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NEW RESULTS ON STABILITY AND BOUNDEDNESS OF THIRD ORDER NONLINEAR DELAY DIFFERENTIAL EQUATIONS

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ABSTRACT. We establish sufficient conditions which guarantee asymptotic stability of the zero solution and boundedness of all the solutions of the following nonlinear differential equation of third order with the variable delay $r(t)$

$$x'''(t) + a(t)x''(t) + b(t)g_1(x'(t - r(t))) + g_2(x'(t)) + h(x(t - r(t))) \\ = p(t, x(t), x'(t), x(t - r(t)), x'(t - r(t)), x''(t)).$$

By defining an appropriate Lyapunov functional, we prove two new theorems on the stability and boundedness of the solutions of the above equation. Our results extend the results obtained in the literature. We also give an example to illustrate our results.

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1. INTRODUCTION

Hara [6] investigated the uniform boundedness of the differential equation

$$x'''(t) + a(t)x''(t) + b(t)x'(t) + c(t)h(x) = p(t, x(t), x'(t), x''(t)).$$

Tunç [15] proved stability result for solutions to the following nonlinear third order differential equations with deviating argument r

$$x'''(t) + a(t)x''(t) + b(t)g_1(x'(t - r)) + g_2(x'(t)) + h(x(t - r)) \\ = p(t, x(t), x'(t), x(t - r), x'(t - r), x''(t)).$$

For several papers published on the qualitative behaviors of solutions of various nonlinear third order differential equations with delay or without delay, we refer the readers to the papers [1, 6–17] and the references therein.

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The object of this paper is to consider nonlinear third order differential equation with variable delay $r(t)$

$$\begin{aligned} &x'''(t)+a(t)x''(t)+b(t)g_1(x'(t-r(t)))+g_2(x'(t))+h(x(t-r(t))) \\ &=p(t,x(t),x'(t),x(t-r(t)),x'(t-r(t)),x''(t)), \quad t\geq 0. \end{aligned} \tag{1.1}$$

Then (1.1) can be written as the following system

$$\begin{cases} x'(t)=y(t), \\ y'(t)=z(t), \\ z'(t)=-a(t)z(t)-b(t)g_1(y(t))+b(t)\int_{t-r(t)}^{t} g_1'(y(s))z(s)ds \\ -g_2(y(t))-h(x(t))+\int_{t-r(t)}^{t} h'(x(s))y(s)ds \\ +p(t,x(t),y(t),x(t-r(t)),y(t-r(t)),z(t)), \end{cases} \tag{1.2}$$

where $0\leq r(t)\leq\rho$, ρ is a positive constant, and $r'(t)\leq\beta$, $0<\beta<1$; the primes in (1.1) denote differentiation with respect to t, $t\in\mathbb{R}^+$, $\mathbb{R}^+=[0,\infty)$; the functions a,b,g_1,g_2,h and p are continuous in their respective arguments on $\mathbb{R}$, $\mathbb{R}$, $\mathbb{R}$, $\mathbb{R}$ and $\mathbb{R}^+\times\mathbb{R}^5$, respectively, with $g_1(0)=g_2(0)=h(0)=0$. The continuity of the functions a,b,g_1,g_2,h and p guarantees the existence of the solution of (1.1) (see [3]). It is assumed that the right hand side of the system (1.2) satisfies a Lipschitz condition in $x(t)$, $y(t)$, $x(t-r(t))$, $y(t-r(t))$ and $z(t)$. This assumption guarantees the uniqueness of solutions of (1.1) (see [3]). It is also supposed that the derivatives $a'(t)\equiv\frac{d}{dt}a(t)$, $b'(t)\equiv\frac{d}{dt}b(t)$, $h'(t)\equiv\frac{d}{dt}h(t)$, and $g_1'(t)\equiv\frac{d}{dt}g_1(t)$, exist and are continuous; throughout the paper $x(t)$, $y(t)$, $z(t)$, are abbreviated as x, y, z, respectively.

Our purpose is to extend and improve the results established by Tunç [15] to equation (1.1) for the asymptotic stability of zero solution and boundedness of all solutions, when $p\equiv 0$ and $p\neq 0$, respectively. We also give an example to illustrate the effectiveness of the used method. Our approach is based on the Lyapunov's second method.

