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ON SOLUTIONS TO STOCHASTIC SET DIFFERENTIAL EQUATIONS OF ITÔ TYPE UNDER THE NON-LIPSCHITZIAN CONDITION

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ABSTRACT. This paper makes a research into a class of stochastic set differential equations (SSDEs) disturbed by l-dimensional Brownian motion with non-Lipschitzian coefficients. The solutions of SSDEs are set-valued stochastic processes. Thus, the existence and uniqueness of solutions to SSDEs with non-Lipschitzian coefficients is first proven. And their continuous dependence on initial conditions and a stability property are then investigated. The main mathematical tool is the Bihari's inequality.

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1. INTRODUCTION

The investigations of dynamic systems have been extensively developed, in connection with, among other things, set differential equations which were started in 1969 by De Blasi and Iervolino [10]. The evidence of set differential equations for such areas as control theory, differential inclusions and fuzzy differential equations can be found in [9, 11, 12, 19, 22, 23, 24, 36, 40], and references therein. The set differential equations are explored in [8, 14, 38]. One of the main advantages of investigating deterministic set differential equations is that they can be used as a tool for studying properties of solutions of differential inclusions. On the other hand, the set-valued random processes were first introduced by Van Cutsem [41]. Since then the subject has attracted the interest of many mathematicians and further contributions are made from both the theoretical and applied viewpoints (see e.g. [5, 7, 15, 37, 42]). In [28, 30, 31, 35], the set valued random differential equations are explored. The strong solution of Itô type set-valued stochastic differential equation is analyzed in [25].

However, although there exists enormous literature where attempts have been made to investigate stochastic differential inclusions (see e.g. [1, 2, 3, 4, 6, 20, 21, 32, 34], and references therein), it seems, as far as we know, that the problem of

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the existence and uniqueness of solutions to the SSDEs hasn't still been solved well. Recently, in [28], a kind of the SDDEs disturbed by Wiener processes is investigated, where under the Lipschitzian condition the existence and uniqueness of solutions to the SSDEs is proven. Moreover, in our current paper, under the non-Lipschitzian condition the existence and uniqueness of solutions the SSDEs driven by Wiener processes as well as other typical properties are studied. The mathematical tool employed in the paper is the notion of the set-valued stochastic integral, which is studied in [16, 18, 26, 27, 28, 33]. By using the Bihari inequality (see e.g. [13, 29]) we prove the existence and uniqueness of solutions to the SSDEs. The work presented here extends results obtained both for deterministic and for random set differential equations.

The paper is organized as follows. Section 2 gives an appropriate framework on a set-valued analysis within which the notion of a set valued stochastic integral is given. In order to prove the existence and uniqueness for the SSDEs, the properties of the set valued stochastic integral are provided. The existence and uniqueness of set-valued solutions to the SSDEs disturbed by Wiener processes is proven in Sections 3. Moreover, the continuous dependence of the solutions for SSDEs on initial conditions and a stability property are discussed. Finally, the conclusions are made in Section 4.

2. PRELIMINARIES

Let $\mathcal{K}(\mathbf{R}^d)$ be the family of all nonempty compact and convex subsets of $\mathbf{R}^d$. In $\mathcal{K}(\mathbf{R}^d)$, we define the Hausdorff metric d_H of two sets $A, B \in \mathcal{K}(\mathbf{R}^d)$ as follows

$$d_H = \max\left(\sup_{a\in A}\inf_{b\in B}\|a-b\|, \sup_{b\in B}\inf_{a\in A}\|a-b\|\right).$$

Throughout this paper, let $(\Omega, \mathcal{A}, P)$ be complete probability space. $\mathcal{A}\times\mathcal{B}_+$ is a product σ-field of $\Omega\times\mathbb{R}$. $\mathcal{M}(\Omega, \mathcal{A}; \mathcal{K}(\mathbf{R}^d))$ denotes the family of $\mathcal{A}$-measurable multifunctions with values in $\mathbf{R}^d$. A multifunction $F \in \mathcal{M}(\Omega, \mathcal{A}; \mathcal{K}(\mathbf{R}^d))$ is said to be L^p-integrably bounded, $p \geq 1$, if there exists $h \in L^p(\Omega, \mathcal{A}, P; \mathbf{R}_+)$ such that $|||F||| \leq h$ a.s., where

$$|||A||| := d_H(A, \{0\}) = \sup_{a\in A}\|a\| \quad \text{for } A \in \mathcal{K}(\mathbf{R}^d).$$

Let us denote

$$\mathcal{L}^p(\Omega, \mathcal{A}, P; \mathcal{K}(\mathbf{R}^d)) := \{F \in \mathcal{M}(\Omega, \mathcal{A}; \mathcal{K}(\mathbf{R}^d)) : |||F||| \in L^p(\Omega, \mathcal{A}, P; \mathbf{R}_+)\}.$$

Let $T \in (0, \infty)$ and denote $I := [0, T]$. Let $(\Omega, \mathcal{A}, \{\mathcal{A}_t\}_{t\in I}, P)$ be a complete, filtered probability space where the sub-σ-field family $(\mathcal{A}_t, t \in I)$ of $\mathcal{A}$ satisfies the usual conditions. We call $X : I\times\Omega \to \mathcal{K}(\mathbf{R}^d)$ a set-valued stochastic process, if for every $t \in I$ a mapping $X(t, \cdot) = X(t) : \Omega \to \mathcal{K}(\mathbf{R}^d)$ is a set-valued random variable. If $X : I\times\Omega \to \mathcal{K}(\mathbf{R}^d)$ is $\{\mathcal{A}_t\}_{t\in I}$-adapted and measurable, then it will be called nonanticipating. Equivalently, the set-valued process X is nonanticipating if and only if X is measurable with respect to the σ-algebra $\mathcal{N}$, which is defined as follows

$$\mathcal{N} := \{A \in \mathcal{B}(I) \otimes \mathcal{A} : A^t \in \mathcal{A}_t \text{ for every } t \in I\},$$

where $A^t = \{\omega : (t, \omega) \in A\}$ for $t \in I$.

