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UNIQUENESS OF SOLUTIONS FOR DISCRETE BOUNDARY VALUE PROBLEMS

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ABSTRACT. We investigate the uniqueness of solutions for second order discrete boundary value problems via monotonicity and critical point theory.

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1. INTRODUCTION

Since difference equations serve as mathematical models in diverse areas, such as economy, biology, physics, mechanics, computer science, finance – see for example [1], [6], [12] – it is of interest to know the conditions which guarantee a) the existence of solutions, b) their uniqueness, c) dependence of solutions on parameters - this is sometimes known as Hadamard's programme and problems satisfying all three conditions are called well-posed.

The question concerning the existence and multiplicity of solutions for discrete BVPs has been investigated thoroughly lately, see for example [2], [4], [9], [10], [11], [13], [14] by the use of various approaches ranging between topological and critical point theory. Monotonicity theory is also applicable, see for example [19], [20]. Concerning the dependence on parameters for discrete BVPs there was some research, see [3], [7]. There are also uniqueness results in the area of boundary value problems for discrete equations, note [16], [17], [21].

In this submission we are going to provide the uniqueness results. The approaches which we employ are different from those used in [16], [17], [21] and involve rather simple assumptions on the nonlinear terms and therefore may account for wider and easier applicability. Since some discrete problems can be written in a form of a nonlinear system, see for example [1], [20], we shall undertake the following problem

$$Au = f(u),\ u \in \mathbb{R}^n \tag{1.1}$$

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in case when the necessarily symmetric $n \times n$ matrix A need not be positive definite. We will assume that f has the following form $f = [f_1, f_2, \dots, f_n]$ and

A1: $f_i : \mathbb{R}^n \to \mathbb{R}$ is continuous for $i = 1, 2, \dots, n$ and $f_i(0) \neq 0$ for at least one $i = 1, 2, \dots, n$.

We recall that a column of vector $u = (u_1, u_2, \dots, u_n)^T \in \mathbb{R}^n$ is a solution if substitution of u into (1.1) renders it an identity. Moreover, 0 is not a solution to (1.1) due to **A1**.

System (1.1) can be treated as a representation of some discrete boundary value problem which in turn arises as discretization of some continuous models. Let us take for example the Emden-Fowler equation

$$\frac{d}{dt}\left(t^{\rho}\frac{du}{dt}\right) + t^{\delta}u^{\gamma} = 0$$

which originated in the gaseous dynamics in astrophysics and further was used in the study of fluid mechanics, relativistic mechanics, nuclear physics and in the study of chemically reacting systems, see [18]. The discrete version of the generalized Emden-Fowler equation $(p(t)y')' + q(t)y = f(t, y)$ received some considerable interest lately mainly by the use of critical point theory, see for example [9], [11], [13]. The discretization of the generalized Emden-Fowler type boundary value problem can be put as follows

$$\Delta\left(p(k-1)\Delta x(k-1)\right) + q(k)x(k) + f(k, x(k)) = 0 \tag{1.2}$$

with boundary conditions

$$x(0) = x(n),\ p(0)\Delta x(0) = p(n)\Delta x(n) \tag{1.3}$$

and where $f \in C([1,n] \times \mathbb{R}, \mathbb{R})$, $p \in C([0, n+1], \mathbb{R})$, $q \in C([1,n], \mathbb{R})$, $p(n) \neq 0$; $[a, b]$ for $a < b$, $a, b \in \mathbb{Z}$ denotes a discrete interval $\{a, a+1, \dots, b\}$; Δ is the forward difference operator defined by $\Delta u(k) = u(k+1) - u(k)$. The realization of the form of (1.1) requires the following matrices, see [11]

$$M = \begin{bmatrix} p(0)+p(1) & -p(1) & 0 & \dots & 0 & -p(0) \\ -p(1) & p(1)+p(2) & -p(2) & \dots & 0 & 0 \\ 0 & -p(2) & p(2)+p(3) & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & p(n-2)+p(n-1) & -p(n-1) \\ -p(0) & 0 & 0 & \dots & -p(n-1) & p(n-1)+p(0) \end{bmatrix}$$

and

$$Q = \begin{bmatrix} -q(1) & 0 & 0 & \dots & 0 & 0 \\ 0 & -q(2) & 0 & \dots & 0 & 0 \\ 0 & 0 & -q(3) & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -q(n-1) & 0 \\ 0 & 0 & 0 & \dots & 0 & -q(n) \end{bmatrix}.$$

Setting $A = M + Q$, $f_k(x) = f(k, x)$ and using the assumption that $p(n) \neq 0$ we see that problem (1.2)–(1.3) has a form of a nonlinear system (1.1). Indeed, in this case there is a $1-1$ correspondence between solutions to (1.1) and solutions to (1.2)–(1.3).

