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NEW OSCILLATION CRITERIA FOR SECOND ORDER SUBLINEAR DYNAMIC EQUATIONS

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ABSTRACT. The purpose of this paper is to establish and improve the main results of a number of recent papers for a more general sublinear dynamic equation

$$\left(r(t)x^{\Delta}(t)\right)^{\Delta}+p(t)f\left(x^{\sigma}\left(t\right)\right)=0,$$

Our results are established for a time scale $\mathbb{T}$ without assuming certain restrictive conditions on $\mathbb{T}$, and where, in addition, r and p are real-valued, rd-continuous functions on $\mathbb{T}$ with no explicit sign assumptions. The function $f\in C\left(\mathbb{R},\mathbb{R}\right)$ is assumed to satisfy $xf\left(x\right)>0$ and $f'(x)>0$, for $x\neq 0$. Some examples are given to illustrate the main results.

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1. PRELIMINARIES

We are concerned with the oscillatory behavior of the following second order sublinear dynamic equation

$$\left(r(t)x^{\Delta}(t)\right)^{\Delta}+p(t)\ f\left(x^{\sigma}\left(t\right)\right)=0, \tag{1.1}$$

on a time scale $\mathbb{T}$ which is unbounded above, where r and p are real-valued, right-dense continuous functions on $\mathbb{T}$. The function $f\in C\left(\mathbb{R},\mathbb{R}\right)$ is assumed to satisfy $xf\left(x\right)>0$ and $f'(x)>0$, for $x\neq 0$. Here we are interested in the oscillation of solutions of (1.1) when $f(x)$ satisfies, in addition, the sublinearity condition

$$0<\int_0^{\epsilon}\frac{dx}{f(x)};\quad \int_{-\epsilon}^{0}\frac{dx}{f(x)}<\infty,\quad \text{for all}\quad \epsilon>0. \tag{1.2}$$

By a solution of (1.1) we mean a nontrivial real-valued function $x\in C_{rd}^{1}[T_x,\infty)$, $T_x\geq t_0$ which has the property that $rx^{\Delta}\in C_{rd}^{1}[T_x,\infty)$ and satisfies equation (1.1) on $[T_x,\infty)$, where C_{rd} is the space of rd-continuous functions. We shall not consider solutions which vanish identically in some neighborhood of infinity. A solution of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise it is said to be nonoscillatory. There has been a great deal of research

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into obtaining criteria for oscillation of all solutions of dynamic equations on time scales. In most papers dealing with oscillation the assumption is usually made that the functions r, p are nonnegative. We refer the reader to the papers [1, 2, 11, 12], and the references cited therein. On the other hand, much less is known for equations when no explicit sign assumptions are made with respect to the coefficient functions r and p. This will be one of our main concerns in this paper. We shall also relax a certain restrictive condition on the time scale, thereby extending the applicability of our results.

Erbe, Baoguo and Peterson established in [9, 4] some oscillation criteria of Belohorec-type and Kamenev- type for the special case of (1.1) where $r(t) = 1$ and $f(x^{\sigma}(t)) = x^{\gamma}(\sigma(t))$, where $0 < \gamma < 1$ is a quotient of odd positive integers. Hassan, Erbe and Peterson [15] improved these results and generalized them to the sublinear dynamic equation (1.1) with damping term, and $f(x^{\sigma}(t)) = |x^{\sigma}(t)|^{\gamma} \operatorname{sgn} x^{\sigma}(t)$, with $0 < \gamma < 1$ and established Belohorec-type oscillation theorem, where the coefficient functions r and p are allowed to change sign for large t. For superlinear dynamic equation, see [3, 6, 10, 13, 14]

However, the results in [3, 4, 9, 14, 15] apply only to time scales satisfying the so-called Condition (C): There exists an $M > 0$ such that $\chi(t) \leq M\mu(t)$, $t \in \mathbb{T}$, where χ is the characteristic function of the set $\hat{\mathbb{T}} = \{t \in \mathbb{T} : \mu(t) > 0\}$. We note that if $\mathbb{T}$ satisfies condition (C), then the graininess function is bounded away from 0, uniformly, wherever it is positive. Moreover, it is easy to see that the subset $\check{\mathbb{T}}$ of $\mathbb{T}$ defined by

$$\check{\mathbb{T}} = \{t \in \mathbb{T} : t > 0 \text{ is right-scattered or left-scattered}\},$$

is necessarily countable and $\hat{\mathbb{T}} \subset \check{\mathbb{T}}$. Then, we can rewrite $\check{\mathbb{T}}$ as

$$\check{\mathbb{T}} = \{t_i \in \mathbb{T} : 0 < t_1 < t_2 < \cdots < t_n < \cdots\},$$

and so

$$\mathbb{T} = \check{\mathbb{T}} \cup [\cup_{n \in A}(t_{n-1}, t_n)],$$

where A is the set of all integers for which the real open interval (t_{n-1}, t_n) is contained in $\mathbb{T}$. Although most of the standard time scales do satisfy condition (C), which is an assumption on the graininess function, there are many time scales which do not. For example, it is easy to see that the time scale given by

$$\mathbb{T} := \bigcup_{k=1}^{\infty} T_k, \text{ where } T_k = \bigcup_{n=1}^{\infty} \left\{k + \frac{n+1}{n}\right\},$$

does not satisfy condition (C).

The purpose of this paper is to obtain oscillation criteria for the general nonlinear dynamic equation (1.1). We note that in our results, the equation involves a more general function f and the coefficient functions r and p may change sign. Our work applies to general time scales without assuming condition (C). Several examples are given to illustrate the main results.

Our work improves and generalizes those established in [9, 4, 15], and many known results on nonlinear oscillation. These results have significant importance for the study of oscillation criteria on discrete time scales such as $\mathbb{T}=\mathbb{Z}$, $\mathbb{T}=h\mathbb{Z}$, $h>0$, $\mathbb{T}=\{t: t=q^k,\ k\in\mathbb{N}_0,\ q>1\}$, and the harmonic time scale $\mathbb{T}=H_n$.

2. MAIN RESULTS

Before stating our main results, we begin with a couple preliminary lemmas which will play an important role in the proof of our main results.