Let $p \geq 1$ and $L^p(I \times \Omega, \mathcal{N}; \mathbf{R}^d)$ denote the set of all nonanticipating $\mathbf{R}^d$-valued stochastic processes $\{h(t)\}_{t \in I}$ such that $E\left(\int_0^T \|h(s)\|^p ds\right) < \infty$. A set-valued stochastic process X is called L^p-integrably bounded, if there exists a real-valued stochastic process $\hbar \in L^p(I \times \Omega, \mathcal{N}; \mathbf{R}_+)$ such that

$$|||X(t, \omega)||| \leq \hbar(t, \omega) \text{ for a.a. } (t, \omega) \in I \times \Omega.$$

By $\mathcal{L}^p(I \times \Omega, \mathcal{N}; \mathcal{K}(\mathbf{R}^d))$ we denote the set of nonanticipating and L^p-integrably bounded set-valued stochastic processes. Let $X \in \mathcal{L}^1(I \times \Omega, \mathcal{N}; \mathcal{K}(\mathbf{R}^d))$. For such X and a fixed $t \in I$, by the Fubini Theorem, we can define the Aumann's integral $\int_0^t X(s, \omega) ds$, for $\omega \in \Omega$. Obviously, for every $t \in I$ and $\omega \in \Omega$ the Aumann integral $\int_0^t X(s, \omega) ds$ belongs to $\mathcal{K}(\mathbf{R}^d)$ (see e.g. [15, 19]).

We say that a set-valued stochastic process X is d_H-continuous, if almost all its trajectories, i.e. the mappings $X(\cdot, \omega) : I \to \mathcal{K}(\mathbf{R}^d)$ are d_H-continuous functions. It is easy to know that if $X \in \mathcal{L}^p(I \times \Omega, \mathcal{N}; \mathcal{K}(\mathbf{R}^d))$, then the set-valued stochastic process $\int_0^t X(s) ds$ is d_H-continuous (see e.g. Corollary 1 in [28]).

For the integral $\int_0^t X(s) ds$ we have the following proposition.

Proposition 2.1. *Let $p \geq 1$. If $X \in \mathcal{L}^p(I \times \Omega, \mathcal{N}; \mathcal{K}(\mathbf{R}^d))$, then $\int_0^t X(s) ds \in \mathcal{L}^p(I \times \Omega, \mathcal{N}; \mathcal{K}(\mathbf{R}^d))$.*

Proof. It is easy to know that the Aumann integral $\int_0^t X(s, \omega) ds$ is a set-valued nonanticipating process. By the assumption we know that there exists a stochastic process $h \in L^p(I \times \Omega, \mathcal{N}; \mathbf{R}_+)$ with a property $|||X(t, \omega)||| \leq h(t, \omega)$ for almost all $(t, \omega) \in I \times \Omega$. It follows by Hölder inequality that

$$\begin{aligned}
&\int_I \int_\Omega d_H^p \left(\int_0^t X(s, \omega) ds, \{0\} \right) P(d\omega) dt \\
&\leq \int_I \int_\Omega \left(\int_0^t d_H(X(s, \omega), \{0\}) ds \right)^p P(d\omega) dt \\
&\leq T^p \int_\Omega \left(\int_I |||(X(s, \omega)|||^p ds \right) P(d\omega) \\
&\leq T^p \int_\Omega \int_I h^p(s, \omega) ds P(d\omega) < \infty,
\end{aligned}$$

which shows $X \in \mathcal{L}^p(I \times \Omega, \mathcal{N}; \mathcal{K}(\mathbf{R}^d))$. Thus the proof is complete. $\square$

Proposition 2.2. *Let $p \geq 1$. Assume that $X, Y \in \mathcal{L}^p(I \times \Omega, \mathcal{N}; \mathcal{K}(\mathbf{R}^d))$. Then for every $s, t \in I$ and $s \leq t$ it holds*

$$E \sup_{u\in[s,t]} d_H^p \left(\int_s^u X(v)dv, \int_s^u Y(v)dv \right) \leq (t-s)^{p-1} \int_s^t E d_H^p(X(v), Y(v))dv.$$

Proof. For every $s, t \in I$ and $s \leq t$, by virtue of the Hölder inequality we have

$$\begin{aligned}
&\sup_{u\in[s,t]} d_H^p\left(\int_s^u X(v)dv, \int_s^u Y(v)dv\right) \\
&\leq \sup_{u\in[s,t]} \left[\int_s^u d_H(X(v), Y(v))dv\right]^p \\
&\leq \sup_{u\in[s,t]} \left[(u-s)^{p-1} \int_s^u d_H^p(X(v), Y(v))dv\right] \\
&\leq (t-s)^{p-1} \int_s^t d_H^p(X(v), Y(v))dv,
\end{aligned}$$

which easily shows the claim. Thus the proof is complete. □

We introduce the Itô type SSDEs driven by a Wiener process, which is slightly different from those in [28]. Let $\{B(t)\}_{t\in I}$ be a one-dimensional $\{\mathcal{A}\}_{t\in I}$-Brownian motion defined on a complete probability space $(\Omega, \mathcal{A}, P)$ with a filtration $\{\mathcal{A}\}_{t\in I}$ satisfying usual hypotheses. For $X \in L^2(I \times \Omega, \mathcal{N}; \mathbf{R}^d)$, let $\int_0^T X(s)dB(s)$ denote the classical stochastic Itô integral (see e.g. [17, 39]). Also, the following property will be useful in the context of SSDEs.

Proposition 2.3. (i) *Let $X \in L^2(I \times \Omega, \mathcal{N}; \mathbf{R}^d)$. Then the Itô type integral $\int_0^t X(s)dB(s)$ belongs to $L^2(I \times \Omega, \mathcal{N}; \mathbf{R}^d)$.* (ii) *For $X \in \mathcal{L}^2(I \times \Omega, \mathcal{N}; \mathcal{K}(\mathbf{R}^d)), Y \in L^2(I \times \Omega, \mathcal{N}; \mathbf{R}^d)$, $\forall s \leq t \in I$ we have*

$$\begin{aligned}
&d_H\left(\int_0^t X(u)du + \int_0^t Y(u)dB(u), \int_0^s X(u)du + \int_0^s Y(u)dB(u)\right) \\
&= d_H\left(\int_s^t X(u)du + \int_s^t Y(u)dB(u), \{0\}\right).
\end{aligned}$$

Proof. The claim (i) is obvious. For the claim (ii), similar to the discussion of Proposition 2.4 (ii) in [13] we complete the proof. □

Due to the Doob inequality and the Itô isometry for the classical Itô integrals (see e.g. [17]) we have the following property.

Proposition 2.4. *Let $X, Y \in L^2(I \times \Omega, \mathcal{N}; \mathbf{R}^d)$. Then for every $t \in I$*

$$E \sup_{u\in[s,t]} \left\| \int_s^u X(v)dB(v) - \int_s^u Y(v)dB(v) \right\|^2 \leq 4E \int_s^t \|X(v) - Y(v)\|^2 dv.$$

The following Bihari' inequality (see e.g. [13, 29]) will be needed in next section.