We point out that equation (1.1) is different from that investigated in [6–15]. Throughout all the above papers, the terms $g_1(x'(t))$ and $h(x(t))$ did not include the variable delay $r(t)\neq 0$. However, equation (1.1) is in the form of $g_1(x'(t-r(t)))$ and $h(x(t-r(t)))$ with $r(t)\neq 0$. This case is a significant difference between our paper and the above papers.

2. PRELIMINARIES AND MAIN RESULTS

We will give some basic information for the general non-autonomous delay differential system. Firstly, we consider the general non-autonomous delay differential system

$$x'=F(t,x_t), \tag{2.1}$$

where $x_t = x(t+\tau)$, for $t \geq 0$, $-r \leq \tau \leq 0$, $F : [0, \infty) \times \mathbb{C}_H \to \mathbb{R}^n$ is a continuous mapping, $F(t, 0) = 0$. We suppose that F take closed bounded sets of $\mathbb{R}^n$. Here $(\mathbb{C}, \|\cdot\|)$ is the Banach space of continuous functions $\psi : [-r, 0] \to R^n$ with supremum norm, $r > 0$; $\mathbb{C}_H$ is the open H-ball in $\mathbb{C}$; $\mathbb{C}_H := \{\psi : [-r, 0] \to \mathbb{R}^n : \|\psi\| < H\}$. Standard existence theory (see [1]), shows that if $\psi \in \mathbb{C}_H$ and $t \geq 0$, then there is at least one continuous solution $x(t, t_0, \psi)$ such that on $[t_0, t_0 + \theta)$ satisfying equation (2.1) for $t > t_0$, $x_t(t, \psi) = \psi$ and θ is a positive constant. If there is a closed subset $B \subset \mathbb{C}_H$ such that the solution remains in B, then $\theta = \infty$. Further, the symbol $|\cdot|$ will denote a convenient norm in $\mathbb{R}^n$ with $|x| = \max_{t-\theta \leq s \leq t} |x_i|$. Let us assume that $\mathbb{C}_t = \{\psi : [t-\theta, t] \to \mathbb{R}^n | \psi \text{ is continuous}\}$ and ψ_t denotes the ψ in the special $\mathbb{C}_t$ and $\|\psi_t\| = \max_{t-\theta \leq s \leq t} |\psi(t)|$. It is clear that equation (1.2) is a special case of (2.1).

Definition 2.1 (Burton [3])**.** A continuous function $W : [0, \infty) \to [0, \infty)$ with $W(0) = 0, W(s) > 0$ if $s > 0$, and W is strictly increacing. (We denote wedges by W or W_i, where i is an integer.)

Definition 2.2 (Burton [3])**.** Let D be an open set in $\mathbb{R}^n$ with $0 \in D$. A function $V : [0, \infty) \times D \to [0, \infty)$ is called positive definite if $V(t, 0) = 0$ and there is a wedge W_1 with $V(t, x) \geq W_1(|x|)$, and is called decrescent if there is a wedge W_2 with $V(t, x) \leq W_2(|x|)$.

Definition 2.3 (Burton [3])**.** Let $F(t, 0) = 0$, then

(i) the zero solution of equation (2.1) is stable if for each $\varepsilon > 0$ and $t_1 \geq t_0$, there exists $\delta > 0$ such that $[\psi \in \mathbb{C}(t_1), \|\psi\| < \delta, t \geq t_1]$ imply that $|x(t, t_0, \psi)| < \varepsilon$;

(ii) the zero solution of equation (2.1) is asymptotically stable if it is stable and for each $t_1 \geq t_0$, there is an $\eta > 0$ such that $[\psi \in C(t_1), \|\psi\| < \delta]$ implies that $x(t, t_0, \psi) \to 0$ as $t \to \infty$.

Definition 2.4 (Burton [3])**.** Let $V(t, \psi)$ be a continuous functional defined for $t \geq 0$, $\psi \in \mathbb{C}_H$. The derivative of V along solutions of equation (2.1) will be denoted by $\dot{V}$ and is defined by the following relation $\dot{V}(t, \psi) = \lim_{h \to 0} \sup \frac{V(t+h, x_{t+h}(t_0, \psi)) - V(t, x_t(t_0, \psi))}{h}$, where $x(t_0, \psi)$ is the solution of equation (2.1) with $x_{t_0}(t_0, \psi)$.