Lemma 2.1 ([8, Theorem 5.45]). *Let h be a bounded function that is integrable on $[a,b]_{\mathbb{T}}$. Let m_H and M_H be the infimum and supremum of the function $H(t) := \int_a^t h(s)\,\Delta s$ on $[a,b]_{\mathbb{T}}$ respectively. Suppose that g is a nonnegative and nonincreasing function on $[a,b]_{\mathbb{T}}$. Then there is some number Λ with $m_H \le \Lambda \le M_H$ such that*

$$\int_a^b h(t)\,g(t)\,\Delta t = g(a)\,\Lambda.$$

Lemma 2.2. *Assume that x is a positive rd-continuous function on $[t_0,\infty)_{\mathbb{T}}$ and $f:(0,\infty)\to(0,\infty)$ is continuous and increasing. Then*

$$\int_{t_0}^{t} \frac{x^{\Delta}(s)}{f(x^{\sigma}(s))}\Delta s \le \int_{x(t_0)}^{x(t)} \frac{d\tau}{f(\tau)}, \tag{2.1}$$

and

$$\int_{t_0}^{t} \frac{x^{\Delta}(s)}{f(x(s))}\Delta s \ge \int_{x(t_0)}^{x(t)} \frac{d\tau}{f(\tau)}. \tag{2.2}$$

Proof. Let

$$F(x(s)) := \int_{x(t_0)}^{x(s)} \frac{d\tau}{f(\tau)} \qquad \text{for } s\in[t_0,\infty)_{\mathbb{T}}.$$

Then, by the Pötzsche chain rule ([7, Theorem 1.90]), we have

$$(F(x(s)))^{\Delta} = \int_0^1 F'(x_h(s))\,dh\; x^{\Delta}(s) = \int_0^1 \frac{1}{f(x_h(s))} dh\; x^{\Delta}(s).$$

Now for any fixed point $s\in[t_0,\infty)_{\mathbb{T}}$, we have

$$x_h(s) = (1-h)\,x(s) + h x^{\sigma}(s) \begin{cases} \ge x^{\sigma}(s), & \text{if } x^{\Delta}(s)\le 0; \\ \le x^{\sigma}(s), & \text{if } x^{\Delta}(s)\ge 0, \end{cases}$$

and therefore this yields

$$\frac{x^{\Delta}(s)}{f(x_h(s))} \ge \frac{x^{\Delta}(s)}{f(x^{\sigma}(s))} \quad \text{for } s\in[t_0,\infty)_{\mathbb{T}}.$$

Therefore we have

$$\left(F\left(x\left(s\right)\right)\right)^{\Delta}=\int_{0}^{1}\frac{1}{f\left(x_{h}\left(s\right)\right)}dh\ x^{\Delta}(s)\geq\int_{0}^{1}\frac{1}{f\left(x^{\sigma}\left(s\right)\right)}dh\ x^{\Delta}(s)=\frac{x^{\Delta}(s)}{f\left(x^{\sigma}\left(s\right)\right)}.$$

and so

$$\int_{t_0}^{t}\frac{x^{\Delta}(s)}{f\left(x^{\sigma}(s)\right)}\Delta s\leq\int_{t_0}^{t}\left(F\left(x\left(s\right)\right)\right)^{\Delta}\Delta s=F\left(x\left(t\right)\right)=\int_{x(t_0)}^{x(t)}\frac{d\tau}{f\left(\tau\right)}.$$

Also for any fixed point $s\in[t_0,\infty)_{\mathbb{T}}$, we have

$$x_{h}\left(s\right)=\left(1-h\right)x\left(s\right)+hx^{\sigma}\left(s\right)\begin{cases}\leq x\left(s\right), & \text{if } x^{\Delta}\left(s\right)\leq 0;\\ \geq x\left(s\right), & \text{if } x^{\Delta}\left(s\right)\geq 0,\end{cases}$$

which implies

$$\frac{x^{\Delta}(s)}{f\left(x_{h}\left(s\right)\right)}\leq\frac{x^{\Delta}(s)}{f\left(x\left(s\right)\right)}\quad\text{for } s\in[t_0,\infty)_{\mathbb{T}}.$$

Then

$$\left(F\left(x\left(s\right)\right)\right)^{\Delta}=\int_{0}^{1}\frac{1}{f\left(x_{h}\left(s\right)\right)}dh\ x^{\Delta}(s)\leq\int_{0}^{1}\frac{1}{f\left(x\left(s\right)\right)}dh\ x^{\Delta}(s)=\frac{x^{\Delta}(s)}{f\left(x\left(s\right)\right)}.$$

and then

$$\int_{t_0}^{t}\frac{x^{\Delta}(s)}{f\left(x(s)\right)}\Delta s\geq\int_{t_0}^{t}\left(F\left(x\left(s\right)\right)\right)^{\Delta}\Delta s=F\left(x\left(t\right)\right)=\int_{x(t_0)}^{x(t)}\frac{d\tau}{f\left(\tau\right)}.$$

This completes the proof. □

2.1. **Kamenev Type.** In this subsection, we establish oscillation criterion of Kamenev-type for Eq. (1.1) where $r\left(t\right)>0$ on $[t_0,\infty)_{\mathbb{T}}$. As in Philos [18], we consider a nonnegative kernel function $h\left(t,s\right)$ defined on $\mathbb{D}:=\{(t,s)\in\mathbb{T}^2:t\geq s\geq t_0\}$. We shall assume that $h\left(t,s\right)$ satisfies the following conditions:

(H_1) $h\left(t,t\right)\equiv 0$ for $t\geq t_0$,
(H_2) $h^{\Delta_s}\left(t,s\right)\leq 0$ for $t\geq s\geq t_0$,
where $h^{\Delta_s}\left(t,s\right)$ denotes the partial delta derivative of h with respect to s.
(H_3) $\left(r\left(s\right)h^{\Delta_s}\left(t,s\right)\right)^{\Delta_s}\geq 0$ for $t\geq s\geq t_0$,
where $h^{\Delta_s^2}\left(t,s\right)$ denotes the second order partial delta derivative of h with respect to s.
(H_4) $-h^{-1}\left(t,t_0\right)\ h^{\Delta_s}\left(t,s\right)\big|_{s=t_0}\leq M$ for large t.

