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LARGE TIME BEHAVIOR FOR THE SOLUTION OF A DELAY HEAT EQUATION

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ABSTRACT. In this paper we consider a heat equation with boundary time-varying delay or distributed delay. Using the energy method, we prove, under suitable assumptions, that the system in each case is uniformly stable. Our results improve earlier results existing in the literature.

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1. INTRODUCTION

In this paper we are concerned with the following problem

$$
(1.1) \quad \begin{cases} \theta_t(x,t) - k\theta_{xx}(x,t) = 0, & \text{in } (0,L)\times(0,\infty) \\ \theta(0,t) = 0, & \theta_x(L,t) + k_1\theta(L,t) + k_2\theta(L,t-\tau(t)) = 0, \quad t \geq 0 \\ \theta(x,0) = \theta_0(x), & x \in (0,L) \\ \theta(L,-t) = f_0(t), & t \in (0,\tau(0)), \end{cases}
$$

a heat equation with boundary time-varying delay associated with initial data θ_0 and history function f_0 in suitable function spaces. Here, θ is the temperature at time t and location x along a rod of length L, k is a positive constant, k_1 and k_2 are nonnegative constants, and the time-varying delay $\tau(t)$ is a positive bounded differentiable function of t, with $\tau'(t) < 1$. This implies that $(t - \tau(t) > -\tau(0))$ which justifies the domain assigned in (1.1) for the history function f_0. We study the asymptotic behavior for the solution of (1.1) and look for sufficient conditions that guarantee the uniform stability of this system.

Time delays arise in many applications because, in most instances, physical, chemical, biological, thermal, and economic phenomena naturally depend not only on the present state but also on some past occurrences. The stability issue of systems

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with delay is, therefore, of theoretical and practical importance. In recent years, the control of PDEs with time delay effects has become an active area of research, see for example [1, 19], and references therein. In many cases it was shown that delay is a source of instability and even an arbitrarily small delay may destabilize a system which is uniformly asymptotically stable in the absence of delay. For instance, contrary to the exponential stability of the thermoelastic system

$$\begin{cases} au_{tt}(x,t) - du_{xx}(x,t-\tau_1) + \beta\theta_x(x,t) = 0, & \text{in } (0,L)\times(0,\infty) \\ b\theta_t(x,t) - k\theta_{xx}(x,t-\tau_2) + \beta u_{xt}(x,t) = 0, & \text{in } (0,L)\times(0,\infty) \\ u(0,t) = u(L,t) = \theta_x(0,t) = \theta_x(L,t) = 0, & t \geq 0 \end{cases}$$

when $\tau_1 = \tau_2 = 0$, Racke [17] proved that, for any constant delays $\tau_1 > 0$ or $\tau_2 > 0$, this system is instable. In [4] and [5], the authors also showed that a small delay in a boundary control of certain hyperbolic systems could be a source of instability, and stabilizing these systems, involving input delay terms, requires additional control terms. In this aspect, Datko, Lagnese and Polis [5] examined the following problem

$$u_{tt}(x,t) - u_{xx}(x,t) + 2au_t(x,t) + a^2u(x,t) = 0, \qquad \text{in } (0,1)\times(0,\infty)$$

$$u(0,t) = 0, \qquad u_x(1,t) = -ku_t(1,t-\tau), \qquad t > 0$$

with a, k, τ positive real numbers. Through a careful spectral analysis, they showed that, for any $a > 0$ and any k satisfying

$$0 < k < \frac{1-e^{-2a}}{1+e^{-2a}},$$

the spectrum of this system lies in Re $w \leq -\beta$, where β is a positive constant depending on the delay τ. Consequently the uniform stability of the system is obtained.

