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EIGENVALUE CRITERIA ON EXISTENCE OF NONTRIVIAL SOLUTIONS FOR A CLASS OF FRACTIONAL BOUNDARY VALUE PROBLEM

JING CHEN[1,2] AND X. H. TANG[2]

[1]School of Mathematics and Computing Sciences, Hunan University of Science and Technology, Xiangtan, Hunan 411201, P. R. China
[2]School of Mathematical and Statistics, Central South University, Changsha, Hunan 410083, P. R. China
cjhnust@yahoo.com.cn

ABSTRACT. In this paper, we study the existence of nontrivial solutions for the following fractional boundary value problem

$$\begin{cases} D^{\nu}_{0+}u(t) + q(t)f(t,u(t)) = 0, & 0 < t < 1, n-1 < \nu \le n, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, & \\ [D^{\alpha}_{0+}u(t)]_{t=1} = 0, & 1 \le \alpha \le n-2, \end{cases}$$

where $n \in \mathbb{N}$ and $n > 3$, D^{ν}_{0+} is the standard Riemann-Liouville derivative, q may be singular at $t = 0, 1$. f may change sign. Our analysis relies on fixed point index theory and Leray-Schauder degree theory. Conditions are given by the growth behavior of $f(t,u)/u$ for u near 0 and $+\infty$ with respect to the first eigenvalue of the related linear operator. Several examples are presented to illustrate our results.

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1. INTRODUCTION

Consider the following fractional boundary value problem (BVP for short)

$$D^{\nu}_{0+}u(t) + q(t)f(t,u(t)) = 0, \quad t \in (0,1), n-1 < \nu \le n, \tag{1.1}$$

$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \tag{1.2}$$

$$[D^{\alpha}_{0+}u(t)]_{t=1} = 0, \quad 1 \le \alpha \le n-2, \tag{1.3}$$

where $n \in \mathbb{N}$ and $n > 3$, D^{ν}_{0+} is the standard Riemann-Liouville derivative, q may be singular at $t = 0$ and(or) $t = 1$. In this paper, we suppose several hypothses on q and f:

i) $q \in L^1[0,1] \cap C(0,1)$ with $q > 0$ on $(0,1)$;
ii) $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous.

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Fractional calculus has gained considerable popularity and importance for its applications in numerous widespread fields such as physics, mechanics, chemistry, biology science, engineering, control systems etc. With the wide uses for many mathematical models in science and engineering, fractional differential equations of various types have emerged as a new branch of applied mathematics. In the past three decades, many studies on fractional calculus and fractional differential equations have appeared. For details, see [1–7] etc. and the references therein.

Recently, the existence and multiplicity of positive solutions for fractional boundary value problems are established and posed (see [8–16]). In [13], C. S. Goodrich discussed the positive solutions for BVP (1.1) by Krasnosel'skii's fixed point theorem with the nonlinearity $f \in C([0,1]\times[0,\infty),[0,\infty))$ and growing sublinearly. Based upon the fixed point index theory on cone, Xu et al. ([15]) also considered the uniqueness of positive solutions for BVP (1.1) through establishing the lower and upper solutions and proving that the unique positive solution can be uniformly approximated by an iterative sequence under the assumptions that $f \in C([0,1]\times[0,\infty),(0,\infty))$ and $q \in C(0,1)\cap L(0,1)$ is nonnegative and may be singular at $t=0$ and (or) $t=1$.

But it is worth pointing out that the nonlinearity is usually assumed to be nonnegative in the literature available. It is natural to ask whether the similar existence results can be obtained if the conditions on the nonlinearity are relaxed, that is, it may change sign or even be unbounded from below. In [16], Wang et al. studied positive solutions for the following fractional boundary value problem with changing sign nonlinearity by applying Krasnosel'skii's fixed point theorem:

$$\begin{cases} D_{0+}^{\alpha}u(t)+\lambda f(t,u(t))=0, \quad 0<t<1,\\ u(0)=u'(0)=u(1)=0, \end{cases}$$

where $2<\alpha\le 3$ is a real number, D_{0+}^{α} is the standard Riemann-Liouville derivative, λ is a positive parameter.

In the present paper, motivated by [13, 14, 15, 16], we will use fixed point index theory combined with Leray-Schauder degree to discuss the existence results of nontrivial solutions for BVP (1.1). Compared to [13, 15], f is not necessary to be nonnegative but bounded even unbounded from below. Our approach and results in this paper improve and extend the corresponding ones in [13, 14, 15].