Theorem 2.3. *Assume that f satisfies* (1.2). *If there exists a nonnegative kernel function $h\left(t,s\right)$ on $\mathbb{D}$ satisfying (H_1)–(H_4) such that*

$$\limsup_{t\to\infty}\frac{1}{h\left(t,t_0\right)}\int_{t_0}^{t}h\left(t,\sigma\left(s\right)\right)\ p\left(s\right)\ \Delta s=\infty, \tag{2.3}$$

then every solution of equation (1.1) *is oscillatory.*

Proof. Assume (1.1) has a nonoscillatory solution x on $[t_0,\infty)_{\mathbb{T}}$. Then, without loss of generality, there is a solution x of (1.1) and a $T\in[t_0,\infty)_{\mathbb{T}}$ such that $x(t)>0$ on $[T,\infty)_{\mathbb{T}}$. From the quotient rule and Pötzsche chain rule ([7, Theorem 1.90]) and then by Eq. (1.1), we get

$$\begin{aligned}\left[\frac{r(t)\,x^{\Delta}(t)}{f(x(t))}\right]^{\Delta} &= \frac{\left(r(t)\,x^{\Delta}(t)\right)^{\Delta}}{f(x^{\sigma}(t))} - \frac{r(t)\,x^{\Delta}(t)}{f(x(t))\,f(x^{\sigma}(t))}\int_0^1 f'(x_h(t))\,dh\;x^{\Delta}(t)\\ &= -p(t) - \frac{r(t)\left(x^{\Delta}(t)\right)^2}{f(x(t))\,f(x^{\sigma}(t))}\int_0^1 f'(x_h(t))\,dh\\ &\le -p(t),\end{aligned}\tag{2.4}$$

since $r(t)>0$ and $x_h(t):=(1-h)x(t)+hx^{\sigma}(t)>0$, for $0\le h\le1,\ t\in[T,\infty)_{\mathbb{T}}$. In (2.4), replace t by s and multiply both sides by $h(t,\sigma(s))$ and integrate with respect to s from T to $t,\ t\ge T$

$$\int_T^t h(t,\sigma(s))\left[\frac{r(s)\,x^{\Delta}(s)}{f(x(s))}\right]^{\Delta}\ \Delta s \le -\int_T^t h(t,\sigma(s))\,p(s)\ \Delta s.\tag{2.5}$$

Now using the integration by parts formula and the second mean value theorem (Lemma 2.1) combined with (H_1) – (H_3), we get

$$\begin{aligned}&\int_T^t h(t,\sigma(s))\left[\frac{r(s)\,x^{\Delta}(s)}{f(x(s))}\right]^{\Delta}\ \Delta s\\ &\quad= -h(t,T)\,\frac{r(T)\,x^{\Delta}(T)}{f(x(T))} - \int_T^t r(s)\,h^{\Delta_s}(t,s)\,\frac{x^{\Delta}(s)}{f(x(s))}\ \Delta s\\ &\quad= -N\ h(t,T) - r(T)\ h^{\Delta_s}(t,s)\big|_{s=T}\,\Lambda,\end{aligned}\tag{2.6}$$

where $N:=\frac{r(T)x^{\Delta}(T)}{f(x(T))}$ and where $m_x\le\Lambda\le M_x$, and where m_x and M_x denote the infimum and supremum, respectively, of the function $\int_T^t\frac{x^{\Delta}(s)}{f(x(s))}\ \Delta s$. By (2.2) in Lemma 2.2, we have

$$\int_T^t\frac{x^{\Delta}(s)}{f(x(s))}\ \Delta s \ge \int_{x(T)}^{x(t)}\frac{d\tau}{f(\tau)} = \int_0^{x(t)}\frac{d\tau}{f(\tau)} - \int_0^{x(T)}\frac{d\tau}{f(\tau)} \ge -\int_0^{x(T)}\frac{d\tau}{f(\tau)}.$$

So

$$\Lambda \ge m_x \ge -\int_0^{x(T)}\frac{d\tau}{f(\tau)}.\tag{2.7}$$

From (2.5), (2.6) and (2.7), we find

$$-N\ h(t,T) + r(T)\ h^{\Delta_s}(t,s)\big|_{s=T}\int_0^{x(T)}\frac{d\tau}{f(\tau)} \le -\int_T^t h(t,\sigma(s))\,p(s)\ \Delta s.$$

Dividing by $h(t,T)$ and using (H_4), we get

$$-N - M\ r(T)\int_0^{x(T)}\frac{d\tau}{f(\tau)} \le -\frac{1}{h(t,T)}\int_T^t h(t,\sigma(s))\,p(s)\ \Delta s.$$

If we now take the lim inf as $t \to \infty$ of both sides, we get, from (1.2) and (2.3) the desired contradiction. This completes the proof. $\square$

Remark 2.4. In the case $r(t) = 1$ and $f(x^{\sigma}(t)) = x^{\gamma}(\sigma(t))$, where $0 < \gamma < 1$ is a quotient of odd positive integers, the additional assumption was imposed in [4] that $\mathbb{T}$, $\mathbb{T}$ satisfies condition (C). Therefore, Theorem 2.3 improves the main results in [4] by removing this assumption.

Let us illustrate the previous result for the case when the coefficient changes sign:

Example 2.5. Consider the second order q-difference equation

$$\left(\frac{1}{q^{na}} x^{\Delta}(q^n)\right)^{\Delta} + \frac{b(-1)^{n+1}}{q^{n(c+1)}} f\left(x\left(q^{n+1}\right)\right) = 0, \tag{2.8}$$

for $n \in \mathbb{N}_0$, where $a \geq 0$, $b > 0$ and $c < 0$. We choose $h(t,s) = t - s = q^m - q^n$ for $m \geq n$ and $m, n \in \mathbb{N}$. It is clear that conditions (H$_1$)–(H$_4$) hold. Since

$$\begin{aligned}
\frac{1}{h(t,t_0)} \int_{t_0}^{t} h(t,\sigma(s))\ p(s)\ \Delta s &= \frac{1}{t-1} \int_{1}^{t} (t - \sigma(s))\ p(s)\ \Delta s \\
&> \int_{1}^{t} p(s)\ \Delta s - \frac{1}{t} \int_{1}^{t} \sigma(s)\ p(s)\ \Delta s \\
&= \sum_{n=0}^{m-1} \frac{b(-1)^{n+1}}{q^{n(c+1)}} (q-1) q^n - \frac{1}{q^m} \sum_{n=0}^{m-1} q q^n \frac{b(-1)^{n+1}}{q^{n(c+1)}} (q-1) q^n \\
&= b(q-1) \left[\sum_{n=0}^{m-1} (-1)^{n+1} q^{-nc} - q^{1-m} \sum_{n=0}^{m-1} (-1)^{n+1} q^{n(1-c)}\right].
\end{aligned}$$

By choosing $m = 2k+1$, we have

$$\begin{aligned}
&\sum_{n=0}^{m-1} (-1)^{n+1} q^{-nc} - q^{1-m} \sum_{n=0}^{m-1} (-1)^{n+1} q^{n(1-c)} \\
&\quad = -\frac{1 + q^{-c(2k+1)}}{1 + q^{-c}} + q^{-2k} \frac{1 + q^{(1-c)(2k+1)}}{1 + q^{1-c}} \\
&\quad = \frac{\left(q^{-2k} + q q^{-c(2k+1)}\right)(1 + q^{-c}) - \left(1 + q^{-c(2k+1)}\right)(1 + q q^{-c})}{(1 + q^{-c})(1 + q^{1-c})} \\
&\quad = \frac{q^{-2k}(1 + q^{-c}) + q^{-c(2k+1)}(q-1) - 1 - q q^{-c}}{(1 + q^{-c})(1 + q^{1-c})} \to \infty \text{ as } k \to \infty.
\end{aligned}$$

Therefore

$$\limsup_{t \to \infty} \frac{1}{h(t,t_0)} \int_{t_0}^{t} h(t,\sigma(s))\ p(s)\ \Delta s = \infty.$$

Then, by Theorem 2.3, every solution of (2.8) is oscillatory.