The heat equation with internal or boundary delay was treated in [3, 6, 7, 16, 20]. In particular, system (1.1), with boundary time-varying delay, was studied by Nicaise *et. al* in [16] and the system

$$\begin{cases} u_t(x,t) - k_1u_{xx}(x,t) - k_2u_{xx}(x,t-\tau(t)) = 0, & \text{in } (0,L)\times(0,\infty) \\ u(0,t) = u(L,t) = 0, & t > 0, \end{cases}$$

with internal time-varying delay, was studied by Caraballo *et. al* in [3]. In both situations, the authors assumed that $\tau(t)$ satisfies, for some positive constants K, M, q and for any $t > 0$,

$$0 < K \leq \tau(t) \leq M \quad \text{and} \quad \tau\prime(t) \leq q < 1$$

and showed, using two different methods, that the condition

$$k_2 < \sqrt{1-q}\,k_1 \tag{1.2}$$

is sufficient in both cases to obtain exponential stability. The same result, under condition (1.2), was also obtained by Zhang *et. al* [22] in the presence of a nonlinear source term in (1.1).

Regarding the systems of wave equations with linear frictional damping term and internal constant delay

$$(1.3)\qquad \begin{cases} u_{tt}(x,t) - \triangle u(x,t) + k_1 u_t(x,t) + k_2 u_t(x,t-\tau) = 0, & \text{in } \Omega \times (0,\infty) \\ u(x,t) = 0, & x \in \Gamma_0,\ t > 0 \\ \frac{\partial u}{\partial v}(x,t) = 0, & x \in \Gamma_1,\ t > 0 \end{cases}$$

or with boundary constant delay

$$(1.4)\qquad \begin{cases} u_{tt}(x,t) - \triangle u(x,t) = 0, & \text{in } \Omega \times (0,\infty) \\ u(x,t) = 0, & x \in \Gamma_0,\ t > 0 \\ \frac{\partial u}{\partial v}(x,t) = -k_1 u_t(x,t) - k_2 u_t(x,t-\tau), & x \in \Gamma_1,\ t > 0 \end{cases}$$

it is well known, in the absence of delay ($k_2 = 0$, $k_1 > 0$), that these systems are exponentially stable, see [9–12, 23, 24]. In the presence of delay ($k_2 > 0$), Nicaise and Pignotti [13] examined systems (1.3) and (1.4) and proved under the assumption $k_2 < k_1$ that the energy is exponentially stable. Otherwise, they produced a sequence of delays for which the corresponding solution is instable. The main approach used there is an observability inequality combined with a Carleman estimate. See also [2] for treatment to these problems in more general abstract form and [15] for analogous results in the case of boundary time-varying delay. We also recall the result by Yung *et. al* [21], where the authors proved the same result as in [13] for the one space dimension by adopting the spectral analysis approach. Said-Houari and Laskri [18] also imposed the same condition ($k_2 < k_1$) to establish the exponential stability of the following Timoshenko system with constant delay

$$\begin{cases} \rho_1 \varphi_{tt}(x,t) - K(\varphi_x + \psi)_x(x,t) = 0, \quad (0,1) \times \mathbb{R}_+ \\ \rho_2 \psi_{tt}(x,t) - b\psi_{xx}(x,t) + K(\varphi_x + \psi)(x,t) + k_1 \psi_t(x,t) + k_2 \psi_t(x,t-\tau) = 0. \end{cases}$$

This result was recently extended to the case of time-varying delay by Kirane *et. al* [8].

When the delay term in (1.3) or (1.4) is replaced by the distributed delay

$$\int_{\tau_1}^{\tau_2} k_2(s) u_t(x,t-s)ds,$$

exponential stability results have been obtained in [14] under the condition

$$\int_{\tau_1}^{\tau_2} k_2(s)ds < k_1.$$

Our aim in this work is to investigate (1.1) and establish exponential decay result under suitable assumption (see (2.4)) on the delay term that is weaker than the assumption (1.2) above. In fact, we show, even if $k_1 = 0$, that we still have uniform stability provided $k_2 < \frac{\sqrt{1-q}}{L}$, where q is such that $\tau\prime(t) \leq q < 1$. This extends the stability region of the system and improves the result obtained in [16]. We also study the heat equation with distributed delay given by

