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# OSCILLATION AND SPECTRAL THEORY OF STURM-LIOUVILLE DIFFERENTIAL EQUATIONS WITH NONLINEAR DEPENDENCE IN SPECTRAL PARAMETER

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**ABSTRACT.** In this paper, we consider the eigenvalue problem for the second order Sturm-Liouville differential equation and the Dirichlet boundary conditions. Our setting is more general than in the current literature in two respects: (i) the coefficients depend on the spectral parameter $\lambda$ in general nonlinearly, and (ii) the potential is merely monotone in $\lambda$ and not necessarily strictly monotone in $\lambda$, so that the usual strict normality assumption is now removed. This general setting leads to new definitions of an eigenvalue and an eigenfunction – called a finite eigenvalue and a finite eigenfunction. With these new concepts we show that the finite eigenvalues are isolated, bounded from below, and establish an oscillation theorem, i.e., a result counting the zeros of the finite eigenfunctions. The traditional theory in which the potential is linear and strictly monotone in $\lambda$ nicely follows from our results.

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## 1. INTRODUCTION

The aim of this paper is to revive the study of classical oscillation and spectral theory for the second order Sturm-Liouville differential equation

$$-(r_0(t)\,x')' - q_0(t)\,x = \lambda\,w(t)x, \quad t \in [a,b], \tag{SL$_0(\lambda)$}$$

where $r_0$, $q_0$, $w$ are real piecewise continuous functions on $[a,b]$, $r_0(t) \geq \alpha > 0$, and $w(t) > 0$ on $[a,b]$. In particular, we generalize some of the very traditional notions related to the simplest eigenvalue problem

$$(\mathrm{SL}_0(\lambda)), \quad x(a) = 0 = x(b). \tag{E$_0$}$$

The main concepts and results of this classical theory are the following, see e.g. [1, 5, 11, 15]:

(i) (definition) a number $\lambda_0$ is an eigenvalue of problem $(\mathrm{E}_0)$, if there exists a solution $x(t,\lambda_0) \not\equiv 0$ of equation $(\mathrm{SL}_0(\lambda_0))$ satisfying $x(a,\lambda_0) = 0 = x(b,\lambda_0)$;

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(ii) (theorem) a number $\lambda_0$ is an eigenvalue of ($E_0$) if and only if $\hat{x}(b, \lambda_0) = 0$, where $\hat{x}(t, \lambda)$ is the principal solution of ($SL_0(\lambda)$), that is, it the solution starting with the initial conditions $\hat{x}(a, \lambda) = 0$ and $\hat{x}'(a, \lambda) = 1/r(a, \lambda)$;

(iii) (theorem) the eigenvalues of ($E_0$) are (a) real, (b) isolated, (c) bounded from below, and (d) unbounded from above;

(iv) (oscillation theorem) for every $k \in \mathbb{N}$, the $k$-th eigenfunction has exactly $k$ zeros in the interval $(a, b]$.

The above list does not contain further properties of ($E_0$), such as the orthogonality and completeness of the eigenfunctions in $L^2_w[a, b]$ with the inner product

$$\langle x, y \rangle_w = \int_a^b w(t)x(t)y(t)\, \mathrm{d}t,$$

and the Rayleigh principle.

In this paper we consider the second order Sturm-Liouville differential equation

$$(\mathrm{SL}(\lambda)) \qquad (r(t, \lambda)\, x')' + q(t, \lambda)\, x = 0, \quad t \in [a, b],$$

whose coefficients depend in general nonlinearly on the spectral parameter $\lambda$. Such a nonlinear dependence on $\lambda$ arises in various branches of science, for example in quantum mechanics, magnetohydrodynamics, and chemistry (see [9, Section 4]). Several authors investigated equation ($\mathrm{SL}(\lambda)$) in papers [2, 9, 10, 12, 16]. The eigenvalue problem

$$(\mathrm{E}) \qquad (\mathrm{SL}(\lambda)), \quad \lambda \in \mathbb{R}, \quad x(a) = 0 = x(b),$$