2.2. **Kiguradze Type.** In the next theorem, we establish oscillation criteria of Kiguradze-type for the sublinear equation (1.1) *and for both cases*

$$\int_{t_0}^{\infty} \frac{\Delta t}{\phi(t)\, r(t)} = \infty, \tag{2.9}$$

and

$$\int_{t_0}^{\infty} \frac{\Delta t}{\phi(t)\, r(t)} < \infty. \tag{2.10}$$

Theorem 2.6. *Assume that f satisfies* (1.2). *If there exists a C_{rd}^1 function ϕ such that* (2.9) *holds and*

$$\phi(t)\, r(t) > 0, \quad \phi^{\Delta}(t)\, r(t) \geq 0, \quad \left(\phi^{\Delta}(t)\, r(t)\right)^{\Delta} \leq 0, \quad \textit{for} \quad t \in [t_0, \infty)_{\mathbb{T}}.$$

In addition, assume

$$\int_{t_0}^{\infty} \phi^{\sigma}(t)\, p(t)\, \Delta t = \infty, \tag{2.11}$$

hen every bounded solution of equation (1.1) *is oscillatory.*

Proof. Assume (1.1) has a bounded nonoscillatory solution x on $[t_0, \infty)_{\mathbb{T}}$. Then, without loss of generality, there is a solution x of (1.1) and a $T \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t) > 0$ on $[T, \infty)_{\mathbb{T}}$. There are two cases to consider: either $x^{\Delta}(t)$ is eventually negative or $x^{\Delta}(t)$ is not eventually negative.

Case (i). $x^{\Delta}(t)$ is eventually negative. Then there exists $T_1 \geq T$ such that $x^{\Delta}(t) < 0$ for $t \geq T_1$. From (2.11)

$$\int_{T_2}^{t} \phi^{\sigma}(s)\, p(s)\, \Delta s \geq 0 \quad \text{for all } t \geq T_2. \tag{2.12}$$

Multiplying both sides of (1.1) by $\phi^{\sigma}(t)$ and integrating from T_2 to t for $t \in \mathbb{T}$, we obtain that

$$\int_{T_2}^{t} \phi^{\sigma}(s) \left(r(s) x^{\Delta}(s)\right)^{\Delta} \Delta s + \int_{T_2}^{t} \phi^{\sigma}(s)\, p(s)\, f(x^{\sigma}(s))\, \Delta s = 0. \tag{2.13}$$

Using integration by parts, we have that for $t \geq T_2$

$$\begin{aligned} &\int_{T_2}^{t} \phi^{\sigma}(s) \left(r(s) x^{\Delta}(s)\right)^{\Delta} \Delta s \\ &= \phi(t)\, r(t) x^{\Delta}(t) - \phi(T_2)\, r(T_2) x^{\Delta}(T_2) - \int_{T_2}^{t} \phi^{\Delta}(s)\, r(s) x^{\Delta}(s) \Delta s \\ &\geq \phi(t)\, r(t) x^{\Delta}(t) - \phi(T_2)\, r(T_2) x^{\Delta}(T_2). \end{aligned} \tag{2.14}$$

Again by integration by parts and using the Pötzsche chain rule, from (2.12), we get that for $t \in [T_2, \infty)_{\mathbb{T}}$

$$\int_{T_2}^{t} \phi^{\sigma}(s)\, p(s)\, f(x^{\sigma}(s))\, \Delta s = f(x(t)) \int_{T_2}^{t} \phi^{\sigma}(\tau)\, p(\tau)\, \Delta\tau$$

$$-\int_{T_2}^{t}\left[\int_0^1 f'(x_h(s))\,dh\right]\ x^{\Delta}(s)\int_{T_2}^{s}\phi^{\sigma}(\tau)\,p(\tau)\,\Delta\tau\ \Delta s$$

$$\geq f(x(t))\int_{T_2}^{t}\phi^{\sigma}(\tau)\,p(\tau)\,\Delta\tau\geq 0. \tag{2.15}$$

Applying (2.14) and (2.15) in (2.13), we obtain that

$$\phi(t)\,r(t)x^{\Delta}(t)\leq\phi(T_2)\,r(T_2)x^{\Delta}(T_2),$$

and hence

$$x(t)-x(T_2)\leq\phi(T_2)\,r(T_2)x^{\Delta}(T_2)\int_{T_2}^{t}\frac{\Delta s}{\phi(s)\,r(s)}.$$

Since $\phi(T_2)\,r(T_2)x^{\Delta}(T_2)<0$, we conclude that $\lim_{t\to\infty}x(t)=-\infty$, which is a contradiction.

Case (ii). $x^{\Delta}(t)$ is not eventually negative. Multiplying both sides of (1.1) by $\left[\frac{\phi(t)}{f(x(t))}\right]^{\sigma}$ and integrating from T to t, $t\geq T$, we get

$$\int_T^t\phi^{\sigma}(s)\,p(s)\ \Delta s=-\int_T^t\left[\frac{\phi(s)}{f(x(s))}\right]^{\sigma}\left(r(s)x^{\Delta}(s)\right)^{\Delta}\ \Delta s$$

By integration by parts we have that for $t\geq T$

$$\begin{aligned}\int_T^t\phi^{\sigma}(s)\,p(s)\ \Delta s &= \left[\frac{\phi(T)}{f(x(T))}\right]r(T)x^{\Delta}(T)-\left[\frac{\phi(t)}{f(x(t))}\right]r(t)x^{\Delta}(t)\\ &\quad+\int_T^t\left[\frac{\phi(s)}{f(x(s))}\right]^{\Delta}r(s)x^{\Delta}(s)\ \Delta s\end{aligned}$$