Secondly, we consider the general autonomous delay differential system

$$\dot{x} = G(x_t), \tag{2.2}$$

which is a special case of equation (2.1), and the following lemma is given.

Lemma 2.5 (Sinha [10])**.** *Suppose $F(0) = 0$. Let V be a continuous functional defined on $\mathbb{C}_H$ with $V(0) = 0$ and let $u(s)$ be a nonnegative and continuous function for $0 \leq s < \infty$ and $u(s) \to \infty$ as $s \to \infty$, with $u(0) = 0$. If for all $\psi \in \mathbb{C}$, $u(\psi(0)) \leq V(\psi)$, $V(\psi) \geq 0$, $\dot{V}(\psi) \leq 0$, then the solution $x = 0$ of equation (2.2) is stable.*

If we define $Y = \{\psi \in \mathbb{C}_H : \dot{V}(\psi) = 0\}$, then the solution $x = 0$ of equation (2.2) is asymptotically stable, provided that the largest invariant set in Y is $Z = \{0\}$.

Let $\Omega = \{(t, x, y, z) \in \mathbb{R}^+ \times \mathbb{R}^3 : 0 \le t < \infty,\ |x| < H_1,\ |y| < H_1,\ |z| < H_1, H_1 < H\}$.

Suppose there are positive constants a, α, β, b_1, b_2, B, c, c_1 and L such that the following assumptions hold for every t, x, y and z in Ω:

(a_1) $a(t) \ge 2\alpha + a$,
(a_2) $B \ge b(t) \ge \beta$,
(a_3) $g_1(0) = g_2(0) = h(0) = 0$,
(a_4) $0 < c_1 \le h'(x) \le c$, $\alpha\beta - c > 0$,
(a_5) $\frac{g_1(y)}{y} \ge b_1 \ge 1$, $\frac{g_2(y)}{y} \ge b_2 \ge 1$, $(y \ne 0)$ and $|g_1'(y)| \le L$,
(a_6) $[ab(t) - c]y^2 \ge 2^{-1}\alpha a'(t)y^2 + b'(t)\int_0^y g_1(\eta)d\eta$,
(a_7) $|p(t, x, y, x(t - r(t)), y(t - r(t)), z)| \le q(t)$,

where $q \in \mathbb{L}^1(0, \infty)$, $\mathbb{L}^1$ is the space of Lebesgue integrable functions. Now we give our main results.

Theorem 2.6. *Suppose that the functions a, b, g_1, g_2 and h satisfy assumptions (a_1)–(a_6). Then the zero solution of equation (1.1) with $p \equiv 0$ is asymptotically stable, provided that*

$$\rho < \min\left\{\frac{2\alpha(1-\beta)b_2}{\alpha(1-\beta)(BL+c)+(\alpha+1)c}, \frac{2(1-\sigma)(\alpha+a)}{(BL+c)(1-\sigma)+(\alpha+1)BL}\right\}.$$

Theorem 2.7. *Suppose that the functions a, b, g_1, g_2, h and p satisfy assumptions (a_1)–(a_7). Then there exists a positive constant M such that the solution $x(t)$ of equation (1.1) with $p \ne 0$ defined by the initial functions*

$$x(t) = \psi(t), \quad x'(t) = \psi'(t), \quad x''(t) = \psi''(t),$$

satisfies the inequalities

$$|x(t)| \le \sqrt{M}, |x'(t)| \le \sqrt{M}, \quad |x''(t)| \le \sqrt{M},$$

for all $t \ge t_0$, where $\psi \in C^2([t_0 - \theta, t_0], R)$, provided that

$$\rho < \min\left\{\frac{2\alpha(1-\beta)b_2}{\alpha(1-\beta)(BL+c)+(\alpha+1)c}, \frac{2(1-\sigma)(\alpha+a)}{(BL+c)(1-\sigma)+(\alpha+1)BL}\right\}.$$

We define the following Lyapunov functional for the proofs of Theorem 2.6 and Theorem 2.7:

$$\begin{aligned}(2.3)\quad V(t, x_t, y_t, z_t) = {}& \frac{1}{2}z^2 + \alpha yz + b(t)\int_0^y g_1(\eta)d\eta + \int_0^y g_2(\eta)d\eta \\ & + \frac{\alpha}{2}a(t)y^2 + h(x)y + \alpha\int_0^x h(\xi)d\xi + \mu_1\int_{-r(t)}^0\int_{t+s}^t y^2(\theta)d\theta ds\end{aligned}$$

$$+\mu_2 \int_{-r(t)}^{0} \int_{t+s}^{t} z^2(\theta)d\theta ds,$$

where μ_1, μ_2 are some positive constants which will be specified later in the proofs.