having the Dirichlet boundary conditions as above, or more general separated or joint boundary conditions, has been studied in [6, Section 8.3] under the following main monotonicity assumptions:

$$(1.1) \qquad r(t, \lambda) \text{ is nonincreasing in } \lambda,$$

$$(1.2) \qquad q(t, \lambda) \text{ is strictly increasing in } \lambda,$$

$$(1.3) \qquad \lim_{\lambda \to -\infty} q(t, \lambda) = -\infty \quad \text{and} \quad \lim_{\lambda \to \infty} q(t, \lambda) = \infty.$$

The same strict monotonicity assumption (1.2) is used in [9, 10] to prove the accumulation or nonaccumulation of (classical) eigenvalues. In the present work we show that replacing assumptions (1.2)–(1.3) by the weaker condition

$$(1.4) \qquad q(t, \lambda) \text{ is nondecreasing in } \lambda$$

leads to new (more special) definitions of an eigenvalue and an eigenfunction for problem (E), called a finite eigenvalue and a finite eigenfunction. With these new concepts we prove that the finite eigenvalues are isolated and bounded from below, and establish the corresponding oscillation theorem. Hence, we address the properties of the eigenvalue problem (E) covering items (i), (ii), (iii–b,c), and (iv) in the list above. As there is no suitable inner product when the dependence on $\lambda$ is nonlinear, one has to require $\lambda$ to be real in (E). Note that even in the classical case of equation ($\mathrm{SL}_0(\lambda)$), i.e., when

$$r(t,\lambda) = r_0(t), \quad q(t,\lambda) = q_0(t) + \lambda w(t), \tag{1.5}$$

the results of this paper are more general than the traditional ones, since under (1.4) we allow $w(t) \geq 0$ on $[a,b]$. However, for such a linear dependence on $\lambda$, the corresponding theory follows from the recently developed results in [7, 8, 13, 14]. The presented results covering the nonlinear dependence of the coefficients on $\lambda$ is an example of a more general theory of linear Hamiltonian systems developed recently in [3].

This paper is divided as follows. In Section 2 we present our new concept of finite eigenvalues for problem (E). We establish the oscillation theorem (Theorem 2.6) and other properties of finite eigenvalues, such as the existence (Theorems 2.10–2.11) and a characterization of the smallest finite eigenvalue. In Section 3 we develop the corresponding geometric characterization of finite eigenvalues, i.e., the concept of finite eigenfunctions. In Section 4 we provide an example of a simple Sturm-Liouville equation which illustrates this theory.

## 2. **FINITE EIGENVALUES**

Let us begin with stating the assumptions about the coefficients of ($\mathrm{SL}(\lambda)$). The functions $r, q : [a,b] \times \mathbb{R} \to \mathbb{R}$ satisfy the following conditions: there exist a partition $a = \tau_0 < \tau_1 < \cdots < \tau_m = b$ of $[a,b]$ and a partition $-\infty < \cdots < \lambda_k < \lambda_{k+1} < \cdots < \infty$ of $\mathbb{R}$ with no finite accumulation point such that

- the functions $r$ and $q$ are continuous on $[\tau_i, \tau_{i+1}] \times \mathbb{R}$ for every $i \in \{0, \ldots, m-1\}$,
- $r(t,\lambda) > 0$ on $[a,b] \times \mathbb{R}$,
- the functions
$$r_\lambda := \frac{\partial r}{\partial \lambda} \quad \text{and} \quad q_\lambda := \frac{\partial q}{\partial \lambda}$$
are continuous on $[\tau_i, \tau_{i+1}] \times [\lambda_k, \lambda_{k+1}]$ for every $i \in \{0, \ldots, m-1\}$ and $k \in \mathbb{Z}$,
- $r_\lambda(t,\lambda) \leq 0$ and $q_\lambda(t,\lambda) \geq 0$ on $[a,b] \times \mathbb{R}$.