Then, from the quotient rule and the Pötzsche chain rule, we get

$$\begin{aligned}\int_T^t\phi^{\sigma}(s)\,p(s)\ \Delta s &= \frac{\phi(T)\,r(T)x^{\Delta}(T)}{f(x(T))}-\frac{\phi(t)\,r(t)x^{\Delta}(t)}{f(x(t))}\\ &\quad+\int_T^t\left[\frac{\phi^{\Delta}(s)}{f(x^{\sigma}(s))}-\frac{\phi(s)\int_0^1 f'(x_h(s))\,dh\ x^{\Delta}(s)}{f(x(s))\,f(x^{\sigma}(s))}\right]r(s)x^{\Delta}(s)\ \Delta s\\ &= \frac{\phi(T)\,r(T)x^{\Delta}(T)}{f(x(T))}-\frac{\phi(t)\,r(t)x^{\Delta}(t)}{f(x(t))}\\ &\quad+\int_T^t\frac{\phi^{\Delta}(s)\,r(s)x^{\Delta}(s)}{f(x^{\sigma}(s))}\ \Delta s-\int_T^t\frac{\phi(s)\,r(s)\int_0^1 f'(x_h(s))\,dh\ \left(x^{\Delta}(s)\right)^2}{f(x(s))\,f(x^{\sigma}(s))}\ \Delta s\end{aligned}$$

where $x_h(t):=(1-h)x(t)+hx^{\sigma}(t)>0$, for $0\leq h\leq 1$, $t\in[T,\infty)_{\mathbb{T}}$. Since $vf(v)>0$, $f'(v)>0$, for all $v\neq 0$ and $\phi(t)\,r(t)>0$, for all $t\geq T$, we get

$$\int_T^t\phi^{\sigma}(s)\,p(s)\ \Delta s\leq\frac{\phi(T)\,r(T)x^{\Delta}(T)}{f(x(T))}-\frac{\phi(t)\,r(t)x^{\Delta}(t)}{f(x(t))}+\int_T^t\frac{\phi^{\Delta}(s)\,r(s)x^{\Delta}(s)}{f(x^{\sigma}(s))}\ \Delta s. \tag{2.16}$$

We claim that $\int_T^t \frac{\phi^{\Delta}(s)r(s)x^{\Delta}(s)}{f(x^{\sigma}(s))}\Delta s$ is bounded above for all $t \geq T$. Since $\phi^{\Delta}(t)\, r(t) \geq 0$ and $\left(\phi^{\Delta}(t)\, r(t)\right)^{\Delta} \leq 0$, we have from Lemma 2.1 that for each $t \in [T,\infty)_{\mathbb{T}}$

$$\int_T^t \frac{\phi^{\Delta}(s)\, r(s) x^{\Delta}(s)}{f(x^{\sigma}(s))}\Delta s = \left(\phi^{\Delta} r\right)(T)\,\Lambda(t),$$

where $m \leq \Lambda(t) \leq M$, and where m and M denote the infimum and supremum, respectively, of the function $\int_T^s \frac{x^{\Delta}(\tau)}{f(x^{\sigma}(\tau))}\Delta\tau$, for $s \in [T,t)_{\mathbb{T}}$. By (2.1) in Lemma 2.2, we have

$$\int_T^t \frac{x^{\Delta}(s)}{f(x^{\sigma}(s))}\Delta s \leq \int_{x(T)}^{x(t)} \frac{d\tau}{f(\tau)} = \int_0^{x(t)} \frac{d\tau}{f(\tau)} - \int_0^{x(T)} \frac{d\tau}{f(\tau)} \leq \int_0^{x(t)} \frac{d\tau}{f(\tau)}.$$

Hence, for all $t \in [T,\infty)_{\mathbb{T}}$, we have

$$\begin{aligned}\int_T^t \frac{\phi^{\Delta}(s)\, r(s) x^{\Delta}(s)}{f(x^{\sigma}(s))}\Delta s &= \phi^{\Delta}(T)\, r(T)\ \Lambda(t) \leq \phi^{\Delta}(T)\, r(T)\ M \\ &\leq \phi^{\Delta}(T)\, r(T)\ \int_0^{x(t)} \frac{d\tau}{f(\tau)} \overset{(1.2)}{\leq} N. \end{aligned} \tag{2.17}$$

From (2.16) and (2.17), we get for $t \in [T,\infty)_{\mathbb{T}}$,

$$\frac{\phi(t)\, r(t) x^{\Delta}(t)}{f(x(t))} \leq \frac{\phi(T)\, r(T) x^{\Delta}(T)}{f(x(T))} + N - \int_T^t \phi^{\sigma}(s)\, q(s)\Delta s.$$

In view of condition (2.11), it follows from the last inequality that there exists a sufficiently large $T_1 \geq T$ such that

$$x^{\Delta}(t) < 0, \quad \text{for } t \in [T_1,\infty)_{\mathbb{T}},$$

which is a contradiction. This completes the proof. □

In the following, we assume that there exists a C^1_{rd} function ϕ such that (2.10) holds and we establish some sufficient conditions for oscillation of all bounded solutions of equation (1.1).

Theorem 2.7. *Assume that f satisfies* (1.2). *If there exists a C^1_{rd} function ϕ such that* (2.11) *and* (2.10) *and*

$$\int_{t_0}^{\infty} \frac{1}{\phi(t)\, r(t)} \left[\int_{t_0}^t \phi^{\sigma}(s)\, p(s)\, \Delta s\right]\ \Delta t = \infty, \tag{2.18}$$

then every bounded solution of equation (1.1) *is oscillatory.*

Proof. Assume (1.1) has a nonoscillatory solution x on $[t_0,\infty)_{\mathbb{T}}$. Then, without loss of generality, there is a solution x of (1.1) and a $T \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t) > 0$ on $[T,\infty)_{\mathbb{T}}$. As in the proof of Theorem 2.6, we have two cases to consider: either $x^{\Delta}(t)$ is eventually negative or $x^{\Delta}(t)$ is not eventually negative.