The following lemmas are needed in the proofs of Theorem 2.6 and Theorem 2.7.

Lemma 2.8. *Assume that all the conditions of Theorem 1 hold. Then there exist positive constants E_i $(i = 1, 2, 3)$ such that*

$$V(t, x_t, y_t, z_t) \geq E_1 x^2 + E_2 y^2 + E_3 z^2, \tag{2.4}$$

for all x, y and z.

Proof. From the assumptions (a_1)–(a_6), $\frac{g_1(y)}{y} \geq b_1 \geq 1$, $\frac{g_2(y)}{y} \geq b_2$, $(y \neq 0)$, and $0 \leq c_1 \leq h'(x) \leq c$, it follows that

$$b(t)\int_0^y g_1(\eta)d\eta = b(t)\int_0^y \frac{g_1(\eta)}{\eta}\eta d\eta \geq \frac{\beta b_1 y^2}{2} \geq \frac{\beta y^2}{2},$$

$$\int_0^y g_2(\eta)d\eta = \int_0^y \frac{g_2(\eta)}{\eta}\eta d\eta \geq \frac{b_2 y^2}{2},$$

$$\frac{h^2(x)}{2} = \int_0^x h(\xi)h'(\xi)d\xi \leq c\int_0^x h(\xi)d\xi.$$

Taking into account the above discussion, we have

$$\begin{aligned}
V(t, x_t, y_t, z_t) \geq\; & \frac{1}{2}(z+\alpha y)^2 + \alpha\int_0^x h(\xi)d\xi - \frac{h^2(x)}{2\beta} + \frac{\beta[y+\beta^{-1}h(x)]^2}{2} + \frac{b_2 y^2}{2} \\
& + \mu_1 \int_{-r(t)}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds + \mu_2 \int_{-r(t)}^{0}\int_{t+s}^{t} z^2(\theta)d\theta ds, \\
\geq\; & \frac{1}{2}(z+\alpha y)^2 + \alpha\int_0^x h(\xi)d\xi - \frac{c\int_0^x h(\xi)d\xi}{\beta} + \frac{\beta[y+\beta^{-1}h(x)]^2}{2} + \frac{b_2 y^2}{2} \\
& + \mu_1 \int_{-r(t)}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds + \mu_2 \int_{-r(t)}^{0}\int_{t+s}^{t} z^2(\theta)d\theta ds.
\end{aligned}$$

It is clear that

$$\alpha\int_0^x h(\xi)d\xi - \frac{c\int_0^x h(\xi)d\xi}{\beta} = \beta^{-1}(\alpha\beta - c)\int_0^x h(\xi)d\xi \geq 2^{-1}c_1\beta^{-1}(\alpha\beta - c)x^2.$$

Hence

$$\begin{aligned}
V(t, x_t, y_t, z_t) \geq\; & \frac{1}{2}(z+\alpha y)^2 + \frac{\beta[y+\beta^{-1}h(x)]^2}{2} + \frac{b_2 y^2}{2} + 2^{-1}c_1\beta^{-1}(\alpha\beta - c)x^2 \\
& + \mu_1 \int_{-r(t)}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds + \mu_2 \int_{-r(t)}^{0}\int_{t+s}^{t} z^2(\theta)d\theta ds.
\end{aligned}$$

It follows from the terms of the above inequality that there exist sufficiently small positive constants E_i $(i = 1, 2, 3)$ such that

$$V(t, x_t, y_t, z_t) \geq E_1 x^2 + E_2 y^2 + E_3 z^2.$$

□

Lemma 2.9. *Assume that all the conditions of Theorem 2.6 hold. Then there exist positive constants E_4 and E_5 such that*

$$\frac{d}{dt}V(t, x_t, y_t, z_t) \leq -E_4 y^2 - E_5 z^2,$$

for any solution $(x(t), y(t), z(t))$ of (1.2).