The remainder of this paper is organized as follows. In section 2, some preliminary results are presented. Various criteria on existence of nontrivial solutions for BVP (1.1) are established and their proofs are given in section 3. Finally, in section 4, some examples are given to illustrate the main results.

2. PRELIMINARIES

Definition 2.1. The fractional derivative of order $\alpha > 0$ of a continuous function $f : (0, +\infty) \to \mathbb{R}$ is given by

$$D_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \left(\int_0^t (t-s)^{n-\alpha-1} f(s) ds \right)$$

where $n-1 \leq \alpha < n$ and $n \in \mathbb{N}$, provided that the right-hand side is pointwise defined on $(0, +\infty)$.

Let $E = C^n[0,1]$ be equipped with the ordering $u \leq v$ if $u(t) \leq v(t)$ for all $t \in [0,1]$, and the usual maximum norm $\|u\| = \max\limits_{0 \leq t \leq 1} |u(t)|$.

Define the cone $P \subset E$ by

$$P = \{u \in E \mid u(t) \geq 0,\ t \in [0,1]\}.$$

Lemma 2.1 ([13])**.** *Let $g \in C^n[0,1]$ be given. Then the unique solution to problem $-D_{0^+}^{\nu} u(t) = g(t)$ together with the boundary conditions* (1.2) *and* (1.3) *is*

$$u(t) = \int_0^1 G(t,s) g(s) ds,$$

where

$$G(t,s) = \begin{cases} \dfrac{t^{\nu-1}(1-s)^{\nu-\alpha-1} - (t-s)^{\nu-1}}{\Gamma(\nu)}, & 0 \leq s \leq t \leq 1, \\ \dfrac{t^{\nu-1}(1-s)^{\nu-\alpha-1}}{\Gamma(\nu)}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{2.1}$$

Observing (2.1), *it is clear that $G(t,s)$ is continuous and nonnegative for $s,t \in [0,1]$.*

Lemma 2.2. *Let $T : P \to E$ be an integral operator defined by*

$$T\varphi(t) = \int_0^1 G(t,s) q(s) \varphi(s) ds, \tag{2.2}$$

then $T : P \to P$ is completely continuous.

Proof of Lemma 2.2. Since $G(t,s) \geq 0$ and $q > 0$, then $T\varphi(t) \geq 0$ for all $\varphi \in P$. Hence if $\varphi \in P$ then $T\varphi \in P$.

Let $\Omega \subset P$ be bounded, i.e. there exists a positive constant $M > 0$ such that $\|\varphi\| \leq M$, for all $\varphi \in \Omega$.

By the use of the monotonicity of $G(t,s)$ (see Theorem 3.2 of [13]), one has

$$\max_{0 \leq t \leq 1} G(t,s) = G(1,s) = \frac{1}{\Gamma(\nu)}[(1-s)^{\nu-\alpha-1} - (1-s)^{\nu-1}], \quad \text{for each } s \in [0,1].$$

For $\varphi \in \Omega$, we have

$$\begin{aligned}
|T\varphi(t)| \le & \left|\int_0^1 G(t,s)q(s)\varphi(s)ds\right| \\
\le & \int_0^1 G(1,s)q(s)\varphi(s)ds \\
\le & \frac{M}{\Gamma(\nu)}\int_0^1 [(1-s)^{\nu-\alpha-1}-(1-s)^{\nu-1}]q(s)ds \\
\le & \frac{M}{\Gamma(\nu)}\int_0^1 q(s)ds < +\infty.
\end{aligned}$$

So, $T(\Omega)$ is bounded.

On the other hand, given $\varepsilon > 0$, setting

$$\delta = \min\left\{1, \left(\frac{\Gamma(\nu)\varepsilon}{2^{\nu+1}M\int_0^1 q(s)ds}\right)^{\frac{1}{\nu-1}}\right\},$$

then for every $\varphi \in \Omega$, $t_1, t_2 \in [0,1]$, $t_1 < t_2$ and $t_2 - t_1 < \delta$, one has $|T\varphi(t_2) - T\varphi(t_1)| < \varepsilon$. That is to say, $T(\Omega)$ is equicontinuous.