2. UNIQUENESS OF SOLUTIONS VIA MONOTONICITY THEORY

Now we recall the general existence and uniqueness principles which we further use.

Let E be a reflexive Banach space with norm $\|\cdot\|_E$. Let $(\cdot,\cdot)$ denotes the duality pairing between E^* and E. Let us recall that an operator $K : E \to E^*$ is bounded when it maps bounded sets in E into bounded sets in E^*; K is demicontinuous if for any sequence $\{u_n\} \subset E$ and any $u_0 \in E$ such that $u_n \to u_0$ it holds that $Ku_n \rightharpoonup Ku_0$ in E^*; K is monotone if for all $u, v \in E$ it holds that

$$(Ku - Kv, u - v) \geq 0;$$

when $(Ku - Kv, u - v) > 0$ for all $u \neq v$ K is called a strictly monotone operator; finally K is called coercive operator when

$$\lim_{\|u\|_E \to \infty} \frac{(Ku, u)}{\|u\|_E} = +\infty.$$

Theorem 2.1 (Strongly monotone operator principle [5])**.** *Suppose that $K : E \to E^*$ is continuous operator and there exist $c > 0$ such that*

$$(Ku - Kv, u - v) \geq c\|u - v\|_E^2, \quad u, v \in E,$$

Then $K : E \to E^*$ is homeomorphism between E and E^*.

Theorem 2.2 (Browder's Theorem [8])**.** *Let $K : E \to E^*$ be a bounded, demicontinuous and coercive operator. Then equation $Ku = f$ for each $f \in E^*$ has a at least one solution. If additionally K is strictly monotone, then the solution is unique.*

We note that in case E is finite dimensional, an operator $K : E \to E^*$ is demicontinuous if and only if it is continuous. Moreover, when E is a finite dimensional Euclidean space, the duality pairing is the usual scalar product.

Firstly, we apply Theorem 2.1. We assume for each $k \in [1, n]$ that

A2: (*sublinear*) there exist $a_k > 0$ such that $(f_k(t_1) - f_k(t_2))(t_1 - t_2) \le a_k |t_1 - t_2|^2$ for $t_1, t_2 \in \mathbb{R}$;

A3: (*superlinear*) there exist $b_k > 0$ such that $(f_k(t_1) - f_k(t_2))(t_1 - t_2) \ge b_k |t_1 - t_2|^2$ for $t_1, t_2 \in \mathbb{R}$.

In what follows let $\lambda_1 \le \cdots \le \lambda_n$ denote the eigenvalues of A; $\|A\| = \max_{1 \le i \le n} |\lambda_i|$ denotes the norm of a matrix A and we denote

$$a = \max_{1 \le i \le n} \{a_i\}, \quad b = \min_{1 \le i \le n} \{b_i\}.$$

Theorem 2.3. *Assume that conditions* **A1**, **A2** *hold and that* $a < \lambda_1$ *or assume that conditions* **A1**, **A3** *hold and that* $\|A\| < b$. *Then problem* (1.1) *has a unique nontrivial solution in* $\mathbb{R}^n$.

Proof. Assume **A1**, **A2**, let $a < \lambda_1$ and define a continuous operator $K : \mathbb{R}^n \to \mathbb{R}^n$ by

$$Kx = Ax - f(x). \tag{2.1}$$

We get for $x, y \in \mathbb{R}^n$ that

$$(Kx - Ky, x - y) \ge (\lambda_1 - a)|x - y|^2.$$

Hence the assumptions of Theorem 2.1 are satisfied and equation $Ku = 0$ has a unique solution $u^* \in \mathbb{R}^n$ which must necessarily be nontrivial. With assumptions **A1**, **A3** and $\|A\| < b$ we define a continuous operator $K_1 : \mathbb{R}^n \to \mathbb{R}^n$ by

$$K_1 x = f(x) - Ax. \tag{2.2}$$

For $x, y \in \mathbb{R}^n$ it follows by a direct calculation that

$$(K_1 x - K_1 y, x - y) \ge (b - \|A\|)|x - y|^2$$

□

We indicate some examples of nonlinearities which can be used in Theorem 2.3. In what follows $h, u : [1, n] \to \mathbb{R}^+$ are arbitrary, $\alpha \in \mathbb{R}$ is fixed, $\alpha \ne 0$; $t_1, t_2 \in \mathbb{R}$ denote variables; $k \in [1, n]$ is arbitrarily fixed.