Proof. Differentiate (2.3) to obtain

$$\begin{aligned}\frac{d}{dt}V(t, x_t, y_t, z_t) = &-[\alpha b(t)g_1(y)y^{-1} + \alpha g_2(y)y^{-1} - h'(x) - 2^{-1}\alpha a'(t)]y^2 \\ &+ b'(t)\int_0^y g_1(\eta)d\eta - [a(t) - \alpha]z^2 + zb(t)\int_{t-r(t)}^t g_1'(y(s))z(s)ds \\ &+ z\int_{t-r(t)}^t h'(x(s))y(s)ds + \alpha y b(t)\int_{t-r(t)}^t g_1'(y(s))z(s)ds \\ &+ \alpha y\int_{t-r(t)}^t h'(x(s))y(s)ds + \mu_1 y^2 r(t) \\ &+ \mu_2 z^2 r(t) - \mu_1\{1 - r'(t)\}\int_{t-r(t)}^t y^2(s)ds \\ &- \mu_2\{1 - r'(t)\}\int_{t-r(t)}^t z^2(s)ds.\end{aligned}$$

Applying the assumptions of Lemma 2.9 and the inequality $2|st| \leq s^2 + t^2$, we get

$$\begin{aligned}&-[ab(t)g_1(y)y^{-1} + \alpha g_2(y)y^{-1} - h'(x) - 2^{-1}\alpha a'(t)]y^2 + b'(t)\int_0^y g_1(\eta)d\eta \\ &\leq -\{[ab(t) - c]y^2 - 2^{-1}\alpha a'(t)y^2 + b'(t)\int_0^y g_1(\eta)d\eta\} - \alpha b_2 y^2 \leq -\alpha b_2 y^2,\end{aligned}$$

$$-[a(t) - \alpha]z^2 \leq -(\alpha + a)z^2,$$

$$zb(t)\int_{t-r(t)}^t g_1'(y(s))z(s)ds \leq \frac{BL}{2}\rho z^2 + \frac{BL}{2}\int_{t-r(t)}^t z^2(s)ds,$$

$$\alpha y b(t)\int_{t-r(t)}^t g_1'(y(s))z(s)ds \leq \frac{\alpha BL}{2}\rho y^2 + \frac{\alpha BL}{2}\int_{t-r(t)}^t z^2(s)ds,$$

$$z\int_{t-r(t)}^t h'(x(s))y(s)ds \leq \frac{c}{2}\rho z^2 + \frac{c}{2}\int_{t-r(t)}^t y^2(s)ds,$$

$$\alpha y\int_{t-r(t)}^t h'(x(s))y(s)ds \leq \frac{\alpha c}{2}\rho y^2 + \frac{\alpha c}{2}\int_{t-r(t)}^t y^2(s)ds,$$

$$\mu_1 y^2 r(t) + \mu_2 z^2 r(t) \leq \mu_1 y^2 \rho + \mu_2 z^2 \rho,$$

$$\begin{aligned}&-\mu_1\{1 - r'(t)\}\int_{t-r(t)}^t y^2(s)ds - \mu_2\{1 - r'(t)\}\int_{t-r(t)}^t z^2(s)ds \\ &\leq -\mu_1\{1 - \sigma\}\int_{t-r(t)}^t y^2(s)ds - \mu_2\{1 - \sigma\}\int_{t-r(t)}^t z^2(s)ds.\end{aligned}$$

These estimates imply that

$$\begin{aligned}\frac{d}{dt}V(t,x_t,y_t,z_t) \leq &-\left\{\alpha b_2 - (\frac{\alpha BL}{2}+\frac{\alpha c}{2}+\mu_1)\rho\right\} y^2\\ &-\left\{\alpha + a - (\frac{\alpha BL}{2}+\frac{c}{2}+\mu_2)\rho\right\} z^2\\ &+\left\{\frac{(\alpha+1)BL}{2}-\mu_2(1-\sigma)\right\}\int_{t-r(t)}^{t} z^2(s)ds\\ &+\left\{\frac{(\alpha+1)c}{2}-\mu_1(1-\sigma)\right\}\int_{t-r(t)}^{t} y^2(s)ds.\end{aligned}$$