The last condition implies that the function $r(t,\lambda)$ is nonincreasing in $\lambda$ and the function $q(t,\lambda)$ is nondecreasing in $\lambda$ for every fixed $t \in [a,b]$. The continuity of the coefficients $r$ and $q$ in $(t,\lambda)$ implies the continuous dependence of solutions of ($\mathrm{SL}(\lambda)$) on $\lambda$, while the continuity of $r_\lambda$ and $q_\lambda$ yields that the solutions of ($\mathrm{SL}(\lambda)$) are continuously differentiable in $\lambda$. In addition, since the functions $r$, $q$, $r_\lambda$, $q_\lambda$ are piecewise continuous ($\mathrm{C_p}$) in $t$ on $[a,b]$ when $\lambda$ is fixed, the solutions $x$ of ($\mathrm{SL}(\lambda)$) and its derivative $x_\lambda := \frac{\partial x}{\partial \lambda}$ are piecewise continuously differentiable ($\mathrm{C}^1_{\mathrm{p}}$) in $t$ on $[a,b]$.

As it is used above, differentiation with respect to $t$ and $\lambda$ will be denoted by prime and subscript $\lambda$, respectively.

For a fixed $\lambda \in \mathbb{R}$, we consider two solutions $y(t,\lambda)$ and $x(t,\lambda)$ of $(\mathrm{SL}(\lambda))$, whose initial conditions

$$\left.\begin{array}{rl}\text{(i)} & y(a,\lambda),\quad r(a,\lambda)\,y'(a,\lambda),\\ \text{(ii)} & x(a,\lambda),\quad r(a,\lambda)\,x'(a,\lambda)\end{array}\right\}\quad \text{do not depend on } \lambda. \tag{2.1}$$

From equation $(\mathrm{SL}(\lambda))$ it then follows that the Wronskian of the solutions $y(t,\lambda)$ and $x(t,\lambda)$ defined by

$$r(t,\lambda)\begin{vmatrix} y(t,\lambda) & x(t,\lambda)\\ y'(t,\lambda) & x'(t,\lambda)\end{vmatrix}$$

is constant in $t$ on $[a,b]$. If this constant is 1, then we say that the two solutions $y(t,\lambda)$ and $x(t,\lambda)$ are *normalized*. An example of such a normalized pair is the *associated solution* $\tilde{x}(t,\lambda)$ and *principal solution* $\hat{x}(t,\lambda)$ , which are given by the initial conditions

$$\tilde{x}(a,\lambda) = \tilde{x}'(a,\lambda) \equiv 0, \quad r(a,\lambda)\,\hat{x}'(a,\lambda) = \tilde{x}(a,\lambda) \equiv 1 \quad \text{for all } \lambda \in \mathbb{R}.$$

In particular, the principal solution $\hat{x}(t,\lambda)$ will play a central role in the present theory. Note that for a given solution $x(t,\lambda)$ there always exists a solution $y(t,\lambda)$ such that $y(t,\lambda)$ and $x(t,\lambda)$ are normalized for all $\lambda \in \mathbb{R}$ and $y(t_0,\lambda_0) \neq 0$ at a given $t_0 \in [a,b]$ and $\lambda_0 \in \mathbb{R}$. In addition, if the solution $x(t,\lambda)$ satisfies (2.1)(ii), then $y(t,\lambda)$ may be chosen so that it satisfies (2.1)(i). This can be seen by taking the initial conditions of $y(t,\lambda)$ to be

$$y(a,\lambda) = r(a,\lambda)\,x'(a,\lambda)/k(a,\lambda), \quad r(a,\lambda)\,y'(a,\lambda) = -x(a,\lambda)/k(a,\lambda),$$

where $k(a,\lambda) := r^2(a,\lambda)\,[x'(a,\lambda)]^2 + x^2(a,\lambda)$. The choice of $y(t_0,\lambda_0) \neq 0$ then follows from [6, Proposition 4.1.1]. In the following result we describe the behavior in $\lambda$ of the quotient of two such normalized solutions.