(1.5)
$$\begin{cases} \theta_t(x,t) - k\theta_{xx}(x,t) = 0, & \text{in } (0,L)\times(0,\infty) \\ \theta(0,t) = 0, & \theta_x(L,t) + k_1\theta(L,t) + \int_{\tau_1}^{\tau_2} k_2(s)\theta(L,t-s)ds = 0,\ t \geq 0 \\ \theta(x,0) = \theta_0(x), & x \in (0,L) \\ \theta(L,-t) = f_0(t), & t \in (0,\tau_2) \end{cases}$$

where τ_2 is a positive constant, τ_1 is a nonnegative constant with $\tau_1 < \tau_2$, and $k_2 : [\tau_1, \tau_2] \to \mathbb{R}^+$ is a bounded function. We similarly prove that the energy of (1.5) decays exponentially. The paper is organized as follows. In section 2, we present our assumptions, treat the well-posedness issue, and state and prove our main result for system (1.1). Then, in sections 3, we establish the exponential stability of system (1.5).

2. TIME-VARYING DELAY

2.1. **The well-posedness of system (1.1).** Let us introduce the following new variable

$$z(\rho,t) = \theta(L, t-\tau(t)\rho), \qquad (\rho,t) \in (0,1)\times(0,\infty).$$

Then, problem (1.1) is equivalent to

$$\begin{cases} \theta_t(x,t) - k\theta_{xx}(x,t) = 0, & \text{in } (0,L)\times(0,\infty) \\ \tau(t)z_t(\rho,t) + (1-\tau'(t)\rho)z_\rho(\rho,t) = 0, & \text{in } (0,1)\times(0,\infty) \\ \theta(0,t) = 0, & t \geq 0 \\ \theta_x(L,t) + k_1\theta(L,t) + k_2 z(1,t) = 0,\ \ z(0,t) = \theta(L,t), & t \geq 0 \\ \theta(x,0) = \theta_0(x), & x \in (0,L) \\ z(p,0) = f_0(\tau(0)\rho), & \rho \in (0,1). \end{cases} \tag{2.1}$$

We consider the following Hilbert space

$$V = \{v \in H^1(0,L) : v(0) = 0\}.$$

Then, with $U = (\theta, z)$, system (2.1) can be written in the following form

$$\begin{cases} U_t + A(t)U = 0, & t > 0 \\ U(0) = U_0 = (\theta_0, f_0), \end{cases} \tag{2.2}$$

where, with the Hilbert space $H = L^2(0,L) \times L^2(0,1)$, the time dependent operator $A(t) : D(A(t)) \to H$ is given by

$$A(t)U = \left(-k\theta_{xx}, \frac{1-\tau'(t)\rho}{\tau(t)} z_\rho \right)$$

with domain

$$D(A(t)) = \left\{ \begin{array}{c} (\theta, z) \in (H^2(0,L) \cap V) \times H^1(0,1) : \\ \theta(L) = z(0), \theta_x(L) + k_1\theta(L) + k_2 z(1) = 0 \end{array} \right\}$$

which is independent of the time t, i.e. $D(A(t)) = D(A(0))$ for all $t > 0$.

We assume, for some constants (K, M, R, q) and all $t > 0$,

$$\begin{cases} 0 < K \leq \tau(t) \leq M, \quad |\tau'(t)| \leq R \\ \tau'(t) \leq q < 1 \end{cases} \tag{2.3}$$

and

$$k_2 < \sqrt{1-q}\left(k_1 + \frac{1}{L}\right) \tag{2.4}$$

Let ξ be a fixed positive constant satisfying

$$\max\left\{k_1, \frac{k_2}{\sqrt{1-q}}\right\} < \xi < k_1 + \frac{1}{L}. \tag{2.5}$$

Then, for $U = (\theta, z), W = (\overline{\theta}, \overline{z}) \in H$, we equip H with the time dependent inner product

$$(U, W)_t = \int_0^L \theta\overline{\theta}dx + k\xi\tau(t)\int_0^1 z(\rho)\overline{z}(\rho)d\rho.$$

The existence and uniqueness result reads as follows.