Let $\mu_1 = \frac{(\alpha+1)c}{2(1-\beta)}$ and $\mu_2 = \frac{(\alpha+1)c}{2(1-\beta)}$. We have

$$\begin{aligned}\frac{d}{dt}V(t,x_t,y_t,z_t) \leq &-\left\{\alpha b_2 - (\frac{\alpha BL}{2}+\frac{\alpha c}{2}+\mu_1)\rho\right\} y^2\\ &-\left\{\alpha + a - (\frac{\alpha BL}{2}+\frac{c}{2}+\mu_2)\rho\right\} z^2.\end{aligned}$$

The preceding inequality implies

$$\frac{d}{dt}V(t,x_t,y_t,z_t) \leq -E_4 y^2 - E_5 z^2,$$

for some positive constants E_4 and E_5 provided that

$$\rho < \min\left\{\frac{2\alpha(1-\sigma)b_2}{\alpha(1-\sigma)(BL+c)+(\alpha+1)c}, \frac{2(1-\sigma)(\alpha+a)}{(BL+c)(1-\sigma)+(\alpha+1)BL}\right\}.$$

□

Lemma 2.10. *Assume that all the conditions of Theorem* 2.7 *hold. Then there exists a positive constant* E_6 *such that*

$$\frac{d}{dt}V(t,x_t,y_t,z_t) \leq E_6(2+y^2+z^2)q(t), \tag{2.5}$$

for any solution $(x(t),y(t),z(t))$ *of* (1.2) *with* $p \neq 0$.

Proof. Since $p \neq 0$, calculating the total derivative of the functional $V(t,x_t,y_t,z_t)$ with respect to t along the trajectories of the system (1.2) and using the conditions of Lemma 2.10, we get

$$\frac{d}{dt}V(t,x_t,y_t,z_t) \leq -E_4y^2 - E_5z^2 + (\alpha y + z)p(t,x,y,x(t-r(t)),y(t-r(t)),z).$$

Hence

$$\frac{d}{dt}V(t,x_t,y_t,z_t) \leq E_6(|y|+|z|)q(t).$$

where $E_6 = \max\{1,\alpha\}$. By $|y| < 1+y^2$ and $|z| < 1+z^2$, we have

$$\frac{d}{dt}V(t,x_t,y_t,z_t) \leq E_6(2+y^2+z^2)q(t).$$

□

3. PROOFS OF MAIN RESULTS

Proof of Theorem 2.6. By utilizing (2.4), it follows that

$$V(t, x_t, y_t, z_t) \geq E_7(x^2 + y^2 + z^2),$$

where $E_7 = \min\{E_1, E_2, E_3\}$. The existence of a continuous function $u(s) \geq 0$ with $u(|\psi(0)|) \geq 0$ such that $u(|\psi(0)|) \leq V(\psi)$ is now readily verified.

It also followed that the largest invariant set in Y is $Z = \{0\}$, where $Y = \{\psi \in \mathbb{C}_H : \dot{V}(\psi) = 0\}$. That is the only solution of equation (1.1) for which $\frac{d}{dt}V(t, x_t, y_t, z_t) = 0$ is the solution $x \equiv 0$. This discussion guarantees that the null solution of equation (1.1) is asymptotically stable. □

Proof of Theorem 2.7. By using the inequality (2.4) and (2.5), it follows that

$$\begin{aligned}\frac{d}{dt}V(t, x_t, y_t, z_t) &\leq E_6(2 + E_7^{-1}V(t, x_t, y_t, z_t))q(t)\\ &= 2E_6 q(t) + E_6 E_7^{-1} V(t, x_t, y_t, z_t) q(t).\end{aligned}$$

Let $L_0 = \int_0^\infty q(s)ds$. Integrating the above inequality from 0 to t and using the assumption $q \in L^1(0, \infty)$ we get

$$V(t, x_t, y_t, z_t) \leq V(x_0, y_0, z_0) + 2E_6 L_0 + E_6 E_7^{-1} \int_0^t V(t, x_s, y_s, z_s) q(s) ds.$$

Hence making use of the Gronwall-Bellman inequality, we obtain

$$\begin{aligned}V(t, x_t, y_t, z_t) &\leq \{V(x_0, y_0, z_0) + 2E_6 L_0\} \exp(E_6 E_7^{-1} \int_0^t q(s)ds)\\ &= \{V(x_0, y_0, z_0) + 2E_6 L_0\} \exp(E_6 E_7^{-1} L_0) = M_1 < \infty,\end{aligned}$$

where $M_1 > 0$ is a constant.