In fact,

$$\begin{aligned}
&|T\varphi(t_2) - T\varphi(t_1)| \\
&= \left|\int_0^1 G(t_2,s)q(s)\varphi(s)ds - \int_0^1 G(t_1,s)q(s)\varphi(s)ds\right| \\
&\le \int_0^{t_1} |G(t_2,s)-G(t_1,s)|\, q(s)\varphi(s)ds + \int_{t_2}^1 |G(t_2,s)-G(t_1,s)|\, q(s)\varphi(s)ds \\
&\quad + \int_{t_1}^{t_2} |G(t_2,s)-G(t_1,s)| q(s)\varphi(s)ds \\
&\le \frac{M}{\Gamma(\nu)}\Bigg\{\int_0^{t_1} \left|(1-s)^{\nu-\alpha-1}({t_2}^{\nu-1}-{t_1}^{\nu-1}) - (t_2-s)^{\nu-1} + (t_1-s)^{\nu-1}\right| q(s)ds \\
&\quad + \int_{t_2}^1 (1-s)^{\nu-\alpha-1}({t_2}^{\nu-1}-{t_1}^{\nu-1})q(s)ds \\
&\quad + \int_{t_1}^{t_2} \left|(1-s)^{\nu-\alpha-1}({t_2}^{\nu-1}-{t_1}^{\nu-1}) - (t_2-s)^{\nu-1}\right| q(s)ds\Bigg\} \\
&\le \frac{M}{\Gamma(\nu)}\Bigg\{\int_0^1 (1-s)^{\nu-\alpha-1}({t_2}^{\nu-1}-{t_1}^{\nu-1})q(s)ds + \int_0^{t_1} \left[(t_2-s)^{\nu-1}-(t_1-s)^{\nu-1}\right] q(s)ds \\
&\quad + \int_{t_1}^{t_2} (t_2-s)^{\nu-1}q(s)ds\Bigg\} \\
&\le \frac{M}{\Gamma(\nu)}\left\{({t_2}^{\nu-1}-{t_1}^{\nu-1})\int_0^1 (1-s)^{\nu-\alpha-1}q(s)ds + (t_2-t_1)^{\nu-1}\int_0^{t_2} q(s)ds\right\}
\end{aligned}$$

$$\leq \frac{M\int_0^1 q(s)ds}{\Gamma(\nu)}\left[{t_2}^{\nu-1} - {t_1}^{\nu-1} + (t_2 - t_1)^{\nu-1}\right].$$

In the following, we divide the proof into two cases.

Case 1. $\delta \leq t_1 < t_2 < 1$, with the use of mean value theorem,

$$ {t_2}^{\nu-1} - {t_1}^{\nu-1} \leq \frac{\nu-1}{\delta^{2-\nu}}(t_2 - t_1) \leq (\nu-1)\delta^{\nu-1}.$$

Case 2. $0 \leq t_1 < \delta$, $t_2 < 2\delta$,

$$ {t_2}^{\nu-1} - {t_1}^{\nu-1} \leq {t_2}^{\nu-1} < (2\delta)^{\nu-1}.$$

Consequently, we have

$$\max\left\{{t_2}^{\nu-1} - {t_1}^{\nu-1}, (t_2 - t_1)^{\nu-1}\right\} < 2^\nu\delta^{\nu-1},$$

and

$$|T\varphi(t_2) - T\varphi(t_1)| < \frac{M\int_0^1 q(s)ds}{\Gamma(\nu)} 2\cdot 2^\nu\delta^{\nu-1} \leq \varepsilon.$$

By means of the Arzela-Ascoli theorem, we have $T : P \to P$ is completely continuous. The proof is complete. □

Lemma 2.3 ([18]). *Suppose that $T : C[0,1] \to C[0,1]$ is a completely continuous linear operator and $T(P) \subset P$. If there exist $\psi \in C[0,1] \setminus (-P)$ and a constant $c > 0$ such that $cT\psi \geq \psi$, then the spectral radius $r(T) \neq 0$ and T has a positive eigenfunction φ_1 corresponding to its first eigenvalue $\lambda_1 = (r(T))^{-1}$.*

Lemma 2.4. *Suppose T is defined by (2.2), then the spectral radius $r(T) > 0$ and T has a positive eigenfunction φ_1 corresponding to its first eigenvalue $\lambda_1 = (r(T))^{-1}$.*

Proof of Lemma 2.4.. The proof is similar to that of Lemma 4.4 in [14] and is omitted. □

Lemma 2.5 ([19]). *Let P be a cone in a Banach space X, and $\Omega(P)$ be a bounded open set in P. Suppose that $T : \Omega(P) \to P$ is completely continuous operator. If there exists $u_0 \in P \setminus \{\theta\}$ such that*

$$u - Tu \neq \mu u_0,\ \forall u \in \partial\Omega(P),\ \mu \geq 0$$

then the fixed point index $i(T, \Omega(P), P) = 0$.