**Lemma 2.1.** *Let $y(t,\lambda)$ and $x(t,\lambda)$ be normalized solutions of equation* $(\mathrm{SL}(\lambda))$ *such that* (2.1) *holds and $y(t,\lambda_0) \neq 0$ at a given $t \in [a,b]$ and $\lambda_0 \in \mathbb{R}$. Then there exists $\varepsilon > 0$ such that for all $\lambda \in (\lambda_0-\varepsilon, \lambda_0+\varepsilon)$ we have*

$$\left(\frac{x}{y}\right)_{\!\lambda}(t,\lambda) = \frac{1}{y^2(t,\lambda)}\int_a^t \left\{ q_\lambda(\tau,\lambda)\begin{vmatrix} x(\tau,\lambda) & x(t,\lambda)\\ y(\tau,\lambda) & y(t,\lambda)\end{vmatrix}^2 - r_\lambda(\tau,\lambda)\begin{vmatrix} x'(\tau,\lambda) & x(t,\lambda)\\ y'(\tau,\lambda) & y(t,\lambda)\end{vmatrix}^2 \right\} \mathrm{d}\tau \geq 0, \tag{2.2}$$

$$\left(\frac{ry'}{y}\right)_{\!\lambda}(t,\lambda) = -\frac{1}{y^2(t,\lambda)}\int_a^t \left\{ q_\lambda(\tau,\lambda)\,y^2(\tau,\lambda) - r_\lambda(\tau,\lambda)\,[y'(\tau,\lambda)]^2 \right\} \mathrm{d}\tau \leq 0. \tag{2.3}$$

*Proof.* The result is derived by the differentiation of equation (SL($\lambda$)) with respect to $\lambda$ and by using the fact that the fundamental matrix

$$\Phi(t,\lambda) := \begin{pmatrix} y(t,\lambda) & x(t,\lambda) \\ r(t,\lambda)\, y'(t,\lambda) & r(t,\lambda)\, x'(t,\lambda) \end{pmatrix}$$

has determinant equal to 1. See the details in [3, Lemma 2.1]. □

Formula (2.2) implies that the function $(x/y)(t,\lambda)$ is nondecreasing in $\lambda$ on the interval $(\lambda_0-\varepsilon,\lambda_0+\varepsilon)$. And since the zeros of the function $(x/y)(t,\lambda)$ coincide with the zeros of the solution $x(t,\lambda)$, we have the following.

**Corollary 2.2.** *If* $x(t,\lambda)$ *is a solution of* (SL($\lambda$)) *such that* (2.1)*(ii) holds, then the kernel of* $x(t,\lambda)$ *is piecewise constant in* $\lambda$ *on* $\mathbb{R}$.

This means that for every fixed $\lambda_0 \in \mathbb{R}$ and $t \in [a,b]$ there exists $\delta > 0$ such that either $x(t,\lambda) \neq 0$ for all $\lambda \in (\lambda_0-\delta,\lambda_0)$, or $x(t,\lambda) \equiv 0$ for all $\lambda \in (\lambda_0-\delta,\lambda_0)$. And similar conclusion also holds in the right neighborhood of $\lambda_0$.

**Remark 2.3.** When (1.2) holds, which happens e.g. in the classical case (1.5) with $w(t) > 0$ on $[a,b]$, then formula (2.2) yields that $(x/y)(t,\lambda)$ is strictly increasing on $(\lambda_0-\varepsilon,\lambda_0+\varepsilon)$. This means that once $x(t,\lambda_0)=0$, then $x(t,\lambda)\neq 0$ in some left and right neighborhoods of $\lambda_0$. Hence, in this case the values of $\lambda_0 \in \mathbb{R}$ where $x(t,\lambda_0)=0$ are isolated. In particular, when $t=b$ and $x(t,\lambda)=\hat{x}(t,\lambda)$ is the principal solution of (SL($\lambda$)), then in this case the classical eigenvalues of (E), i.e., the zeros of $\hat{x}(b,\lambda)$, are isolated. However, when (1.4) is only assumed instead of (1.2), then the classical eigenvalues may accumulate at a finite $\lambda_0$, or even there may be a whole interval $[\alpha,\beta] \subseteq \mathbb{R}$ of classical eigenvalues; see the example in Section 4.