It follows that

$$x^2 + y^2 + z^2 \leq E_7^{-1} V(t, x_t, y_t, z_t) \leq M,$$

where $M = E_7^{-1} M_1$. This inequality implies that $|x| \leq \sqrt{M}$, $|y| \leq \sqrt{M}$, $|z| \leq \sqrt{M}$, for all $t \geq t_0$.

Hence $|x| \leq \sqrt{M}$, $|x'| \leq \sqrt{M}$, $|x''| \leq \sqrt{M}$ for all $t \geq t_0$. □

4. EXAMPLE

In order to illustrate our main results, we consider the nonlinear third order delay differential equation

$$\begin{aligned}(4.1)\quad & x'''(t) + (13 + (2 + t^2)^{-1})x''(t) + 4(1 + e^{-t})x'(t - r(t)) + 8x''(t) + x(t - r(t))\\ &= p(t, x(t), x'(t), x(t - r(t)), x'(t - r(t)), x''(t)).\end{aligned}$$

Equation (4.1) may be expressed as the following system

$$x' = y,$$

$$y' = z,$$
$$z' = -(13 + (2+t^2)^{-1})z - 4(1+e^{-t})y - 8y - x$$
$$+ 4(1+e^{-t})\int_{t-r(t)}^{t} z(s)ds + \int_{t-r(t)}^{t} y(s)ds + p(t, x, y, x(t-r(t)), y(t-r(t)), z).$$

Let

$$p(t, x, y, x(t-r(t)), y(t-r(t)), z)$$
$$= \frac{8}{1 + t^2 + x^2(t) + x^2(t-r(t)) + x'^4(t) + x'^4(t-r(t)) + x''^2(t)}.$$

We have

$$a(t) = 13 + (2+t^2)^{-1} \geq 2 \times 6 + 1, \quad \alpha = 6, \quad a = 1,$$
$$1 \leq b(t) = 1 + e^{-t} \leq 2, \quad \beta = 1, \quad B = 2,$$
$$g_1(y) = 4y, \quad g_0(y) = 0,$$
$$\frac{g_1(y)}{y} = 4 = b_1 > 1, \quad (y \neq 0), \quad g_1'(y) = 4 = L,$$
$$\int_0^y g_1(\eta)d\eta = 2y^2,$$
$$g_2(y) = 8y, g_2(0) = 0, \quad \frac{g_2(y)}{y} = 8 = b_2, \quad (y \neq 0),$$
$$h(x) = x, \quad h(0) = 0, \quad h'(x) = 1,$$
$$0 < 2^{-1} < h'(x) \leq 1, \quad c_1 = 2^{-1}, \quad c = 1,$$
$$a'(t) = \frac{-2t}{(2+t^2)^2}, \quad (t \geq 0), \quad b'(t) = \frac{-1}{e^t}, \quad (t \geq 0),$$
$$p(t, x, y, x(t-r(t)), y(t-r(t)), z) = p(t) \leq \frac{1}{1+t^2} = q(t).$$

In view of the above discussions, it follows that

$$[\alpha b(t) - c]y^2 = [6 + 5e^{-t}]y^2, \quad (t \geq 0), \quad \alpha\beta - c = 5 > 0,$$
$$2^{-1}\alpha a'(t)y^2 + b'(t)\int_0^y g(\eta)d\eta = -\frac{6t}{(2+t^2)}y^2 - e^{-t}y^2, \quad (t \geq 0),$$
$$[\alpha b(t) - c]y^2 = [6 + 5e^{-t}]y^2 \geq -\frac{6t}{(2+t^2)}y^2 - e^{-t}y^2 = 2^{-1}\alpha a'(t)y^2 + b'(t)\int_0^y g(\eta)d\eta,$$
$$\int_0^\infty q(s)ds = \int_0^\infty \frac{8}{1+s^2}ds = 4\pi < \infty,$$

that is $q \in \mathbb{L}^1(0, \infty)$.

Hence, all the assumptions of Theorem 2.6 and Theorem 2.7 hold. That is the zero solution of (4.1) with $p \equiv 0$ is asymptotically stable and all the solutions of (4.1) with $p \neq 0$ are bounded.

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