Theorem 2.1. *For any $U_0 \in H$, problem (2.2) has a unique solution $U \in C([0,+\infty); H)$. Moreover, if $U_0 \in D(A(0))$, then*

$$U \in C([0,+\infty); D(A(0)) \cap C^1([0,+\infty); H).$$

Proof. By making the transformation

$$U = e^{\frac{R}{2K}t}\overline{U},$$

we need to look at the problem

$$\begin{cases} \overline{U}_t + \overline{A}(t)\overline{U} = 0 \\ \overline{U}(0) = U_0 \end{cases} \tag{2.6}$$

where

$$\overline{A}(t) = A(t) + \frac{R}{2K}I$$

with the same domain of $A(t)$. Using the semigroup method, the well-posedness of system (2.6) can be established by following similar steps as those used in [16] provided we prove, for a fixed t, that the operator $\overline{A}(t)$ is monotone under the weaker assumption (2.4). For this purpose, we find, for any $U \in D(\overline{A}(t))$, that

$$\begin{aligned}(\overline{A}(t)U, U)_t &= k\int_0^L \theta_x^2 dx + kk_1\theta^2(L) + kk_2\theta(L)z(1) \\ &\quad + k\xi\int_0^1 (1-\tau'(t)\rho)z(\rho)z_\rho(\rho)d\rho + \frac{R}{2K}(U,U)_t \\ &= k\int_0^L \theta_x^2 dx + kk_1\theta^2(L) + kk_2\theta(L)z(1) + \frac{k\xi}{2}(1-\tau'(t))z^2(1) \\ &\quad - \frac{k\xi}{2}\theta^2(L) + \frac{k\xi\tau'(t)}{2}\int_0^1 z^2(\rho)d\rho + \frac{R}{2K}(U,U)_t.\end{aligned}$$

Notice that

$$\frac{\tau'(t)}{2\tau(t)}k\xi\tau(t)\int_0^1 z^2(\rho)d\rho + \frac{R}{2K}(U,U)_t \geq \left[\frac{R}{2K} - \frac{|\tau'(t)|}{2\tau(t)}\right]k\xi\tau(t)\int_0^1 z^2(\rho)d\rho \geq 0.$$

Also, using Hölder and Young's inequalities gives

$$\int_0^L \theta_x^2 dx \geq \frac{1}{L}\left(\int_0^L \theta_x dx\right)^2 = \frac{1}{L}\theta^2(L) \tag{2.7}$$

and

$$k_2\theta(L)z(1) \geq -\frac{\xi}{2}\theta^2(L) - \frac{1}{2\xi}k_2^2 z^2(1).$$

Combining all the above, we deduce that

$$(\overline{A}(t)U, U)_t \geq k\left[k_1 + \frac{1}{L} - \xi\right]\theta^2(L) + \frac{k(1-q)}{2\xi}\left[\xi^2 - \frac{k_2^2}{(1-q)}\right]z^2(1)$$

which, in view of (2.5), implies that $(\overline{A}(t)U, U)_t \geq 0$. Therefore, $\overline{A}$ is monotone. □

2.2. **Uniform stability.** In this subsection we state and prove our decay result for the energy of the system (2.1).

Lemma 2.2. *Assume that* (2.3) *and* (2.4) *hold and* (θ, z) *is the solution of* (2.1). *Then the energy functional* E *defined by*

$$E(t) = \frac{1}{2}\int_0^L \theta^2 dx + \frac{k\xi}{2}\tau(t)\int_0^1 z^2(\rho, t)d\rho, \tag{2.8}$$

satisfies, for some positive constant m,

$$E'(t) \leq -m\int_0^L \theta_x^2 dx. \tag{2.9}$$

Proof. Using equations (2.1) and integrating by parts yield

$$\begin{aligned} E'(t) &= k\int_0^L \theta\theta_{xx}dx + \frac{k\xi}{2}\tau'(t)\int_0^1 z^2(\rho,t)d\rho - k\xi\int_0^1 (1-\tau'(t)\rho)z(\rho,t)z_\rho(\rho,t)d\rho \\ &= -k\int_0^L \theta_x^2 dx - kk_1\theta^2(L,t) - kk_2\theta(L,t)z(1,t) - \frac{k\xi}{2}(1-\tau'(t))z^2(1,t) \\ &\quad + \frac{k\xi}{2}\theta^2(L,t). \end{aligned}$$