Example 2.4 (**A1, A2**)**.** Put $f_k(t) = h(k)(t - \alpha)$ and observe

$$(f_k(t_1) - f_k(t_2))(t_1 - t_2) = h(k)(t_1 - t_2)^2.$$

Example 2.5 (**A1, A3**)**.** Put $f_k(t) = h(k)t^3 + u(k)t - \alpha$ and note that

$$\begin{aligned}(f_k(t_1) - f_k(t_2))(t_1 - t_2) &= h(k)\left(t_1^2 + t_1 t_2 + t_2^2\right)(t_1 - t_2)^2 + u(k)(t_1 - t_2)(t_1 - t_2) \\ &\ge u(k)(t_1 - t_2)^2.\end{aligned}$$

Next, we use Theorem 2.2 and we assume for each $k \in [1, n]$ that

A4: $(f_k(t_1) - f_k(t_2))(t_1 - t_2) < 0$ for $t_1, t_2 \in \mathbb{R}$, $t_1 \ne t_2$;

A5: $(f_k(t_1)) - f_k(t_2))(t_1 - t_2) > 0$ for $t_1, t_2 \in \mathbb{R}$, $t_1 \neq t_2$.

Theorem 2.6. *Assume that conditions* **A1, A4** *hold and that A is positive definite or else assume that conditions* **A1**, **A5** *hold and that A is negative definite. Then problem* (1.1) *has a unique nontrivial solution in $\mathbb{R}^n$.*

Proof. In case of assumptions **A1, A4** and that A is positive definite we use a continuous operator K given by (2.1). We get for $x, y \in \mathbb{R}^n$, $x \neq y$ that

$$(Kx - Ky, x - y) \geq \lambda_1 |x - y|^2 - (f(x) - f(y))(x - y) \geq \lambda_1 |x - y|^2 > 0, \tag{2.3}$$

so K is strictly monotone. Since for all $x \in \mathbb{R}^n$

$$\frac{(Kx, x)}{|x|} \geq \frac{\lambda_1 |x|^2 - (f(x) - f(0), x)}{|x|} \geq \lambda_1 |x|$$

it also follows that K is coercive. The application of Theorem 2.2 provides the assertion. With the second set of assumptions we use operator (2.2) and we get for $x, y \in \mathbb{R}^n$, $x \neq y$ that

$$(K_1 x - K_1 y, x - y) \geq (f(x) - f(y))(x - y) - \lambda_n |x - y|^2 \geq -\lambda_n |x - y|^2 > 0,$$

so K is strictly monotone and coercive. □

The application of Theorem 2.6 puts no restriction on the value the eigenvalues of A or on the norm of A as required by Theorem 2.3. However instead it requires the definiteness of A. We see that with **A4** function f is strictly decreasing while with **A5** it is strictly increasing.

3. UNIQUENESS OF SOLUTIONS VIA CRITICAL POINT THEORY

In the application of critical point theory, we connect critical points to a certain action functional to the solution of (1.1). Hence, the uniqueness of a solution is implied by the uniqueness of a critical point and this in turn is guaranteed by strict convexity.

Theorem 3.1 ([15])**.** *Let E be a reflexive Banach space. If the functional $J : E \to \mathbb{R}$, $J \in C^1(E, \mathbb{R})$ is weakly lower semi-continuous and coercive, i.e.* $\lim_{\|x\| \to \infty} J(x) = +\infty$, *then there exist x_0 such that*

$$\inf_{x \in E} J(x) = J(x_0)$$

and x_0 is also a critical point of J, i.e. $J'(x_0) = 0$. Moreover, if J is strictly convex, then a critical point is unique.

When E is finite dimensional, any continuous functional is necessarily weakly l.s.c. Let $F_k(x) = \int_0^x f_k(t)\,dt$. With problem (1.1) we can connect the following action functionals J, $J_D : \mathbb{R}^n \to \mathbb{R}$ given by

$$J(x) = \frac{1}{2}(Ax, x) - \sum_{k=1}^{n} F_k(x(k)),$$

$$J_D(x) = \sum_{k=1}^{n} F_k(x(k)) - \frac{1}{2}(Ax, x).$$

Functionals J and J_D correspond to (1.1) as follows: any solution to (1.1) is a critical point to either J or J_D and next any critical point to either of the functionals solves (1.1). Since $\mathbb{R}^n$ is finite dimensional we do not distinguish between the weak and the strong solutions. As far as J is concerned, we get strict convexity when either A is positive definite and f is non-increasing or when A is positive semidefinite and f is decreasing. We provide results for both action functionals since these involve different assumptions on matrix A.

