\psi_0. \tag{4.37}$$

By Theorem 2.1, the proof is completed.

□

5. CONCLUSION

In this paper, we establish the upper semicontinuity of random attractors for the stochastically perturbed reaction-diffusion equation and damped wave equation with multiplicative noise defined in the entire space $\mathbb{R}^n$ as the coefficient of the white noise term tends to zero. The method here can also be applied to other types of stochastic evolution equations.

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