Lemma 2.6 ([19]). *Let P be a cone in a Banach space X. Suppose that $T : P \to P$ is a completely continuous operator. If there exists a bounded open set $\Omega(P)$ such that each solution of*

$$u = \sigma Tu, \quad u \in P,\ \sigma \in [0,1]$$

satisfies $u \in \Omega(P)$, then the fixed point index $i(T, \Omega(P), P) = 1$.

3. MAIN RESULTS AND PROOFS

Define

$$A\varphi(t)=\int_0^1 G(t,s)q(s)f(s,\varphi(s))ds,\quad \varphi\in E, \tag{3.1}$$

Similarly, it is clear that $A:E\to E$ is a completely continuous operator. By Lemma 2.1, it means that the solutions for BVP (1.1)–(1.3) correspond to the fixed points of A.

Theorem 3.1. *Suppose there exists a real function $b\in C([0,1],(0,+\infty))$, such that the following conditions are met:*

(A1) $f(t,u)\geq -b(t)$;

(A2) $\liminf\limits_{u\to 0}\min\limits_{t\in[0,1]}\dfrac{f(t,u)}{|u|}>\lambda_1$;

(A3) $\limsup\limits_{u\to+\infty}\max\limits_{t\in[0,1]}\dfrac{f(t,u)}{u}<\lambda_1$,

where λ_1 is the first eigenvalue of the operator T defined by (2.2). *Then BVP* (1.1)–(1.3) *has at least a nontrivial solution.*

Proof of Theorem 3.1. By (A2), there exists $r_1>0$, such that

$$f(t,u)\geq\lambda_1|u|,$$

for all $|u|\leq r_1, t\in[0,1]$.

For $\varphi\in\bar{B}_{r_1}$, we have $A(\bar{B}_{r_1})\subset P$. In fact,

$$(A\varphi)(t)\geq\lambda_1\int_0^1 G(t,s)q(s)|\varphi(s)|ds\geq 0,$$

for all $t\in[0,1]$.

For every $\varphi\in\partial B_{r_1}\cap P$, we have

$$(A\varphi)(t)\geq\lambda_1\int_0^1 G(t,s)q(s)\varphi(s)ds=\lambda_1(T\varphi)(t)\geq 0. \tag{3.2}$$

Suppose that A has no fixed point on $\partial B_{r_1}\cap P$ (otherwise, the proof is completed). Let φ^* be the positive eigenfunction of T corresponding to λ_1, thus $\lambda_1 T\varphi^*=\varphi^*$.

Now we claim

$$\varphi-A\varphi\neq\mu\varphi^*,\quad \text{for all } \varphi\in\partial B_{r_1}\cap P,\quad \mu\geq 0. \tag{3.3}$$

Indeed, if the claim is false, then there exist $\varphi_1\in\partial B_{r_1}\cap P$ and $\tau_0\geq 0$, such that

$$\varphi_1-A\varphi_1=\tau_0\varphi^*,$$

and thus $\varphi_1=A\varphi_1+\tau_0\varphi^*\geq\tau_0\varphi^*$.

Let $\tau^*=\sup\{\tau|\ \varphi_1\geq\tau\varphi^*\}$. Clearly, $+\infty>\tau^*\geq\tau_0>0$ and $\varphi_1\geq\tau^*\varphi^*$.

In view that T is a linear operator and $T(P) \subset P$, we have

$$\lambda_1 T\varphi_1 \geq \tau^* \lambda_1 T\varphi^* = \tau^* \varphi^*.$$

Therefore, by (3.2),

$$\varphi_1 = A\varphi_1 + \tau_0 \varphi^* \geq \lambda_1 T\varphi_1 + \tau_0 \varphi^* \geq \tau^* \varphi^* + \tau_0 \varphi^* = (\tau^* + \tau_0)\varphi^*,$$

which contradicts the definition of φ^*. Hence (3.3) is ture and by Lemma 2.5 we have $i(A, B_{r_1} \cap P, P) = 0$.