We assume that

A6: *there exist constants $\varepsilon_1 > 0, \varepsilon_2 \in \mathbb{R}$ and $r > 1$ such that*

$$-\sum_{k=1}^{n} F_k(x) \geq \varepsilon_1 \sum_{k=1}^{n} |x(k)|^r + \varepsilon_2$$

for $k \in [1, n]$ and $|x| \geq B$, where $B > 0$ is certain (possibly large) constant.

Theorem 3.2. *Assume that conditions* **A1**, **A4, A6** *hold and that A is positively semidefinite. Then problem* (1.1) *has a unique nontrivial solution in $\mathbb{R}^n$.*

Proof. We observe that functional $J_1 : \mathbb{R}^n \to \mathbb{R}$ defined by $J_1(x) = \frac{1}{2}(Ax, x)$ is convex, while $J_2 : \mathbb{R}^n \to \mathbb{R}$ defined by $J_2(x) = -\sum_{k=1}^{n} F_k(x(k))$ is strictly convex. Then, $J = J_1 + J_2$ is strictly convex. Since for $x \in \mathbb{R}^n$ such that $|x| \geq B$ we have

$$J(x) \geq \varepsilon_1 \sum_{k=1}^{n} |x(k)|^r + \varepsilon_2 n$$

it is also coercive and by continuity we get the assertion. □

Note that functional J_1 is not coercive.

Corollary 3.3. *Assume that conditions* **A1**, **A4** *hold and that A is positively definite. Then problem* (1.1) *has a unique nontrivial solution in $\mathbb{R}^n$ which is a critical point to functional J.*

Proof. Indeed, in this case functional J_1 is coercive and since J_2 is strictly convex it is bounded from the below, functional $J_1 + J_2$ must also be coercive. □

The requirement that F is convex with respect to the second variable for each $k \in [1, n]$, or in other words that f in nondecreasing with respect to the second variable for each $k \in [1, n]$, can be weakened. Recalling that λ_1 denotes the first eigenvalue of matrix A we assume

A7: *there exists a constant* $\varepsilon \in (0, 1)$ *such that*

$$\left(f_k(t_1) + \varepsilon\lambda_1 t_1^2 - f_k(t_2) - \varepsilon\lambda_1 t_2^2\right)(t_1 - t_2) \leq 0$$

for $t_1, t_2 \in R$ *and each* $k \in [1, n]$.

Theorem 3.4. *Assume that conditions* **A1**, **A7** *hold and that* A *is positively definite. Then problem* (1.1) *has a unique nontrivial solution in* $\mathbb{R}^n$.

Proof. We put $J_1 : \mathbb{R}^n \to \mathbb{R}$, $J_2 : \mathbb{R}^n \to \mathbb{R}$ by

$$J_1(x) = \frac{1}{2}(Ax, x) - \varepsilon\lambda_1 \sum_{k=1}^{n} x^2(k)$$

and

$$J_2(x) = \varepsilon\lambda_1 \sum_{k=1}^{n} x^2(k) - \sum_{k=1}^{n} F_k(x(k)).$$

We observe that J_1 is coercive and strictly convex, while J_2 is convex. Hence, $J_1 + J_2$ must also be coercive and strictly convex. □

Similarly as with Theorems 3.2, 3.4 and Corollary 3.3 we can argue with functional J_D which would require somehow opposite monotonicity, i.e. f being nondecreasing and A negative definite, or else f being increasing and A negative semidefinite with some growth assumptions on the term f in the latter case. We state the results omitting their proofs. Instead of **A6, A7** we assume that

A8: *there exist constants* $\varepsilon_1 > 0, \varepsilon_2 \in \mathbb{R}$ *and* $r > 1$ *such that*

$$\sum_{k=1}^{n} F_k(x) \geq \varepsilon_1 \sum_{k=1}^{n} |x(k)|^r + \varepsilon_2$$

for $k \in [1, n]$ *and* $|x| \geq B$, *where* $B > 0$ *is certain (possibly large) constant;*

A9: *there exists* $\varepsilon \in (0, 1)$ *such that for* $k \in [1, n]$

$$\left(f_k(t_1) + \varepsilon\lambda_n t_1^2 - f_k(t_2) - \varepsilon\lambda_n t_2^2\right)(t_1 - t_2) \geq 0$$

for $t_1, t_2 \in R$.

Proposition 3.5. *Assume that conditions* **A1**, **A5, A8** *hold and that* A *is negatively semidefinite or assume that conditions* **A1**, **A5** *hold and that* A *is negatively definite or else assume that conditions* **A1**, **A9** *hold and that* A *is negatively definite. Then problem* (1.1) *has a unique nontrivial solution in* $\mathbb{R}^n$.