By (2.5) and (2.7), we find that $\mu := L\left[\xi - k_1\right]$ satisfies $0 < \mu < 1$ and

$$\begin{aligned} -k\int_0^L \theta_x^2 dx &= -k(1-\mu)\int_0^L \theta_x^2 dx - k\mu\int_0^L \theta_x^2 dx \\ &= -k(1-\mu)\int_0^L \theta_x^2 dx - kL\left[\xi - k_1\right]\int_0^L \theta_x^2 dx \\ &\leq -k(1-\mu)\int_0^L \theta_x^2 dx - k\left[\xi - k_1\right]\theta^2(L,t). \end{aligned}$$

This leads to

$$\begin{aligned} E'(t) &\leq -k(1-\mu)\int_0^L \theta_x^2 dx - \frac{k\xi}{2}\theta^2(L,t) - kk_2\theta(L,t)z(1,t) - \frac{k\xi}{2}(1-q)z^2(1,t) \\ &= -k(1-\mu)\int_0^L \theta_x^2 dx - \frac{k\xi}{2}\left[\theta(L,t) + \frac{k_2}{\xi}z(1,t)\right]^2 \\ &\quad - \frac{k(1-q)}{2\xi}\left[\xi^2 - \frac{k_2^2}{(1-q)}\right]z^2(1,t) \\ &\leq -k(1-\mu)\int_0^L \theta_x^2 dx - \frac{k(1-q)}{2\xi}\left[\xi^2 - \frac{k_2^2}{(1-q)}\right]z^2(1,t). \end{aligned}$$

Hence, using (2.5), our conclusion holds. □

Theorem 2.3. *Assume that* (2.3) *and* (2.4) *hold and* (θ, z) *is the solution of* (2.1). *Then, there exist positive constants* c_0, c_1 *such that the energy functional satisfies*

$$E(t) \leq c_0 e^{-c_1 t}. \tag{2.10}$$

Proof. Let us define the functional F by

$$F(t) := \tau(t)\int_0^1 e^{-\tau(t)\rho}z^2(\rho,t)d\rho.$$

Then,

$$\begin{aligned} F'(t) &= \tau'(t)\int_0^1 e^{-\tau(t)\rho}z^2(\rho,t)d\rho - \tau(t)\tau'(t)\int_0^1 \rho e^{-\tau(t)\rho}z^2(\rho,t)d\rho \\ &\quad + 2\tau(t)\int_0^1 e^{-\tau(t)\rho}z(\rho,t)z_t(\rho,t)d\rho. \end{aligned}$$

Using the second equation of (2.1) and integrating by parts give

$$2\tau(t)\int_0^1 e^{-\tau(t)\rho}z(\rho,t)z_t(\rho,t)d\rho = -2\int_0^1 [1-\tau'(t)\rho]e^{-\tau(t)\rho}z(\rho,t)z_\rho(\rho,t)d\rho$$
$$= -\int_0^1 [1-\tau'(t)\rho]e^{-\tau(t)\rho}\frac{\partial}{\partial\rho}z^2(\rho,t)d\rho$$
$$= \theta^2(L,t) - [1-\tau'(t)]e^{-\tau(t)}z^2(1,t) - \tau'(t)\int_0^1 e^{-\tau(t)\rho}z^2(\rho,t)d\rho$$
$$+ \tau(t)\tau'(t)\int_0^1 \rho e^{-\tau(t)\rho}z^2(\rho,t)d\rho - \tau(t)\int_0^1 e^{-\tau(t)\rho}z^2(\rho,t)d\rho.$$

Therefore, we obtain

$$F'(t) = \theta^2(L,t) - [1-\tau'(t)]e^{-\tau(t)}z^2(1,t) - \tau(t)\int_0^1 e^{-\tau(t)\rho}z^2(\rho,t)d\rho$$

Next, we make use of (2.3) and (2.7) to infer

$$F'(t) \le L\int_0^L \theta_x^2 dx - e^{-M}\tau(t)\int_0^1 z^2(\rho,t)d\rho \tag{2.11}$$

Now, for $N > 0$, let

$$\mathcal{L}(t) := NE(t) + F(t).$$

By combining (2.9) and (2.11), we obtain

$$\mathcal{L}'(t) \le -(mN - L)\int_0^L \theta_x^2 dx - e^{-M}\tau(t)\int_0^1 z^2(\rho,t)d\rho.$$