Since cone P of Banach space E is a retract of E and $A(\bar{B}_{r_1}) \subset P$, by the permanence of the fixed point index, we have

$$i(A, B_{r_1}, E) = i(A, B_{r_1} \cap P, P) = 0.$$

On the other hand, by the definition of fixed point index,

$$i(A, B_{r_1}, E) = \deg(I - A, B_{r_1}, \theta) = 0.$$

This means that

$$\deg(I - A, B_{r_1}, \theta) = i(A, B_{r_1} \cap P, P) = 0.$$

Define the function

$$\tilde{\varphi}(t) = b(t) \int_0^1 G(t,s) q(s) ds.$$

Clearly, $\tilde{\varphi} \in P$ and $A : E \to P - \tilde{\varphi}$. Also define the integral operator

$$\widetilde{A}\varphi = A(\varphi - \tilde{\varphi}) + \tilde{\varphi}.$$

Obviously, $\widetilde{A} : E \to P$.

By (A3), there exist $0 < \sigma < 1$ and $r_2 > 0$, such that $f(t,u) \leq \sigma \lambda_1 u$, for all $u > r_2, t \in [0,1]$, where $r_2 > r_1 + \|\tilde{\varphi}\|$. Let

(3.4) $$T_1 \varphi = \sigma \lambda_1 T\varphi, \ \varphi \in E.$$

Then $T_1 : E \to E$ is a bounded linear operator and $T_1(P) \subset P$.

Denote

$$\frac{1}{2}M = \max\left\{ c \int_0^1 G(1,s) q(s) ds, \|\tilde{\varphi}\| \right\},$$

where

$$c := \max_{\substack{t \in [0,1] \\ -r_2 \leq u \leq r_2}} |f(t,u)|.$$

Let $W = \{\varphi \in P | \ \varphi = \mu \widetilde{A}\varphi, \ 0 \leq \mu \leq 1\}$, we show that W is bounded.

For $\varphi \in W$, denote $e(\varphi) = \{t \in [0,1] \ \big| \ \varphi(t) - \tilde{\varphi}(t) > r_2\}$, then for all $\varphi \in W$,

$$\begin{aligned} \varphi(t) = \mu(\widetilde{A}\varphi)(t) &\leq \int_0^1 G(t,s) q(s) f(s, \varphi(s) - \tilde{\varphi}(s)) ds + \tilde{\varphi}(t) \\ &= \int_{e(\varphi)} G(t,s) q(s) f(s, \varphi(s) - \tilde{\varphi}(s)) ds \end{aligned}$$

$$
\begin{aligned}
&+\int_{[0,1]\setminus e(\varphi)} G(t,s)q(s)f(s,\varphi(s)-\tilde{\varphi}(s))ds+\tilde{\varphi}(t)\\
&\le \sigma\lambda_1\int_{e(\varphi)} G(t,s)q(s)(\varphi(s)-\tilde{\varphi}(s))ds + c\int_0^1 G(t,s)q(s)ds+\|\tilde{\varphi}\|\\
&\le \sigma\lambda_1\int_0^1 G(t,s)q(s)\varphi(s)ds + c\int_0^1 G(t,s)q(s)ds+\|\tilde{\varphi}\|\\
&\le \sigma\lambda_1\int_0^1 G(t,s)q(s)\varphi(s)ds + M, \quad t\in[0,1],
\end{aligned}
$$

hence

$$(I-T_1)\varphi \le M.$$

Since λ_1 is the first eigenvalue of T and $0<\sigma<1$, by (3.4), then $r(T_1)=\sigma$. Therefore, the inverse operator $(I-T_1)^{-1}$ exists and

$$(I-T_1)^{-1} = I + T_1 + T_1^2 + \cdots + T_1^n + \cdots .$$

So we have $\|\varphi\| \le \|(I-T_1)^{-1}\|M$, $t\in[0,1]$ and W is bounded.