Lemma 2.5. *Let r be Borel measurable, bounded nonnegative and left limit function on I and $c > 0$. Let $K : \mathbf{R}_+ \to \mathbf{R}_+$ be a continuous nondecreasing function such that $K(u) > 0$ for all $u > 0$.*

If $\mu(t)$ is a continuous nonnegative nondecreasing function on I, then the inequality

$$r(t) \leq c + \int_0^t K(r(s-))d\mu(s), \quad \forall t \in I$$

implies that

$$r(t) \leq G^{-1}(G(c) + \mu(t))$$

for all $t \in I$ such that

$$G(c) + \mu(t) \in Dom(G^{-1}),$$

where

$$G(u) = \int_1^u \frac{1}{K(v)} dv, \quad u > 0,$$

G^{-1} is the inverse function of G.

3. EXISTENCE AND UNIQUENESS THEOREM OF SOLUTIONS TO SSDEs

In this section, by $\{B(t)\}_{t\in I}$ we denote an l-dimensional $\{\mathcal{A}_t\}_{t\in I}$-Brownian motion defined on $(\Omega, \mathcal{A}, \{\mathcal{A}_t\}_{t\in I}, P)$, $l \in \mathbf{N}$. The process B is defined as follows $B = (B^1, \ldots, B^l)^\top$, where $\{B^1(t)\}_{t\in I}, \ldots, \{B^l(t)\}_{t\in I}$ are the independent, one-dimensional $\{\mathcal{A}_t\}_{t\in I}$-Brownian motions.

Let us consider the following SSDE of Itô type:

$$dX(t) \overset{I\ P.1}{=} f(t, X(t))dt + g(t, X(t))dB(t), \quad X(0) \overset{P.1}{=} x_0, \tag{3.1}$$

with

$$f : I \times \Omega \times \mathcal{K}(\mathbf{R}^d) \to \mathcal{K}(\mathbf{R}^d),$$
$$g : I \times \Omega \times \mathcal{K}(\mathbf{R}^d) \to \mathbf{R}^d \times \mathbf{R}^l,$$
$$x_0 : \Omega \to \mathcal{K}(\mathbf{R}^d) \text{ being a set-valued random variable.}$$

Since $g = (g^1, \ldots, g^l)$ where $g^k : I \times \Omega \times \mathcal{K}(\mathbf{R}^d) \to \mathbf{R}^d$ $(k = 1, \ldots, l)$, one can write (3.1) as follows

$$dX(t) \overset{I\ P.1}{=} f(t, X(t))dt + \sum_{k=1}^{l} g^k(t, X(t))dB^k(t), \ X(0) \overset{I\ P.1}{=} x_0, \tag{3.2}$$

where $\sum$ denote the addition of d-dimensional vectors. One can observe that such equations generalize the classical stochastic differential equations.

Definition 3.1. By a solution to the equation (3.1) or (3.2) we mean a set-valued stochastic process $X : I \times \Omega \to \mathcal{K}(\mathbf{R}^d)$ such that

(i) $X \in \mathcal{L}^2(I \times \Omega, \mathcal{N}; \mathcal{K}(\mathbf{R}^d))$,

(ii) X is a d_H-continuous set-valued stochastic process,

(iii) it holds

$$X(t) \overset{I\ P.1}{=} x_0 + \int_0^t f(s, X(s))ds + \sum_{k=1}^{l} \int_0^t g^k(s, X(s))dB^k(s). \tag{3.3}$$

Moreover, a solution $X : I \times \Omega \to \mathcal{K}(\mathbf{R}^d)$ to the equation (3.1) is said to be unique, if $X(t) \overset{I\ P.1}{=} Y(t)$, where $Y : I \times \Omega \to \mathcal{K}(\mathbf{R}^d)$ is any solution of (3.1).

In [28], the existence and uniqueness of solution to the SSDE driven by a Wiener process is stated under the assumption that the coefficients f, g satisfy both the uniform Lipschitzian condition and the boundedness condition. In the present paper, we will study the existence and uniqueness theorem under assumptions that the coefficients only satisfy a non-Lipschitzian condition and a boundedness condition, where the disturbed term in (3.1) slightly differs from the one in [28].

Throughout this paper we will assume that $f : I \times \Omega \times \mathcal{K}(\mathbf{R}^d) \to \mathcal{K}(\mathbf{R}^d)$, $g^k : I \times \Omega \times \mathcal{K}(\mathbf{R}^d) \to \mathbf{R}^d$ $(k = 1, \dots, l)$ satisfy: the mapping $f : (I \times \Omega) \times \mathcal{K}(\mathbf{R}^d) \to \mathcal{K}(\mathbf{R}^d)$ is $\mathcal{N} \times \mathcal{B}_{d_H} \setminus \mathcal{B}_{d_H}$-measurable and $g^k : (I \times \Omega) \times \mathcal{K}(\mathbf{R}^d) \to \mathbf{R}^d$ is $\mathcal{N} \otimes \mathcal{B}_{d_H} \setminus \mathcal{B}(\mathbf{R}^d)$-measurable.

We will give the non-Lipschitzian condition of coefficients of SSDE (3.1). Let us first introduce the following hypotheses:

(H3.1) x_0 is an $\mathcal{A}_0$-measurable set-valued random variable such that $Ed_H^2(x_0, \{0\}) < \infty$.

(H3.2) Both $f(t, \{0\})$ and $g^k(t, \{0\})$ are bounded, i.e.,

$$\max\{d_H^2(f(t, \{0\}), \{0\}), \|g^k(t, \{0\})\|^2\} \le C, \quad k = 1, \dots, l,$$

where C is a constant.

(H3.3) The non-Lipschitzian condition, i.e., for $\forall x, y \in \mathcal{K}(\mathbf{R}^d)$,

$$\begin{aligned} &\max\{d_H^2(f(t, x), f(t, y)), \|g^k(t, x) - g^k(t, y)\|^2\} \\ &\quad \le CK(d_H^2(x, y)), \qquad k = 1, \dots, l \end{aligned}$$

with C as in (H3.2). And $K(u)$ is a continuous increasing concave function on $\mathbf{R}_+$ such that (i) $\int_0^1 \frac{du}{K(u)} = +\infty$; (ii) $K(0) = 0$ and $K(u) > 0$, $\forall u > 0$.

From the definition of $K(u)$ in (H3.3), we easily show that $G(u)$ defined in Lemma 2.5 is strictly increasing, $G(u) \to -\infty$ as $u \downarrow 0$ and $G^{-1}(u) \to 0$ as $u \to -\infty$.