We choose N large enough so that

$$\gamma := (mN - L) > 0.$$

So, we arrive at

$$\mathcal{L}'(t) \le -\gamma\int_0^L \theta_x^2 dx - e^{-M}\tau(t)\int_0^1 z^2(\rho,t)d\rho$$

which, using Poincaré's inequality, yields

$$\mathcal{L}'(t) \le -c'E(t) \tag{2.12}$$

for some constant $c' > 0$. On the other hand, we find that

$$NE(t) \le \mathcal{L}(t) \le NE(t) + \tau(t)\int_0^1 z^2(\rho,t)d\rho \le NE(t) + \frac{2}{k\xi}E(t) = \left(N + \frac{2}{k\xi}\right)E(t).$$

Therefore,

$$\mathcal{L}(t) \sim E(t). \tag{2.13}$$

Hence, (2.12) and (2.13) lead to

$$\mathcal{L}'(t) \le -c'\mathcal{L}(t)$$

A simple integration on $(0,t)$, then another use of (2.13) give (2.10). □

Remark. For the special case of a constant delay, $\tau(t) \equiv \alpha$ and α is a positive constant, we conclude that system (1.1) is exponentially stable under the unique condition

$$k_2 < k_1 + \frac{1}{L}.$$

3. DISTRIBUTED DELAY

This section is devoted to investigating the asymptotic behavior of system (1.5). Here, we assume that

$$\int_{\tau_1}^{\tau_2} k_2(s)ds < k_1 + \frac{1}{L} \tag{3.1}$$

and ξ is a positive constant satisfying

$$k_1 < \int_{\tau_1}^{\tau_2} k_2(s)ds + \xi(\tau_2 - \tau_1) < k_1 + \frac{1}{L}. \tag{3.2}$$

By introducing the variable

$$z(\rho, s, t) = \theta(L, t - \rho s), \qquad (\rho, s, t) \in (0,1) \times (\tau_1, \tau_2) \times (0, \infty)$$

we obtain the equivalent system

$$\begin{cases} \theta_t(x,t) - k\theta_{xx}(x,t) = 0, & \text{in } (0,L) \times (0,\infty) \\ sz_t(\rho,s,t) + z_\rho(\rho,s,t) = 0, & \text{in } (0,1) \times (\tau_1,\tau_2) \times (0,\infty) \\ \theta(0,t) = 0, & t \geq 0 \\ \theta_x(L,t) + k_1\theta(L,t) + \int_{\tau_1}^{\tau_2} k_2(s)z(1,s,t)ds = 0, & \\ \qquad z(0,s,t) = \theta(L,t), & t \geq 0 \\ \theta(x,0) = \theta_0(x), & x \in (0,L) \\ z(\rho,s,0) = f_0(\rho s), & (\rho,s) \in (0,1) \times (\tau_1,\tau_2). \end{cases} \tag{3.3}$$

Then, writing (3.3) in abstract form and following similar arguments as in [14], one can deduce the well posedness of this system.

On the other hand, we define the energy functional by

$$E(t) = \frac{1}{2}\int_0^L \theta^2 dx + \frac{k}{2}\int_0^1 \int_{\tau_1}^{\tau_2} s(k_2(s) + \xi)z^2 ds d\rho. \tag{3.4}$$

Using equations (3.3) and integrating by parts yield

$$\begin{aligned} E'(t) = -k\int_0^L \theta_x^2 dx - kk_1\theta^2(L,t) - k\theta(L,t)\int_{\tau_1}^{\tau_2} k_2(s)z(1,s,t)ds \\ - \frac{k}{2}\int_{\tau_1}^{\tau_2}(k_2(s)+\xi)z^2(1,s,t)ds + \frac{k}{2}\theta^2(L,t)\left[\int_{\tau_1}^{\tau_2}(k_2(s)ds + \xi(\tau_2-\tau_1)\right]. \end{aligned}$$