We give some examples of nonlinear terms investigated with the critical point theory.

Example 3.6 (A1, A4, A6). Assumption **A4** pertains to strict monotonicity (it requires f to be decreasing), while **A6** puts some restrictions on its anti-derivative. Put

$$f_k(x) = -h(k)\, x^3 + w(k),$$

where $w : [1, n] \to \mathbb{R}$, $h : [1, n] \to \mathbb{R}^+$. Then f_k for $k \in [1, n]$ satisfies **A4** and $x \to \sum_{k=1}^{n} \int_0^x f_k(t)\, dt$ satisfies **A6**.

Example 3.7 (A1, A7). Let $h, u : [1, n] \to \mathbb{R}^+$. We observe that function

$$f_k(x) = -h(t)\, x^3 + u(t)\, x - \frac{\lambda_1}{2} x^2$$

is not monotone for while it satisfies **A7** for $k \in [1, n]$.

REFERENCES

[1] R. P. Agarwal, *Difference Equations and Inequalities*, Marcel Dekker, New York, 1992.

[2] R. P. Agarwal and D. O'Regan, A fixed-point approach for nonlinear discrete boundary value problems, *Comput. Math. Appl.*, 36:115–121, 1998.

[3] G. Apreutesei and N. Apreutesei, A note on the continuous dependence on data for second-order difference inclusions, *J. Difference Equ. Appl.*, 17:637–641, 2001.

[4] G. Bonanno and P. Candito, Nonlinear difference equations investigated via critical point methods, *Nonlinear Anal., Theory Methods Appl.*, 70:3180–3186 2009.

[5] K. Deimling, *Nonlinear Functional Analysis*, Springler-Verlag, Berlin 1985.

[6] S. N. Elaydi, *An Introduction to Difference Equations*, Springer-Verlag, New York, 1999.

[7] M. Galewski, Dependence on parameters for discrete second order boundary value problems, *J. Difference Equ. Appl.*, 17:1441–1453, 2011.

[8] S. Fucik and A. Kufner, *Nonlinear differential equations*, Elsevier Scientific Publishing Company, Amsterdam, Oxford, New York, 1980.

[9] Z. Guo and J. Yu, On boundary value problems for a discrete generalized Emden–Fowler equation, *J. Differ. Equations*, 231:18–31, 2006.

[10] Y. Guo, W. Wei and Y. Chen, Existence of three positive solutions for m-point discrete boundary value problems with p-Laplacian, *Discrete Dyn. Nat. Soc.*, Article ID 538431, 2009.

[11] X. He and X. Wu, Existence and multiplicity of solutions for nonlinear second order difference boundary value problems, *Comput. Math. Appl.*, 57:1–8, 2009.

[12] V. Lakshmikantham and D. Trigiante, *Theory of Difference Equations: Numerical Methods and Applications*, Academic Press, New York, 1988.

[13] F. Lian and Y. Xu, Multiple solutions for boundary value problems of a discrete generalized Emden-Fowler equation, *Appl. Math. Lett.*, 23:8–12, 2010.

[14] H. Liang and P. Weng, Existence and multiple solutions for a second-order difference boundary value problem via critical point theory, *J. Math. Anal. Appl.*, 326:511–520, 2007.

[15] J. Mawhin, *Problèmes de Dirichlet variationnels non linéaires*, Les Presses de l'Université de Montréal, Montréal, 1987.

[16] I. Rachůnková and C.C. Tisdell, Existence of non-spurious solutions to discrete Dirichlet problems with lower and upper solutions, *Nonlinear Anal., Theory Methods Appl.*, 67:1236–1245, 2007.

[17] C.C. Tisdell, The uniqueness of solutions to discrete, vector, two-point boundary value problems, *Appl. Math. Lett.*, 16:1321–1328, 2003.

[18] J. S. W. Wong, On the generalized Emden-Fowler equations, *SIAM Review*, 17:339–360, 1975.

[19] Y. Yang and Z. Zhang, Existence results for a nonlinear system with a parameter, *J. Math. Anal. Appl.*, 340:658–668, 2008.

[20] Y. Yang and J. Zhang, Existence and multiple solutions for a nonlinear system with a parameter, *Nonlinear Anal., Theory Methods Appl.*, 70:2542–2548, 2009.

[21] Ch. Yuan, D. Jiang and Y. Zhang, Existence and uniqueness of solutions for singular higher order continuous and discrete boundary value problems, *Bound. Value Probl.*, Article ID 123823, 2008.