We state the following existence and uniqueness theorem of SSDE (3.1) or (3.3).

Theorem 3.2. *Let* (H3.1)–(H3.3) *hold. Then there exists a unique solution* $X(t)$ *to equation* (3.1)*, and*

$$E \sup_{t \in I} d_H^2(X(t), \{0\}) < \infty. \tag{3.4}$$

Moreover

$$\lim_{n\to\infty} E \sup_{t\in I} d_H(X_n(t), X(t)) = 0, \tag{3.5}$$

where $X_n(t)$, $t \geq 0$ *and* $n \geq 1$, *are defined as follows:*

$$\begin{aligned} &X_n(0) = x_0, \\ &X_n(t) = X_n(\frac{i}{n}) + \int_{i/n}^t f(s, \widetilde{X}_n(s))ds + \sum_{k=1}^{l} \int_{i/n}^t g^k(s, \widetilde{X}_n(s))dB^k(s) \\ &\qquad \text{if } \frac{i}{n} < t \leq \frac{i+1}{n}, \quad i = 0, 1, \ldots, \\ &\widetilde{X}_n(t) = \sum_{i=0}^{\infty} X_n(\frac{i}{n})\mathbf{1}_{[i/n,(i+1)/n)}(t) \text{ being a simple function.} \end{aligned} \tag{3.6}$$

In order to prove the theorem, we need to prepare several lemmas. We notice that (3.6) can be rewritten as

$$X_n(t) = x_0 + \int_0^t f(s, \widetilde{X}_n(s))ds + \sum_{k=1}^{l} \int_0^t g^k(s, \widetilde{X}_n(s))dB^k(s). \tag{3.7}$$

Lemma 3.3. *Let* (H3.1) *hold. Assume that* $f(t, x)$ *and* $g^k(t, x)$ $(k = 1, \ldots, l)$ *satisfy the linear growth condition, i.e., there exists a positive constant* L *such that for all* $x \in \mathcal{K}(\mathbf{R}^d), \forall t \in I$,

$$\max\left(d_H^2(f(t,x), \{0\}), \|g^k(t,x)\|^2\right) \leq L(1 + d_H^2(x, \{0\})) \text{ a.s.} \tag{3.8}$$

Then there exists a $\bar{C} > 0$ *such that*

$$Ed_H^2(X_n(t), \widetilde{X}_n(t)) \leq \frac{\bar{C}}{n}, \quad \forall n \in \mathbf{N}.$$

Proof. By virtue of the definition of $\widetilde{X}_n(t)$, we have $\widetilde{X}_n(t) \in \mathcal{L}^2(I \times \Omega, \mathcal{N}; \mathcal{K}(\mathbf{R}^d))$. Thus, we have:

- the composition $f(\cdot, \cdot, \widetilde{X}_n(\cdot, \cdot)) : I \times \Omega \to \mathcal{K}(\mathbf{R}^d)$ is nonanticipating set-valued stochastic process, and the composition $g^k(\cdot, \cdot, \widetilde{X}_n(\cdot, \cdot)) : I \times \Omega \to \mathbf{R}^d$ is nonanticipating $\mathbf{R}^d$-valued stochastic process,
- the composition $f(\cdot, \cdot, \widetilde{X}_n(\cdot, \cdot)) \in \mathcal{L}^2(I \times \Omega, \mathcal{N}; \mathcal{K}(\mathbf{R}^d))$, $g^k(\cdot, \cdot, \widetilde{X}_n(\cdot, \cdot)) \in L^2(I \times \Omega, \mathcal{N}; \mathbf{R}^d)$, $k = 1, \ldots, l$,
- due to Propositions 2.1 and 2.3 that the process X_n defined as in (3.6) belongs to $\mathcal{L}^2(I \times \Omega, \mathcal{N}; \mathcal{K}(\mathbf{R}^d))$, and is d_H-continuous.

In terms of Propositions 2.2 and 2.4, it follows from (3.7) and (3.8) that

$$\begin{aligned} &E \sup_{0\leq s\leq t} d_H^2(X_n(s), \{0\}) \\ &\quad \leq (l+2)\Big[Ed_H^2(x_0, \{0\}) + E \sup_{0\leq s\leq t} d_H^2(\int_0^s f(u, \widetilde{X}_n(u))du, \{0\}) \end{aligned}$$

$$+\sum_{k=1}^{l} E \sup_{0\le s\le t} \| \int_0^s g^k(u,\widetilde{X}_n(u))dB^k(u)\|^2 \Big]$$

$$\le (l+2)\left[Ed_H^2(x_0,\{0\})+tL\int_0^t (1+Ed_H^2(\widetilde{X}_n(u),\{0\}))du\right.$$

$$\left.+4lL\int_0^t (1+Ed_H^2(\widetilde{X}_n(u),\{0\}))du\right]$$

$$\le c_1+c_2\int_0^t E\sup_{0\le s\le u} d_H^2(X_n(s),\{0\})du,$$

where $c_1=(l+2)[Ed_H^2(x_0,\{0\})+T^2L+4lLT]$, $c_2=(l+2)(T+4l)L$, the last inequality is from the definition of the simple function $\widetilde{X}_n(t)$. Thus, by using Gronwall inequality, we obtain

$$E\sup_{0\le s\le t} d_H^2(X_n(s),\{0\})\le c_1e^{c_2t},\quad \forall t\in I. \tag{3.9}$$

From Propositions 2.2-2.4, (3.8) and (3.9), we therefore have that, if $0\le s\le t\le T$, $t-s\le \frac{1}{n}$,

$$Ed_H^2(X_n(t),X_n(s))$$

$$=Ed_H^2\left(\int_s^t f(u,\widetilde{X}_n(u))du+\sum_{k=1}^{l}\int_s^t g^k(u,\widetilde{X}_n(u))dB^k(u),\{0\}\right)$$

$$\le (l+1)\left[Ed_H^2(\int_s^t f(u,\widetilde{X}_n(u))du,\{0\})\right.$$

$$\left.+\sum_{k=1}^{l}E\left\|\int_s^t g^k(u,\widetilde{X}_n(u))dB^k(u)\right\|^2\right]$$

$$\le (l+1)\left[(t-s)\int_s^t Ed_H^2(f(u,\widetilde{X}_n(u)),\{0\})du\right.$$

$$\left.+4\sum_{k=1}^{l}E\int_s^t \left\|g^k(u,\widetilde{X}_n(u))\right\|^2du\right]$$

$$\le (l+1)L\left[(t-s)\int_s^t(1+Ed_H^2(\widetilde{X}_n(u),\{0\}))du+4l\int_s^t(1+Ed_H^2(\widetilde{X}_n(u),\{0\}))du\right]$$

$$\le (l+1)L\left[(t-s)^2+4l(t-s)+(t-s+4l)\int_s^t Ed_H^2(\widetilde{X}_n(u),\{0\})du\right]$$

$$\le (l+1)L[T+4l+(T+4l)c_1e^{c_2T}](t-s):=\bar{C}(t-s)\le\frac{\bar{C}}{n}.$$

Thus, the required inequality follows by the definition of $\widetilde{X}_n(t)$. The proof is complete. □