By (2.7) and (3.2), we find that $\mu := L\left[\int_{\tau_1}^{\tau_2} k_2(s)ds + \xi(\tau_2-\tau_1) - k_1\right]$ satisfies $0 < \mu < 1$ and

$$\begin{aligned}-k\int_0^L \theta_x^2 dx &= -k(1-\mu)\int_0^L \theta_x^2 dx - k\mu\int_0^L \theta_x^2 dx\\ &= -k(1-\mu)\int_0^L \theta_x^2 dx - kL\left[\int_{\tau_1}^{\tau_2}(k_2(s)ds + \xi(\tau_2-\tau_1) - k_1\right]\int_0^L \theta_x^2 dx\\ &\leq -k(1-\mu)\int_0^L \theta_x^2 dx - k\left[\int_{\tau_1}^{\tau_2}(k_2(s)ds + \xi(\tau_2-\tau_1) - k_1\right]\theta^2(L,t).\end{aligned}$$

Also, Young's inequality gives

$$-k\theta(L,t)\int_{\tau_1}^{\tau_2} k_2(s)z(1,s,t)ds \leq \frac{k}{2}\theta^2(L,t)\int_{\tau_1}^{\tau_2} k_2(s)ds + \frac{k}{2}\int_{\tau_1}^{\tau_2} k_2(s)z^2(1,s,t)ds.$$

Combining all the above, we conclude that

$$E'(t) \leq -k(1-\mu)\int_0^L \theta_x^2 dx - \frac{k}{2}\xi(\tau_2-\tau_1)\theta^2(L,t) - \frac{k\xi}{2}\int_{\tau_1}^{\tau_2} z^2(1,s,t)ds.$$

Therefore, for some constant $d > 0$,

$$E'(t) \leq -d\int_0^L \theta_x^2 dx. \tag{3.5}$$

We are now ready to state and prove the following exponential decay result.

Theorem 3.1. *Assume that* $\left(\int_{\tau_1}^{\tau_2} k_2(s)ds < k_1 + \frac{1}{L}\right)$. *Then, there exist positive constants* d_0, d_1 *such that the energy functional of system* (3.3) *satisfies*

$$E(t) \leq d_0 e^{-d_1 t}. \tag{3.6}$$

Proof. Let us define the functional I by

$$I(t) := \int_0^1\int_{\tau_1}^{\tau_2} se^{-s\rho}(k_2(s)+\xi)z^2(\rho,s,t)dsd\rho.$$

Then, we exploit the second equation of (3.3) to get

$$\begin{aligned}I'(t) &= -2\int_{\tau_1}^{\tau_2}(k_2(s)+\xi)\int_0^1 e^{-s\rho}zz_\rho d\rho ds = -\int_{\tau_1}^{\tau_2}(k_2(s)+\xi)\int_0^1 e^{-s\rho}\frac{\partial}{\partial\rho}z^2 d\rho ds\\ &= -\int_{\tau_1}^{\tau_2}(k_2(s)+\xi)\left[e^{-s}z^2(1,s,t) - z^2(0,s,t) + s\int_0^1 e^{-s\rho}z^2 d\rho\right]ds\\ &\leq \theta^2(L,t)\int_{\tau_1}^{\tau_2}(k_2(s)+\xi)ds - \int_0^1\int_{\tau_1}^{\tau_2} se^{-s\rho}(k_2(s)+\xi)z^2 dsd\rho.\end{aligned}$$

This, using (2.7), yields, for some positive constants c, δ,

$$I'(t) \leq c\int_0^L \theta_x^2 dx - \delta\int_0^1\int_{\tau_1}^{\tau_2} s(k_2(s)+\xi)z^2 dsd\rho. \tag{3.7}$$

Now, For $M > 0$, let

$$R(t) := ME(t) + I(t).$$

Making use of (3.5) and (3.7), we infer

$$R'(t) \leq -(dM-c)\int_0^L \theta_x^2 dx - \delta \int_0^1 \int_{\tau_1}^{\tau_2} s(k_2(s)+\xi) z^2 ds d\rho.$$

Next, choosing M large enough so that $(dM-c) > 0$ and using Poincaré's inequality imply

$$R'(t) \leq -d' E(t) \tag{3.8}$$

for some constant $d' > 0$. On the other hand, we easily find that

$$R(t) \sim E(t). \tag{3.9}$$

Then, (3.8) and (3.9) clearly lead to (3.6). □

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