Based on the above analysis, we present the following algebraic definition of finite eigenvalues of problem (E).

**Definition 2.4** (Finite eigenvalue)**.** A number $\lambda_0 \in \mathbb{R}$ is a *finite eigenvalue* of (E) if $\hat{x}(b,\lambda_0)=0$ and at the same time $\hat{x}(b,\lambda)\neq 0$ in some left neighborhood of $\lambda_0$, where $\hat{x}(t,\lambda)$ is the principal solution of (SL($\lambda$)).

The name "finite" eigenvalue is motivated by its discrete time counterpart in the theory of matrix pencils (see [4, Remark 1(iv)]). From Corollary 2.2 and Definition 2.4 we immediately obtain the following.

**Corollary 2.5.** *The finite eigenvalues of* (E) *are isolated.*

The main result of this section is contained in the next statement.

**Theorem 2.6** (Oscillation theorem)**.** *Let $\hat{x}(t,\lambda)$ be the principal solution of* (SL($\lambda$))*. Denote by*

$$n_1(\lambda) := \text{ the number of zeros of } \hat{x}(t,\lambda) \text{ in } (a,b],$$

$$n_2(\lambda) := \text{ the number of finite eigenvalues of (E) in } (-\infty,\lambda].$$

*Then there exists $m \in \mathbb{N} \cup \{0\}$ such that*

$$n_1(\lambda) = n_2(\lambda) + m \quad \text{for all } \lambda \in \mathbb{R}. \tag{2.4}$$

*Moreover, for a suitable $\lambda_0 < 0$ we have $n_2(\lambda) \equiv 0$ and $n_1(\lambda) \equiv m$ for all $\lambda \leq \lambda_0$.*

*Proof.* This statement is a special case of a more general result for linear Hamiltonian systems in [3, Theorem 3.5]. □

Since $n_2(\lambda) \equiv 0$ for all $\lambda \leq \lambda_0$ means that there are no finite eigenvalues of (E) in $(-\infty,\lambda]$, we have the following.

**Corollary 2.7.** *The finite eigenvalues of* (E) *are bounded from below (provided there is a finite eigenvalue at all).*

The total number of finite eigenvalues of (E) will now depend on the behavior of the functions $r(t,\lambda)$ and $q(t,\lambda)$.

In the standard theory of second order linear differential equations, see e.g. [5,11], it is known that $n_1(\lambda_0) = 0$, i.e., the nonexistence of zeros of the principal solution $\hat{x}(t,\lambda_0)$ in $(a,b]$, is equivalent with the positivity of the quadratic functional

$$\mathcal{F}(\eta,\lambda_0) := \int_a^b \left\{ r(t,\lambda_0)\,[\eta'(t)]^2 - q(t,\lambda_0)\,\eta^2(t) \right\} \mathrm{d}t.$$

More precisely, $n_1(\lambda_0) = 0$ if and only if $\mathcal{F}(\eta,\lambda_0) > 0$ for all $\eta \in \mathrm{C}^1_\mathrm{p}$ with $\eta(a) = 0 = \eta(b)$ and $\eta(t) \not\equiv 0$ (we write $\mathcal{F}(\cdot,\lambda_0) > 0$). Hence, we obtain from Theorem 2.6 the traditional result in the spirit of item (iv) in the list in Section 1.