Lemma 3.4. *Let* (H3.1)-(H3.3) *hold. Then we have*

$$E\sup_{t\in I} d_H^2(X_m(t),X_n(t))\to 0 \text{ as } m,n\to\infty.$$

Proof. By the definition of $K(u)$, we can choose two positive constants κ and ρ such that $K(u) \leq \kappa + \rho u, \forall u \geq 0$. From (H3.2), we obtain

$$\begin{aligned} d_H^2(f(t,x),\{0\}) &\leq 2(d_H^2(f(t,x),f(t,\{0\})) + d_H^2(f(t,\{0\}),\{0\})) \\ &\leq 2CK(d_H^2(x,\{0\})) + 2C \leq 2C\kappa + 2C + 2C\rho d_H^2(x,\{0\}), \end{aligned}$$

which shows that $f(t,x)$ satisfies the linear growth condition (3.8). Similarly, $g^k(t,x)$, $(k = 1 \ldots, l)$, satisfy (3.8). Therefore, we have

$$\begin{aligned} &E \sup_{0\leq s\leq t} d_H^2(X_m(s), X_n(s)) \\ &= E \sup_{0\leq s\leq t} d_H^2\left(\int_0^s f(u,\widetilde{X}_m(u))du + \sum_{k=1}^l \int_0^s g^k(u,\widetilde{X}_m(u))dB^k(u), \int_0^s f(u,\widetilde{X}_n(u))du \right. \\ &\quad \left. + \sum_{k=1}^l \int_0^s g^k(u,\widetilde{X}_n(u))dB^k(u)\right) \\ &\leq (1+l)E \sup_{0\leq s\leq t}\left[d_H^2\left(\int_0^s f(u,\widetilde{X}_m(u))du, \int_0^s f(u,\widetilde{X}_n(u))du\right)\right. \\ &\quad \left. + \sum_{k=1}^l \left\|\int_0^s g^k(u,\widetilde{X}_m(u))dB^k(u)\right\|^2\right] \\ &\leq (1+l)t\int_0^t Ed_H^2\left(f(u,\widetilde{X}_m(u)), f(u,\widetilde{X}_n(u))\right)du \\ &\quad + 4(1+l)\sum_{k=1}^l \int_0^t E\|g^k(u,\widetilde{X}_m(u))\|^2 du := I_1 + I_2. \end{aligned}$$

From (H3.3) and the concavity of the function $K(u)$, we deduce

$$\begin{aligned} I_1 &\leq 3t(1+l)E\int_0^t [d_H^2(f(u,\widetilde{X}_m(u)), f(u,X_m(u))) \\ &\quad + d_H^2(f(u,X_m(u)), f(u,X_n(u))) + d_H^2(f(u,X_n(u)), f(u,\widetilde{X}_n(u)))]du \\ &\leq 3(1+l)TC\int_0^t \Big[EK(d_H^2(\widetilde{X}_m(u),X_m(u))) \\ &\quad + EK(d_H^2(X_m(u),X_n(u))) + EK(d_H^2(X_n(u),\widetilde{X}_n(u)))\Big]du \\ &\leq 3(1+l)TC\int_0^t \Big[K(Ed_H^2(\widetilde{X}_m(u),X_m(u))) \\ &\quad + K(Ed_H^2(X_m(u),X_n(u))) + K(Ed_H^2(X_n(u),\widetilde{X}_n(u)))\Big]du, \end{aligned}$$

which, from Lemma 3.3 and the function $K(u)$ being increasing, shows that

$$I_1 \leq 3(1+l)T^2C\left(K\left(\frac{\bar{C}}{m}\right) + K\left(\frac{\bar{C}}{n}\right)\right) + 3(1+l)TC\int_0^t K(Ed_H^2(X_m(u),X_n(u)))du.$$

Similarly, we have

$$\begin{aligned}
I_2 \le\ & 12(1+l)\sum_{k=1}^{l}\int_0^t E[\|g^k(u,\widetilde{X}_m(u)) - g^k(u,X_m(u))\|^2 \\
& + \|g^k(u,X_m(u)) - g^k(u,X_n(u))\|^2 \\
& + \|g^k(u,X_n(u)) - g^k(u,\widetilde{X}_n(u))\|^2]du \\
\le\ & 12(1+l)lC\int_0^t \Big[EK(d_H^2(\widetilde{X}_m(u),X_m(u))) \\
& + EK(d_H^2(X_m(u),X_n(u))) \\
& + EK(d_H^2(X_n(u),\widetilde{X}_n(u)))\Big]du \\
\le\ & 12(1+l)lC\int_0^t \Big[K(Ed_H^2(\widetilde{X}_m(u),X_m(u))) \\
& + K(Ed_H^2(X_m(u),X_n(u))) \\
& + K(Ed_H^2(X_n(u),\widetilde{X}_n(u)))\Big]du \\
\le\ & 12(1+l)lCT\left(K(\frac{\bar{C}}{m}) + K(\frac{\bar{C}}{n})\right) \\
& + 12(1+l)lC\int_0^t K(Ed_H^2(X_m(u),X_n(u)))du.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
E\sup_{0\le s\le t} d_H^2(X_m(s),X_n(s)) &\le I_1 + I_2 \\
&\le C_1^{m,n} + C_2\int_0^t K(Ed_H^2(X_m(u),X_n(u)))du \\
&\le C_1^{m,n} + C_2\int_0^t K(E\sup_{0\le s\le u} d_H^2(X_m(s),X_n(s)))du,
\end{aligned}$$

where

$$\begin{aligned}
C_1^{m,n} &= (3(1+l)T^2C + 12(1+l)lTC)\left(K(\frac{\bar{C}}{m}) + K(\frac{\bar{C}}{n})\right), \\
C_2 &= 3(1+l)TC + 12(1+l)lC.
\end{aligned}$$

Obviously, $C_1^{m,n} \to 0$ as $m,n\to\infty$.