Case (i). $x^{\Delta}(t)$ is eventually negative. Then there exists $T_1 \geq T$ such that $x^{\Delta}(t) < 0$ for $t \geq T_1$. Multiplying both sides of (1.1) by $\left[\frac{\phi(t)}{f(x(t))}\right]^{\sigma}$ and integrating from T_1 to t, $t \geq T_1$, we get

$$\int_{T_1}^{t} \phi^{\sigma}(s)\, p(s)\ \Delta s = -\int_{T_1}^{t} \left[\frac{\phi(s)}{f(x(s))}\right]^{\sigma} \left(r(s)x^{\Delta}(s)\right)^{\Delta}\ \Delta s$$

By integration by parts we have that for $t \geq T_1$

$$\begin{aligned}\int_{T_1}^{t} \phi^{\sigma}(s)\, p(s)\ \Delta s &= \left[\frac{\phi(T_1)}{f(x(T_1))}\right] r(T_1)x^{\Delta}(T_1) - \left[\frac{\phi(t)}{f(x(t))}\right] r(t)x^{\Delta}(t) \\ &\quad + \int_{T_1}^{t} \left[\frac{\phi(s)}{f(x(s))}\right]^{\Delta} r(s)x^{\Delta}(s)\ \Delta s\end{aligned}$$

Then, from the quotient rule and the Pötzsche chain rule, we get

$$\begin{aligned}\int_{T_1}^{t} \phi^{\sigma}(s)\, p(s)\ \Delta s &= \frac{\phi(T_1)\, r(T_1)x^{\Delta}(T_1)}{f(x(T_1))} - \frac{\phi(t)\, r(t)x^{\Delta}(t)}{f(x(t))} \\ &\quad + \int_{T_1}^{t} \left[\frac{\phi^{\Delta}(s)}{f(x^{\sigma}(s))} - \frac{\phi(s)\int_0^1 f'(x_h(s))\, dh\ x^{\Delta}(s)}{f(x(s))\, f(x^{\sigma}(s))}\right] r(s)x^{\Delta}(s)\ \Delta s \\ &= \frac{\phi(T_1)\, r(T_1)x^{\Delta}(T_1)}{f(x(T_1))} - \frac{\phi(t)\, r(t)x^{\Delta}(t)}{f(x(t))} \\ &\quad + \int_{T_1}^{t} \frac{\phi^{\Delta}(s)\, r(s)x^{\Delta}(s)}{f(x^{\sigma}(s))}\ \Delta s - \int_{T_1}^{t} \frac{\phi(s)\, r(s)\int_0^1 f'(x_h(s))\, dh\ \left(x^{\Delta}(s)\right)^2}{f(x(s))\, f(x^{\sigma}(s))}\ \Delta s,\end{aligned}$$

which implies

$$\int_{T_1}^{t} \phi^{\sigma}(s)\, p(s)\ \Delta s \leq -\frac{\phi(t)\, r(t)x^{\Delta}(t)}{f(x(t))},$$

since $vf(v) > 0$, $f'(v) > 0$, for all $v \neq 0$ and $x^{\Delta}(t) < 0$, and $\phi(t)\, r(t) > 0$, for all $t \geq T$. Therefore

$$\begin{aligned}\int_{T_1}^{t} \frac{1}{\phi(s)\, r(s)} \left[\int_{T_1}^{s} \phi^{\sigma}(u)\, p(u)\, \Delta u\right]\ \Delta s &\leq -\int_{T_1}^{t} \frac{x^{\Delta}(t)}{f(x(t))} \Delta s \overset{(2.2)}{\leq} -\int_{x(T_1)}^{x(t)} \frac{d\tau}{f(\tau)} \\ &= \int_{x(t)}^{x(T_1)} \frac{ds}{f(s)} \leq \int_{0}^{x(T_1)} \frac{ds}{f(s)} \overset{(1.2)}{<} \infty,\end{aligned}$$

since $x^{\Delta}(t) < 0$, then $\lim_{t\to\infty} x(t) \geq 0$, which is a contradiction to (2.18).

Case (iii). The proof is similar to that of Theorem 2.6 and hence is omitted. □

2.3. **Belohorec type.** In the following theorem, we prove oscillation criteria of Belohorec-type for equation (1.1) where the assumption is made that σ is differentiable on $\mathbb{T}$ and $r(t) > 0$ on $[t_0, \infty)_{\mathbb{T}}$ and where $f(x^{\sigma}(t)) = |x^{\sigma}(t)|^{\gamma} \operatorname{sgn} x^{\sigma}(t)$, where $0 < \gamma < 1$.

Theorem 2.8. *If there exists a positive C^1_{rd} function ϕ such that,for $t \in [t_0,\infty)_{\mathbb{T}}$,*

$$\phi^{\Delta}(t) \geq 0, \quad \left(r(t)\,\phi^{\Delta}(t)\right)^{\Delta} \leq 0, \tag{2.19}$$

and

$$\int_{t_0}^{\infty} \frac{\Delta t}{r(t)\,\phi(t)} = \int_{t_0}^{\infty} \phi^{\gamma\sigma}(t)\,p(t)\,\Delta t = \infty, \tag{2.20}$$

then every solution of equation (1.1) *is oscillatory.*

Proof. Assume (1.1) has a nonoscillatory solution x on $[t_0,\infty)_{\mathbb{T}}$. Then, without loss of generality, there is a solution x of (1.1) and a $T \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t) > 0$ on $[T,\infty)_{\mathbb{T}}$. We can write the equation (1.1) in the form (suppressing arguments)

$$\left(r\phi^{\Delta}\right)^{\Delta} y^{\sigma} + \left(r\phi^{\Delta}\right)^{\sigma} (y^{\sigma})^{\Delta} + (r\phi)^{\Delta} y^{\Delta} + (r\phi)^{\sigma} y^{\Delta\Delta} + \phi^{\gamma\sigma} p y^{\gamma\sigma} = 0, \tag{2.21}$$

where $x = \phi y$ and so, from the product rule, we get

$$x^{\Delta} = \phi^{\Delta} y^{\sigma} + \phi y^{\Delta},$$

and

$$\left(r x^{\Delta}\right)^{\Delta} = \left(r\phi^{\Delta}\right)^{\Delta} y^{\sigma} + \left(r\phi^{\Delta}\right)^{\sigma} (y^{\sigma})^{\Delta} + (r\phi)^{\Delta} y^{\Delta} + (r\phi)^{\sigma} y^{\Delta\Delta}.$$

Multiplying both sides of (2.21) by $\dfrac{1}{y^{\gamma\sigma}(t)}$ and integrating from T to t, $t \geq T$, we get

$$\begin{aligned}
\int_T^t \phi^{\gamma\sigma}(s)\,p(s)\,\Delta s &= -\int_T^t \frac{\left(r(s)\,\phi^{\Delta}(s)\right)^{\Delta} y^{\sigma}(s)}{y^{\gamma\sigma}(s)}\Delta s \\
&\quad - \int_T^t \frac{\left(r(s)\,\phi^{\Delta}(s)\right)^{\sigma} (y^{\sigma}(s))^{\Delta}}{y^{\gamma\sigma}(s)}\Delta s \\
&\quad - \int_T^t \frac{(r(s)\,\phi(s))^{\Delta} y^{\Delta}(s)}{y^{\gamma\sigma}(s)}\Delta s \\
&\quad - \int_T^t \frac{(r(s)\,\phi(s))^{\sigma} y^{\Delta\Delta}(s)}{y^{\gamma\sigma}(s)}\Delta s.
\end{aligned} \tag{2.22}$$