**Corollary 2.8** (Oscillation theorem)**.** *Let $\hat{x}(t,\lambda)$ be the principal solution of equation* (SL($\lambda$))*. The number $m$ in Theorem* 2.6 *is zero, i.e., $n_1(\lambda) = n_2(\lambda)$ for all $\lambda \in \mathbb{R}$, if and only if there exists $\lambda_0 < 0$ such that $\mathcal{F}(\cdot,\lambda_0) > 0$.*

**Remark 2.9.** Under (1.2)–(1.3), or under (1.5) with $w(t) > 0$ on $[a,b]$, there always exists $\lambda_0 < 0$ such that $\mathcal{F}(\cdot,\lambda_0) > 0$. This follows from [6, pg. 246] or from [7, Lemma 2.10]. Therefore, in this classical case the oscillation theorem is known as the traditional equality $n_1(\lambda) = n_2(\lambda)$ for all $\lambda \in \mathbb{R}$.

Next we present conditions in terms of $\mathcal{F}(\cdot,\lambda)$ related to the existence of finite eigenvalues.

**Theorem 2.10** (Existence of finite eigenvalues: necessity)**.** *If* (E) *has a finite eigenvalue, then there exist* $\lambda_0, \lambda_1 \in \mathbb{R}$ *with* $\lambda_0 < \lambda_1$ *and* $m \in \mathbb{N} \cup \{0\}$ *such that* $n_1(\lambda) \equiv m$ *for all* $\lambda \leq \lambda_0$ *and* $\mathcal{F}(\cdot, \lambda_1) \not> 0$*.*

*Proof.* The result follows from the oscillation theorem (Theorem 2.6; compare with [3, Theorem 4.5]). If there is a finite eigenvalue of (E), then $n_2(\lambda_1) \geq 1$ for some $\lambda_1 \in \mathbb{R}$. From Theorem 2.6 we know that $n_1(\lambda) \equiv m$ for all $\lambda \leq \lambda_0$ for some $\lambda_0 < 0$ and $m \in \mathbb{N} \cup \{0\}$. By shifting $\lambda_0$ to the sufficiently negative numbers, if needed, we may choose $\lambda_0 < \lambda_1$. Finally, from equality (2.4) of Theorem 2.6 we get $n_1(\lambda_1) = n_2(\lambda_1) + m \geq 1$, so that the principal solution of (SL($\lambda_1$)) has at least one zero in the interval $(a, b]$. This is however equivalent with $\mathcal{F}(\cdot, \lambda_1) \not> 0$, which completes the proof. ☐

**Theorem 2.11** (Existence of finite eigenvalues: sufficiency)**.** *If there exist* $\lambda_0, \lambda_1 \in \mathbb{R}$ *with* $\lambda_0 < \lambda_1$ *such that* $\mathcal{F}(\cdot, \lambda_0) > 0$ *and* $\mathcal{F}(\cdot, \lambda_1) \not> 0$*, then the eigenvalue problem* (E) *has at least one finite eigenvalue. Moreover, the smallest finite eigenvalue* $\lambda_{\min}$ *lies in the interval* $(\lambda_0, \lambda_1]$*, and*

$$-\infty < \lambda_{\min} = \sup\{\lambda \in \mathbb{R},\ \mathcal{F}(\cdot, \lambda) > 0\} = \min\{\lambda \in \mathbb{R},\ \mathcal{F}(\cdot, \lambda) \not> 0\}.$$

*Proof.* This result follows from the oscillation theorem (Corollary 2.8); see the details in [3, Theorem 4.6]. ☐

## 3. FINITE EIGENFUNCTIONS

In this section we develop the geometric notion – a finite eigenfunction – corresponding to a finite eigenvalue $\lambda_0$ of (E). It is clear from Definition 2.4 that the definition of such a finite eigenfunction must depend on the behavior of the solutions of (E) for $\lambda$ near (to the left of) $\lambda_0$. In particular, the notion of a classical eigenfunction needs to be narrowed.