By Bihari's inequality (see Lemma 2.6), we have

$$E\sup_{0\le s\le t} d_H^2(X_m(s),X_n(s)) \le G^{-1}(G(C_1^{m,n}) + C_2 t), \quad \forall t\in I.$$

Due to the properties of the functions G and G^{-1}, we deduce

$$G(C_1^{m,n}) + C_2T \to -\infty \text{ as } m,n\to\infty,$$

from which we know

$$E\sup_{s\in I} d_H^2(X_m(s), X_n(s)) \le G^{-1}(G(C_1^{m,n}) + C_2T) \to 0 \quad \text{as } m, n \to \infty.$$

Hence, the proof is complete. □

We now can start to prove Theorem 3.2.

Proof of Theorem 3.2. From the definition of $\widetilde{X}_n(t)$, Propositions 2.1 and 2.3 we know that the process X_n defined as in (3.6) belongs to $\mathcal{L}^2(I\times\Omega, \mathcal{N}; \mathcal{K}(\mathbf{R}^d))$, and is d_H-continuous.

By Lemma 3.4, we know that $X_n(t)$ is a Cauchy sequence in $\mathcal{L}^2(I\times\Omega, \mathcal{N}; \mathcal{K}(\mathbf{R}^d))$. Thus, there exists a $X(t,\omega) \equiv X(t)$ such that $X_n(t) \overset{I\ P,1}{\to} X(t)$ and (3.5) holds.

Observing that P-a.s. for every $t \in I$ it holds

$$d_H\left(X_n(t,\omega), X(t,\omega)\right) \to 0 \quad \text{as } n \to \infty,$$

we deduce that $X_n(t,\cdot) : \Omega \to \mathcal{K}(\mathbf{R}^d)$ is an $\mathcal{A}_t$-measurable multifunction. Thus, X is a continuous $\{A\}_{t\in I}$-adapted set-valued stochastic process, and hence nonanticipating.

As $X_n \in \mathcal{L}^2(I\times\Omega, \mathcal{N}; \mathcal{K}(\mathbf{R}^d))$, we know that for every fixed $t \in I$ the set-valued random variable $X_n(t) \in \mathcal{L}^2(\Omega, \mathcal{A}, P; \mathcal{K}(\mathbf{R}^d))$, which, from (3.9), implies that $E\sup_{t\in I} d_H^2(X(t), \{0\}) < \infty$. Thus (3.4) holds. Further, $E\int_0^T d_H^2(X(t), \{0\})dt \le T\sup_{t\in I} E d_\infty^2(X(t), \{0\}) < \infty$, which means that $X \in \mathcal{L}^2(I\times\Omega, \mathcal{N}; \mathcal{K}(\mathbf{R}^d))$.

In what follows we shall show that X is a solution to equation (3.1). Since the function $K(u)$ is increasing and concave, due to (H3.3), Propositions 2.2 and 2.4, we have

$$\begin{aligned}
(3.10)\qquad & E\sup_{u\in[0,t]} d_H^2\left(\int_0^u f(s,\widetilde{X}_n(s))ds, \int_0^u f(s,X(s))ds\right) \\
&\quad + \sum_{k=1}^{l} E\sup_{u\in[0,t]}\left\|\int_0^u g^k(s,\widetilde{X}_n(s))dB^k(s) - \int_0^u g^k(s,X(s))dB^k(s)\right\|^2 \\
&\le 2\left[E\sup_{u\in[0,t]} d_H^2\left(\int_0^u f(s,\widetilde{X}_n(s))ds, \int_0^u f(s,X_n(s))ds\right)\right. \\
&\quad + E\sup_{u\in[0,t]} d_H^2\left(\int_0^u f(s,X_n(s))ds, \int_0^u f(s,X(s))ds\right) \\
&\quad + \sum_{k=1}^{l} E\sup_{u\in[0,t]}\left\|\int_0^u g^k(s,\widetilde{X}_n(s))dB^k(s) - \int_0^u g^k(s,X_n(s))dB^k(s)\right\|^2 \\
&\quad \left. + \sum_{k=1}^{l} E\sup_{u\in[0,t]}\left\|\int_0^u g^k(s,X_n(s))dB^k(s) - \int_0^u g^k(s,X(s))dB^k(s)\right\|^2\right] \\
&\le 2(t+4l)C\int_0^t \left[E\sup_{u\in[0,s]} K\left(d_H^2(\widetilde{X}_n(u), X_n(u))\right)\right.
\end{aligned}$$

$$
\begin{aligned}
&+E\sup_{u\in[0,s]}K\left(d_H^2(X_n(u),X(u))\right)\Big]\,ds\\
&\le 2(T+4l)C\int_0^T K\left(E\sup_{u\in[0,s]}d_H^2(\widetilde{X}_n(u),X_n(u))\right)ds\\
&\quad+2(T+4l)C\int_0^t K\left(E\sup_{u\in[0,s]}d_H^2(X_n(u),X(u))\right)ds\\
&:= I_n.
\end{aligned}
$$

From Lemma 3.3 and the definition of $X(t)$, we deduce $I_n \to 0$. Hence, we have, for $\forall t \in I$,

$$
\begin{aligned}
&E\sup_{u\in[0,t]}d_H^2\left(\int_0^u f(s,\widetilde{X}_n(s))ds,\int_0^u f(s,X(s))ds\right)\to 0,\\
&\sum_{k=1}^l E\sup_{u\in[0,t]}\left\|\int_0^u g^k(s,\widetilde{X}_n(s))dB^k(s)-\int_0^u g^k(s,X(s))dB^k(s)\right\|^2\to 0 \text{ as } n\to\infty,
\end{aligned}
$$

from which we can let $n \to \infty$ in (3.6) to obtain that, for $\forall t \in I$,

$$
X(t)=x_0+\int_0^t f(s,X(s))ds+\sum_{k=1}^l\int_0^t g^k(s,X(s))dB^k(s).
$$

In other words, $X(t)$ is a solution to equation (3.1).