Integrating by parts, we obtain

$$\begin{aligned}
\int_T^t \frac{\left(r(s)\,\phi^{\Delta}(s)\right)^{\sigma} (y^{\sigma}(s))^{\Delta}}{y^{\gamma\sigma}(s)}\Delta s &= \frac{r(t)\,\phi^{\Delta}(t)\,y^{\sigma}(t)}{y^{\gamma}(t)} \\
&\quad - \frac{r(T)\,\phi^{\Delta}(T)\,y^{\sigma}(T)}{y^{\gamma}(T)} - \int_T^t \left[\frac{r(s)\,\phi^{\Delta}(s)}{y^{\gamma}(s)}\right]^{\Delta} y^{\sigma}(s)\,\Delta s \\
&\geq -\frac{r(T)\,\phi^{\Delta}(T)\,y^{\sigma}(T)}{y^{\gamma}(T)} - \int_T^t \frac{\left(r(s)\,\phi^{\Delta}(s)\right)^{\Delta} y^{\sigma}(s)}{y^{\gamma\sigma}(s)}\Delta s \\
&\quad + \int_T^t \frac{r(s)\,\phi^{\Delta}(s)\,(y^{\gamma}(s))^{\Delta}\,y^{\sigma}(s)}{y^{\gamma}(s)\,y^{\gamma\sigma}(s)}\Delta s,
\end{aligned} \tag{2.23}$$

It is easy to see by the Pötzsche chain rule ([7, Theorem 1.90]) that

$$(y^{\gamma}(s))^{\Delta} = \gamma \int_0^1 \left[y(s) + h\mu(s) y^{\Delta}(s)\right]^{\gamma-1} dh\; y^{\Delta}(s)$$

$$= \gamma \int_0^1 [(1-h)\,y(s) + hy^\sigma(s)]^{\gamma-1}\,dh\;y^\Delta(s)$$

$$\geq \gamma\,[y^\sigma(s)]^{\gamma-1}\,y^\Delta(s), \tag{2.24}$$

since for $0 < \gamma < 1$ and for a fixed point $s \in [T,\infty)_{\mathbb{T}}$, we have

$$[(1-h)\,y\,(s) + hy^\sigma\,(s)]^{\gamma-1} \begin{cases} \leq [y^\sigma\,(s)]^{\gamma-1}, & y^\Delta\,(s) \leq 0 \\ \geq [y^\sigma\,(s)]^{\gamma-1}, & y^\Delta\,(s) \geq 0, \end{cases}$$

and so

$$[(1-h)\,y\,(s) + hy^\sigma\,(s)]^{\gamma-1}\;\; y^\Delta\,(s) \geq [y^\sigma\,(s)]^{\gamma-1}\;\; y^\Delta(s), \quad \text{for } s \in [T,\infty)_{\mathbb{T}}.$$

By using (2.24) in (2.23), we get

$$\int_T^t \frac{\left(r\,(s)\,\phi^\Delta\,(s)\right)^\sigma \left(y^\sigma\,(s)\right)^\Delta}{y^{\gamma\sigma}\,(s)}\Delta s$$

$$\geq -\frac{r\,(T)\,\phi^\Delta\,(T)\,y^\sigma\,(T)}{y^\gamma\,(T)} - \int_T^t \frac{\left(r\,(s)\,\phi^\Delta\,(s)\right)^\Delta y^\sigma\,(s)}{y^{\gamma\sigma}\,(s)}\Delta s + \gamma\int_T^t \frac{r\,(s)\,\phi^\Delta\,(s)\,y^\Delta(s)}{y^\gamma\,(s)}\Delta s, \tag{2.25}$$

Again by integrating by parts, we have

$$\begin{aligned}
&\int_T^t \frac{\left(r\,(s)\,\phi\,(s)\right)^\sigma y^{\Delta\Delta}\,(s)}{y^{\gamma\sigma}\,(s)}\Delta s \\
&= \frac{r\,(t)\,\phi\,(t)\,y^\Delta\,(t)}{y^\gamma\,(t)} - \frac{r\,(T)\,\phi\,(T)\,y^\Delta\,(T)}{y^\gamma\,(T)} - \int_T^t \left[\frac{r\,(s)\,\phi\,(s)}{y^\gamma\,(s)}\right]^\Delta y^\Delta\,(s)\,\Delta s \\
&= \frac{r\,(t)\,\phi\,(t)\,y^\Delta\,(t)}{y^\gamma\,(t)} - \frac{r\,(T)\,\phi\,(T)\,y^\Delta\,(T)}{y^\gamma\,(T)} - \int_T^t \frac{\left(r\,(s)\,\phi\,(s)\right)^\Delta y^\Delta\,(s)}{y^{\gamma\sigma}\,(s)}\Delta s \\
&\quad + \gamma\int_T^t \frac{r\,(s)\,\phi\,(s)\int_0^1 y_h^{\gamma-1}\,(s)\,dh\;\left(y^\Delta\,(s)\right)^2}{y^\gamma\,(s)\,y^{\gamma\sigma}\,(s)}\Delta s
\end{aligned}$$

$$\geq \frac{r\,(t)\,\phi\,(t)\,y^\Delta\,(t)}{y^\gamma\,(t)} - \frac{r\,(T)\,\phi\,(T)\,y^\Delta\,(T)}{y^\gamma\,(T)} - \int_T^t \frac{\left(r\,(s)\,\phi\,(s)\right)^\Delta y^\Delta\,(s)}{y^{\gamma\sigma}\,(s)}\Delta s, \tag{2.26}$$

where $y_h\,(t) := (1-h)y\,(t) + hy^\sigma\,(t) \geq 0$, for $0 \leq h \leq 1$, $t \in [T,\infty)_{\mathbb{T}}$. From (2.22), (2.25) and (2.26), we get

$$\int_T^t \phi^{\gamma\sigma}\,(s)\,Q\,(s)\,\Delta s \leq K_1 - \frac{r\,(t)\,\phi\,(t)\,y^\Delta\,(t)}{y^\gamma\,(t)} - \gamma\int_T^t \frac{r\,(s)\,\phi^\Delta\,(s)\,y^\Delta(s)}{y^\gamma\,(s)}\Delta s, \tag{2.27}$$