**Definition 3.1** (Degenerate solution)**.** Let $\lambda_0 \in \mathbb{R}$ be given. A solution $x(t, \lambda_0)$ of (E) with $\lambda = \lambda_0$ is said to be *degenerate at* $\lambda_0$ (or it is a *degenerate solution at* $\lambda_0$) if there exists $\delta > 0$ such that for all $\lambda \in (\lambda_0 - \delta, \lambda_0]$ the solution $x(t, \lambda)$ of equation (SL($\lambda$)) given by the initial conditions

$$x(a, \lambda) = x(a, \lambda_0), \quad r(a, \lambda)\, x'(a, \lambda) = r(a, \lambda_0)\, x'(a, \lambda_0) \tag{3.1}$$

satisfies

$$r_\lambda(t, \lambda)\, x'(t, \lambda) \equiv 0, \quad q_\lambda(t, \lambda)\, x(t, \lambda) \equiv 0 \quad \text{for all } t \in [a, b]. \tag{3.2}$$

In the opposite case we say that $x(t, \lambda_0)$ is *nondegenerate at* $\lambda_0$.

This means that, while fixing the initial conditions of $x(t,\lambda)$ at $t=a$ to be those of $x(t,\lambda_0)$ as in (3.1), we examine the solutions $x(t,\lambda)$ of a family of equations (SL($\lambda$)) in which $\lambda$ varies in some small left neighborhood of $\lambda_0$. If those solutions $x(t,\lambda)$ are compared with $r_\lambda(t,\lambda)$ and $q_\lambda(t,\lambda)$ so that the conditions in (3.2) are satisfied, then we say that the solution $x(t,\lambda_0)$ is degenerate at $\lambda_0$.

**Remark 3.2.** It is easy to see that the trivial solution $x(t,\lambda_0)\equiv 0$ on $[a,b]$ is degenerate at every $\lambda_0$.

**Definition 3.3** (Finite eigenfunction)**.** Every nondegenerate solution $x(t,\lambda_0)$ at $\lambda_0$ of (E) is called a *finite eigenfunction* corresponding to the finite eigenvalue $\lambda_0$.

**Remark 3.4.** (i) Under (1.5), a degenerate solution $x(t,\lambda_0)$ at $\lambda_0$ is determined by the condition $w(t)\,x(t,\lambda_0)\equiv 0$ on $[a,b]$. Therefore, the finite eigenfunctions are in this case given by the condition $w(t)\,x(t,\lambda_0)\not\equiv 0$ on $[a,b]$. This definition can be found in [7, 8, 14].

(ii) Under (1.2)–(1.3), the trivial solution $x(t,\lambda_0)\equiv 0$ on $[a,b]$ is the only degenerate solution at each $\lambda_0\in\mathbb{R}$. Thus, the finite eigenfunctions coincide in this case with the usual eigenfunctions of problem (E), and in view of the second part of (3.2) they are determined by the condition $x(t,\lambda_0)\not\equiv 0$ on $[a,b]$.

The meaning of the above definition is justified by the following result, which provides a geometric interpretation of the finite eigenvalues.

**Theorem 3.5** (Geometric characterization of finite eigenvalues)**.** *A number $\lambda_0\in\mathbb{R}$ is a finite eigenvalue of* (E) *if and only if there exists a corresponding finite eigenfunction $x(t,\lambda_0)$.*

*Proof.* The proof utilizes the monotonicity formula (2.2) from Lemma 2.1 with $t=b$ and a uniqueness argument for solutions of equation (SL($\lambda$)). In particular, a solution $x(t,\lambda)$ of (E) satisfies $x(b,\lambda)\equiv 0$ in some left neighborhood of $\lambda_0$ if and only if condition (3.2) holds for $\lambda$ in this neighborhood. The details can be found in [3, Theorem 5.5]. ☐

## 4. EXAMPLE

The aim of this section is to provide a simple example illustrating the new notions of finite eigenvalues and finite eigenfunctions, which were developed in the previous sections.