Finally, we prove that X is strongly unique. Let us assume that $X, Y : I \times \Omega \to \mathcal{K}(\mathbf{R}^d)$ are strong solutions to equations (3.1). Define $\xi(t) = E\sup_{u\in[0,t]} d_H^2(X(u), Y(u))$. From Propositions 2.2 and 2.4, we obtain

$$
\begin{aligned}
\xi(t)&\le(1+l)\left[E\sup_{u\in[0,t]}d_H^2\left(\int_0^u f(s,X(s))ds,\int_0^u f(s,Y(s))ds\right)\right.\\
&\quad\left.+\sum_{k=1}^l E\sup_{u\in[0,t]}\left\|\int_0^u g^k(s,X(s))dB^k(s)-\int_0^u g^k(s,Y(s))dB^k(s)\right\|^2\right]\\
&\le(1+l)\left[t\int_0^t Ed_H^2(f(s,X(s)),f(s,Y(s)))ds\right.\\
&\quad\left.+4\sum_{k=1}^l\int_0^t E\|g^k(s,X(s))-g^k(s,Y(s))ds\|^2\right]\\
&\le(1+l)(T+4l)C\int_0^t EK\left(d_H^2(X(s),Y(s))\right)ds\\
&\le(1+l)(T+4l)C\int_0^t E\left(\sup_{u\in[0,s]}K\left(d_H^2(X(u),Y(u))\right)\right)ds\\
&\le(1+l)(T+4l)C\int_0^t K\left(E\sup_{u\in[0,s]}d_H^2(X(u),Y(u))\right)ds
\end{aligned}
$$

$$= (1+l)(T+4l)C\int_0^t K(\xi(s))ds := C_1\int_0^t K(\xi(s))ds.$$

By Bihari's inequality (see Lemma 2.5), we have

$$\xi(t) \le G^{-1}(G(0+) + C_1 t) = 0,\ \forall t \ge 0,$$

which shows that $X(t) \overset{I\ P.1}{=} Y(t)$. Thus, the proof is complete. □

4. STABILITY OF SOLUTIONS TO SSDEs

A stability of the solution with respect to initial value is a desired property. This kind of stability ensures that in the case of replacement of x_0 by its approximate value y_0, the solution of equation with initial value y_0 does not differ much from the solution of equation with initial value x_0. We will show that such the property holds for solutions of SSDEs. Let X, Y denote strong solutions to SSDEs

$$dX(t) \overset{I\ P.1}{=} f(t, X(t))dt + g(t, X(t))dB(t), \quad X(0) \overset{P.1}{=} x_0, \tag{4.1}$$

$$dY(t) \overset{I\ P.1}{=} f(t, Y(t))dt + g(t, Y(t))dB(t), \quad Y(0) \overset{P.1}{=} y_0, \tag{4.2}$$

respectively.

Proposition 4.1. *Assume that* $x_0, y_0 \in \mathcal{L}^2(\Omega, \mathcal{A}_0, P; \mathcal{K}(\mathbf{R}^d))$, *and* $f : I \times \Omega \times \mathcal{K}(\mathbf{R}^d) \to \mathcal{K}(\mathbf{R}^d)$, $g^k : I \times \Omega \times \mathcal{K}(\mathbf{R}^d) \to \mathbf{R}^d$ $(k = 1, \ldots, l)$ *satisfy* (H3.1)–(H3.3). *Then we have*

$$E \sup_{u\in[0,t]} d_H^2(X(u), Y(u)) \le G^{-1}(G(C_0)) + C_1 t), \quad \forall t \in I,$$

where $C_0 = (2+l)Ed_H^2(x_0, y_0), C_1 = (2+l)(T+4l)C$. *Especially, if* $x_0 \overset{P.1}{=} y_0$, *then* $X(t) \overset{I\ P.1}{=} Y(t)$.

Proof. Let us assume that $X, Y : I \times \Omega \to \mathcal{K}(\mathbf{R}^d)$ are solutions to the equations (4.1) and (4.2), respectively. Define $r(t) = E \sup_{u\in[0,t]} d_H^2(X(u), Y(u))$. Since $K(u)$ is increasing and concave, from Propositions 2.2 and 2.4 we obtain

$$\begin{aligned}
r(t) \le (2+l)\Bigg[&Ed_H^2(x_0, y_0) + E\sup_{u\in[0,t]} d_H^2\left(\int_0^u f(s, X(s))ds, \int_0^u f(s, Y(s))ds\right) \\
&+ \sum_{k=1}^{l} E\sup_{u\in[0,t]}\left\|\int_0^u g^k(s, X(s))dB^k(s) - \int_0^u g^k(s, Y(s))dB^k(s)\right\|^2\Bigg] \\
\le (2+l)\Bigg[&Ed_H^2(x_0, y_0) + t\int_0^t Ed_H^2(f(s, X(s)), f(s, Y(s)))ds \\
&+4\sum_{k=1}^{l}\int_0^t E\|g^k(s, X(s)) - g^k(s, Y(s))\|^2 ds\Bigg]
\end{aligned}$$

$$\le (2+l)Ed_H^2(x_0,y_0) + (2+l)(T+4l)C\int_0^t EK\left(d_H^2(X(s),Y(s))\right)ds$$
$$\le (2+l)Ed_H^2(x_0,y_0)$$
$$+ (2+l)(T+4l)C\int_0^t EK\left(\sup_{u\in[0,s]} d_H^2(X(u),Y(u))\right)ds$$
$$\le (2+l)Ed_H^2(x_0,y_0)$$
$$+ (2+l)(T+4l)C\int_0^t K\left(E\sup_{u\in[0,s]} d_H^2(X(u),Y(u))\right)ds$$
$$= (2+l)Ed_H^2(x_0,y_0) + (2+l)(T+4l)C\int_0^t K(r(s))ds := C_0 + C_1\int_0^t K(r(s))ds.$$

By Lemma 2.5 we have

$$r(t) \le G^{-1}(G(C_0)+C_1 t),\ \forall t \ge 0.$$

If $x_0 \overset{P.1}{=} y_0$, then $C_0 = 0$. By the properties of the functions G and G^{-1}, we know $r(t) = 0$ which shows that $X(t) \overset{I\ P.1}{=} Y(t)$. Thus, the proof is complete. □

The exponential stability of stochastic differential equation driven by semimartingale is discussed in [29]. Now, the second kind of stability for strong solutions to SSDEs which is stability with respect to the equation coefficients f, g^k $(k = 1\dots, l)$ is explored. We will show that if approximations $f_n, g_n^1, \dots, g_n^l$ of the coefficients $f, g^1, \dots, g^l$ converge to the exact coefficients, then approximate solutions converge to the solution of the equation with exact coefficients, too. Therefore, let X, X_n denote solutions of the following SSDEs

$$dX(t) \overset{I\ P.1}{=} f(t,X(t))dt + g(t,X(t))dB(t), \quad X(0) \overset{P.1}{=} x_0, \tag{4.3}$$

$$dX_n(t) \overset{I\ P.1}{=} f_n(t,X_n(t))dt + g_n(t,X_n(t))dB(t), \quad X_n(0) \overset{P.1}{=} x_0, \tag{4.4}$$

respectively.