where $K_1 := \frac{r(T)\phi^\Delta(T)y^\sigma(T)}{y^\gamma(T)} + \frac{r(T)\phi(T)y^\Delta(T)}{y^\gamma(T)}$. We claim that $\int_T^t \frac{r(s)\phi^\Delta(s)y^\Delta(s)}{y^\gamma(s)}\Delta s$ is bounded below for $t \geq T$. Since $r\,(s)\,\phi^\Delta\,(s) \geq 0$ and $\left(r\,(s)\,\phi^\Delta\,(s)\right)^\Delta \leq 0$, we have from Second Mean Value Theorem Lemma [8, Theorem 5.45] that for each $t \in [T,\infty)_{\mathbb{T}}$

$$\int_T^t \frac{r\,(s)\,\phi^\Delta\,(s)\,y^\Delta(s)}{y^\gamma\,(s)}\Delta s = r\,(T)\,\phi^\Delta\,(T)\,\Lambda\,(t)\,,$$

where $m \leq \Lambda(t) \leq M$, and where m and M denote the infimum and supremum, respectively, of the function $\int_T^s \frac{y^{\Delta}(\tau)}{y^{\gamma}(\tau)}\Delta\tau$, for $s \in [T,t)_{\mathbb{T}}$. Then$\int_{t_0}^t \frac{x^{\Delta}(s)}{f(x(s))}\Delta s \geq \int_{x(t_0)}^{x(t)} \frac{d\tau}{f(\tau)}$

$$\int_T^t \frac{y^{\Delta}(s)}{y^{\gamma}(s)}\Delta s \overset{(2.2)}{\geq} \int_{y(T)}^{y(t)} \frac{d\tau}{\tau^{\gamma}} = \frac{1}{1-\gamma}\left[y^{1-\gamma}(t) - y^{1-\gamma}(T)\right] \geq \frac{y^{1-\gamma}(T)}{\gamma-1}.$$

Hence, for all $t \geq T$, we have

(2.28)
$$\int_T^t \frac{r(s)\,\phi^{\Delta}(s)\,y^{\Delta}(s)}{y^{\gamma}(s)}\Delta s = r(T)\,\phi^{\Delta}(T)\,\Lambda(t) \geq \frac{1}{\gamma-1}\left[r(T)\,\phi^{\Delta}(T)\,y^{1-\gamma}(T)\right].$$

From (2.27) and (2.28), we get, for all $t \in [T,\infty)_{\mathbb{T}}$,

$$\int_T^t \phi^{\gamma\sigma}(s)\,q(s)\,\Delta s \leq K_1 - \frac{r(t)\,\phi(t)\,y^{\Delta}(t)}{y^{\gamma}(t)} + \frac{\gamma}{1-\gamma}\left[y^{1-\gamma}(T)\,r(T)\,\phi^{\Delta}(T)\right].$$

In view of condition (2.20), it follows from the last inequality that there exists a sufficiently large $T_1 \geq T$ and a positive constant K_2 such that

$$\frac{r(t)\,\phi(t)\,y^{\Delta}(t)}{y^{\gamma}(t)} < -K_2, \quad \text{for } t \in [T_1,\infty)_{\mathbb{T}},$$

and so

$$\int_{T_1}^t \frac{y^{\Delta}(s)}{y^{\gamma}(s)}\Delta s < -K_2 \int_{T_1}^t \frac{\Delta s}{r(s)\,\phi(s)}, \quad \text{for } t \in [T_1,\infty)_{\mathbb{T}}.$$

Then, from (2.20), we get

$$\lim_{t\to\infty} \int_{T_1}^t \frac{y^{\Delta}(s)}{y^{\gamma}(s)}\Delta s = -\infty. \tag{2.29}$$

Again by (2.2), we have

$$\int_{T_1}^t \frac{y^{\Delta}(s)}{y^{\gamma}(s)}\Delta s \geq \frac{\gamma}{1-\gamma}\left[y^{1-\gamma}(t) - y^{1-\gamma}(T_1)\right], \quad \text{for } t \in [T_1,\infty)_{\mathbb{T}}. \tag{2.30}$$

Hence from (2.29) and (2.30), we have $\lim_{t\to\infty} y(t) = -\infty$, which is a contradiction. This completes the proof. □

Remark 2.9. In the case $\mathbb{T} = \mathbb{R}$ and $r(t) = 1$ Theorem 2.8 is due to Wong [21] and, when $\phi(t) = t^{\beta},\ 0 \leq \beta \leq 1$ Theorem 2.8 is due to Belohorec [5]. If $\mathbb{T} = \mathbb{Z}$, $r(t) = 1$ and $p(t) \geq 0$ and $\phi(t) = (t-1)^{\beta},\ 0 \leq \beta \leq 1$; then Theorem 2.8 includes Theorem 4.3 in Hooker and Patula [17, Theorem 4.1] and Mingarelli [20].

Example 2.10. Let $t_0 > 0$ and $\mathbb{T}$ is a discrete time scale, i.e. $\mathbb{T} = \{t_n : n \in \mathbb{N}_0\}$ such that $t_n \to \infty$, and consider the dynamic equation

$$\Delta\left(r(t_n)\,\Delta x(t_n)\right) + p(t_n)\,|x(t_{n+1})|^{\gamma}\ \operatorname{sgn} x(t_{n+1}) = 0, \tag{2.31}$$

where $0 < \gamma < 1$ is a positive real number. Define

$$r(t_n) := t_n^{\beta_1},\ p(t_n) := \frac{1}{t_{n+1}^{\gamma}}\left(\frac{1}{t_n^{\beta_2}} + \frac{(-1)^n}{t_n t_{n+1}}\right),$$

with $\beta_i \in \mathbb{R}$, $i = 1, 2$ such that $\beta_1 \leq 0$ and $\beta_2 \leq 1$. Let $\phi(t_n) := t_n$. Then (2.19) is satisfied. By Example 5.60 in [8], we see that

$$\int_{t_0}^{\infty} \frac{\Delta t}{r(t)\,\phi(t)} = \sum_{n=n_0}^{\infty} \frac{\Delta t_n}{t_n^{\beta_1+1}} = \infty$$

and

$$\int_{t_0}^{\infty} \phi^{\gamma\sigma}(t)\, p(t)\, \Delta t = \sum_{n=n_0}^{\infty} \left(\frac{1}{t_n^{\beta_2}} + \frac{(-1)^n}{t_n t_{n+1}} \right) \Delta t_n = \infty,$$

i.e., (2.20) is satisfied. Then, by Theorem 2.8, Eg. (2.31) is oscillatory.

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