**Example 4.1.** Let $[a,b]=[0,\pi]$ and define for $t\in[0,\pi]$ the coefficients as follows

$$r(t,\lambda):=\frac{1}{q(t,\lambda)},\qquad \text{where}\qquad q(t,\lambda):=\begin{cases}1, & \text{for } \lambda\in(-\infty,0),\\ \lambda+1, & \text{for } \lambda\in[0,3],\\ 4, & \text{for } \lambda\in(3,\infty).\end{cases}$$

Then for $t \in [0, \pi]$ we have $r_\lambda(t, \lambda) = q_\lambda(t, \lambda) = 0$ when $\lambda \in (-\infty, 0) \cup (3, \infty)$, while for $\lambda \in (0, 3)$ we have $r_\lambda(t, \lambda) = -1/(\lambda + 1)^2$ and $q_\lambda(t, \lambda) = 1$. Thus, we can see that the main assumptions on the functions $r(t, \lambda)$ and $q(t, \lambda)$ stated at the beginning of Section 2 are satisfied. The principal solution of equation (SL($\lambda$)) is in this case

$$\hat{x}(t, \lambda) = \begin{cases} \sin t, & \text{for } \lambda \in (-\infty, 0], \\ \sin(\lambda + 1)\, t, & \text{for } \lambda \in (0, 3], \\ \sin 4t, & \text{for } \lambda \in (3, \infty). \end{cases}$$

Therefore, at $t = \pi$ we have

$$\hat{x}(\pi, \lambda) = \begin{cases} 0, & \text{for } \lambda \in (-\infty, 0] \cup (3, \infty), \\ \sin(\lambda + 1)\, \pi, & \text{for } \lambda \in (0, 3]. \end{cases}$$

Consequently, the finite eigenvalues are, according to Definition 2.4, located at the points $\lambda \in \{1, 2, 3\}$. Note that the number $\lambda = 0$ is not a finite eigenvalue of (E), since $\hat{x}(\pi, \lambda) \equiv 0$ for $\lambda < 0$.

The number $m$ in the oscillation theorem (Theorem 2.6) is $m = 1$. Indeed, for example, for $\lambda = \frac{1}{2}$, we have $\hat{x}(t, \frac{1}{2}) = \sin \frac{3}{2}\, t$, which has just one zero at $t = \frac{2}{3}\pi$ in $(0, \pi]$, that is, $n_1(\frac{1}{2}) = 1$, while there is no finite eigenvalue of (E) in the interval $(-\infty, \frac{1}{2}]$, that is, $n_2(\frac{1}{2}) = 0$. With the value $\lambda = 1$, we have $\hat{x}(t, 1) = \sin 2t$, which has two zeros at $t = \frac{\pi}{2}$ and $t = \pi$ in $(0, \pi]$, that is, $n_1(1) = 2$, while the problem (E) has one finite eigenvalue at $\lambda = 1$ in the interval $(-\infty, 1]$, that is, $n_2(1) = 1$.

According to Definition 3.1, we can see that the solution $x(t, \lambda_0) = \sin t$ of (SL($\lambda$)) is degenerate at $\lambda_0 = 0$ as well as at every $\lambda_0 < 0$. Similarly, the solution $x(t, \lambda_0) = \sin 4t$ of (SL($\lambda$)) is degenerate at every $\lambda_0 > 3$. Therefore, these functions do not represent finite eigenfunctions of problem (E) according to Definition 3.3. On the other hand, for $n \in \{1, 2, 3\}$ the functions $\hat{x}(t, n) = \sin(n + 1)\, t$ are the finite eigenfunctions corresponding to the finite eigenvalues $\lambda = n$ of (E).

**Remark 4.2.** The above example shows that this theory works also in the case when $\lambda$ is restricted to some compact interval $[\alpha, \beta] \subseteq \mathbb{R}$. In this case we can extend the coefficients $r(t, \lambda)$ and $q(t, \lambda)$ for $\lambda$ outside of the interval $[\alpha, \beta]$ as constants and apply the results to this new modified eigenvalue problem. The resulting finite eigenvalues are then located inside the interval $(\alpha, \beta]$. In Example 4.1 this would be $[\alpha, \beta] = [0, 3]$.

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