Proposition 4.2. *Let $x_0 \in \mathcal{L}^2(\Omega, \mathcal{A}_0, P; \mathcal{K}(\mathbf{R}^d))$, and $f, f_n : I\times\Omega\times\mathcal{K}(\mathbf{R}^d) \to \mathcal{K}(\mathbf{R}^d)$, $g^k, g_n^k : I\times\Omega\times\mathcal{K}(\mathbf{R}^d) \to \mathbf{R}^d$ $(n \in \mathbf{N},\ k = 1,\dots,l)$ satisfy* (H3.1)–H(3.3). *Assume that for every $x \in \mathcal{K}(\mathbf{R}^d)$ it holds*

$$E\int_0^T d_H^2(f_n(t,x), f(t,x))dt \to 0, \tag{4.5}$$

$$E\int_0^T \|g_n^k(t,x) - g^k(t,x))dt\|^2 dt \to 0, \quad k = 1,\dots,l, \tag{4.6}$$

as $n \to \infty$. Then, for the solutions $X, X_n : I\times\Omega \to \mathcal{K}(\mathbf{R}^d)$ of the equations (4.3), (4.4) *we have*

$$E\sup_{t\in I} d_H^2(X_n(t), X(t)) \to 0 \quad \text{as } n\to\infty.$$

Proof. The solutions X to (4.3) and X_n to (4.4) exist and are unique due to Theorem 3.2. By (H3.1)–H(3.3), Propositions 2.2 and 2.4 we deduce

$$\begin{aligned}
&E \sup_{v\in[0,t]} d_H^2(\int_0^v f_n(s,X_n(s))ds, \int_0^v f(s,X(s))ds)\\
&\le 2[tE\int_0^t d_H^2(f_n(s,X_n(s)), f_n(s,X(s)))\\
&\quad + tE\int_0^t d_H^2(f_n(s,X(s)), f(s,X(s)))ds]\\
&\le 2[CT\int_0^t EK(d_H^2(X_n(s),X(s)))ds\\
&\quad + TE\int_0^T d_H^2(f_n(s,X(s)), f(s,X(s)))ds],
\end{aligned}$$

and

$$\begin{aligned}
&E \sup_{v\in[0,t]} \left\| \int_0^v g_n^k(s,X_n(s))dB^k(s) - \int_0^v g^k(s,X(s))dB^k(s) \right\|^2\\
&\le 8[E\int_0^t \|g_n^k(s,X_n(s)) - g_n^k(s,X(s))\|^2 ds\\
&\quad + E\int_0^t \|g_n^k(s,X(s)) - g^k(s,X(s))\|^2 ds]\\
&\le 8[C\int_0^t EK(d_H^2(X_n(s),X(s)))\\
&\quad + \int_0^t E\|g_n^k(s,X(s)) - g^k(s,X(s))\|^2 ds],
\end{aligned}$$

from which, we have, for $t \in I$,

$$\begin{aligned}
&E \sup_{v\in[0,t]} d_H^2(X_n(v),X(v))\\
&\le (1+l)\left[E \sup_{v\in[0,t]} d_H^2\left(\int_0^v f_n(s,X_n(s))ds, \int_0^v f(s,X(s))ds\right)\right.\\
&\quad \left. + \sum_{k=1}^{l} E \sup_{v\in[0,t]} \left\| \int_0^v g_n^k(s,X_n(s))dB^k(s) - \int_0^v g^k(s,X(s))dB^k(s) \right\|^2\right]\\
&\le 2(1+l)\left[CT\int_0^t EK\left(d_H^2(X_n(s),X(s))\right)ds\right.\\
&\quad + T\int_0^T Ed_H^2(f_n(s,X(s)), f(s,X(s)))ds\\
&\quad + 4lC\int_0^t EK\left(d_H^2(X_n(s),X(s))\right)ds\\
&\quad \left. +4\sum_{k=1}^{l}\int_0^t E\|g_n^k(s,X(s)) - g^k(s,X(s))\|^2 ds\right]
\end{aligned}$$

$$\leq 2(1+l)\left[C(T+4l)\int_0^t EK\left(d_H^2(X_n(s),X(s))\right)ds\right.$$
$$+T\int_0^T Ed_H^2(f_n(s,X(s)),f(s,X(s)))ds$$
$$\left.+4\sum_{k=1}^{l}\int_0^T E\|g_n^k(s,X(s))-g^k(s,X(s))\|^2ds\right]$$
$$\leq 2(1+l)\left[C(T+4l)\int_0^t K\left(E\sup_{v\in[0,s]} d_H^2(X_n(v),X(v))\right)ds\right.$$
$$+T\int_0^T Ed_H^2(f_n(s,X(s)),f(s,X(s)))ds$$
$$\left.+4\sum_{k=1}^{l}\int_0^T E\|g_n^k(s,X(s))-g^k(s,X(s))\|^2ds\right]$$
$$:= C_1^n + C_2\int_0^t K\left(E\sup_{v\in[0,s]} d_H^2(X_n(v),X(v))\right)ds,$$

where

$$C_1^n = 2(1+l)\left[T\int_0^T Ed_H^2(f_n(s,X(s)),f(s,X(s)))ds\right.$$
$$\left.+4\sum_{k=1}^{l}\int_0^T E\|g_n^k(s,X(s))-g^k(s,X(s))\|^2ds\right],$$
$$C_2 = 2C(1+l)(T+4l).$$

Again by Bihari's inequality (see Lemma 2.5) we obtain

$$E\sup_{v\in[0,T]} d_H^2(X_n(v),X(v)) \leq G^{-1}(G(C_1^n)+C_2T),$$

Hence, from assumptions (4.5) and (4.6), we know that $C_1^n \to 0$ as $n\to\infty$, from which we have $G(C_1^n)\to-\infty$ as $n\to\infty$. By the definition of G^{-1}, we have $G^{-1}(G(C_1^n)+C_2T)\to 0$ as $n\to\infty$. Hence, we prove immediately the claim. □

5. CONCLUSION

In real dynamic systems, we are often faced with random experiments whose outcomes might be multi-valued. We analyze this phenomenon by using the set-valued calculus. The stochastic set differential equations characterize a large class of physically important dynamic systems which can be applied in such areas as control, economics and finance, etc. In this paper, we discuss the behavior of solutions to SSDEs disturbed by the Brownian motion with non-Lipschitzian coefficients. First, the existence and uniqueness theorem of solutions to the SSDEs of Itô type is proven by employing the well-known Cauchy-Maruyama approximation procedure (c.f. [29]).

Second, the stability of solutions to the SSDEs is discussed. Main mathematical tool is Bihari's inequality. Moreover, the present case can be in future extended to the SSDEs driven by one-dimensional continuous local martingales.

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