Choose $r_3 > \max\{r_2, \sup\{W\}+\|\tilde{\varphi}\|\}$, then by Lemma 2.6, we have

$$i(\widetilde{A}, B_{r_3}\cap P, P) = 1,$$

then

$$\deg(I-\widetilde{A}, B_{r_3}, \theta) = i(\widetilde{A}, B_{r_3}\cap P, P) = 1. \tag{3.5}$$

Define

$$H(t,\varphi) = A(\varphi - t\tilde{\varphi}) + t\tilde{\varphi}.$$

It is easy to know $H(t,\varphi)$ is completely continuous and suppose

$$\varphi - H(t,\varphi) \ne \theta, \quad \forall (t,\varphi)\in(0,1)\times\partial B_{r_3}.$$

In fact, if there exists $(t_0,\varphi_0)\in[0,1]\times\partial B_{r_3}$ such that $H(t_0,\varphi_0)=\varphi_0$, then

$$A(\varphi_0 - t_0\tilde{\varphi}) = \varphi_0 - t_0\tilde{\varphi}.$$

This shows that A has a fixed point $\varphi_0 - t_0\tilde{\varphi}$.

In view of

$$\|\varphi_0 - t_0\tilde{\varphi}\| \ge \|\varphi_0\| - t_0\|\tilde{\varphi}\| \ge r_3 - r_2,$$

hence, $\varphi_0 - t_0\tilde{\varphi}$ is a nontrivial solution of BVP (1.1)–(1.3).

If

$$H(t_0,\varphi_0) \ne \varphi_0,$$

using (3.5) and homotopy invariance property of Leray-Schauder degree, one has

$$
\begin{aligned}
\deg(I-A, B_{r_3}, \theta) &= \deg(I-H(0,\cdot), B_{r_3}, \theta)\\
&= \deg(I-H(1,\cdot), B_{r_3}, \theta)
\end{aligned}
$$

$$= \deg(I - \widetilde{A}, B_{r_3}, \theta)$$
$$= 1.$$

By the additivity and the solution property of Leray-Schauder degree, we have

$$\deg(I - A, B_{r_3} \backslash \bar{B}_{r_1}, \theta) = \deg(I - A, B_{r_3}, \theta) - \deg(I - A, B_{r_1}, \theta) = 1,$$

the operator A has a fixed point in $B_{r_3} \backslash \bar{B}_{r_1}$, this means that BVP (1.1)–(1.3) has at least a nontrivial solution. The proof is complete. □

Corollary 3.1. *Suppose that* (A2)–(A3) *hold and* $f(t,u)$ *satisfies the following assumption: there exists a real constant* $b^* > 0$*, such that*

$$f(t,u) \geq -\frac{b^*}{M^*}, \quad \forall\ u \geq -b^*, \quad t \in [0,1],$$

where

$$M^* := \max_{t\in[0,1]} \int_0^1 G(t,s)q(s)ds.$$

Then BVP (1.1)–(1.3) *has at least a nontrivial solution.*

Remark 3.1. Compared to Theorem 3.1, the result of Corollary 3.1 can be applied to some cases where f is unbounded from below, for example,

$$f(u) = \left(1 - \frac{2u^2}{1+u^2}\right) e^{1-u} \tag{3.6}$$

and $q(t) = 1 - t$, for $\nu = \frac{5}{2}$ and $\alpha = 1$ in BVP (1.1)–(1.3).

Proof of Corollary 3.1. Let $f_1 \in C([0,1] \times \mathbb{R}, \mathbb{R})$ and define

$$f_1(t,u) = \begin{cases} f(t,u), & u \geq -b^*, \\ f(t,-b^*), & u < -b^*, \end{cases}$$

and

$$(A_1\varphi)(t) = \int_0^1 G(t,s)q(s)f_1(s,\varphi(s))ds. \tag{3.7}$$

As a consequence of Theorem 3.1, A_1 has a nontrivial solution ψ. Combined with (3.7), we have

$$\psi \geq \frac{-b^*}{M^*} \int_0^1 G(t,s)q(s)ds \geq -b^*. \tag{3.8}$$

Notice (3.8), for every $t \in [0,1]$, we have

$$f_1(t,\psi) = f(t,\psi). \tag{3.9}$$

Therefore, we have

$$\psi(t) = \int_0^1 G(t,s)q(s)f_1(s,\psi(s))ds \tag{3.10}$$

$$= \int_0^1 G(t,s)q(s)f(s,\psi(s))ds.$$

Hence, $\psi(t)$ is the fixed point of A. The proof is complete. □

Theorem 3.2. *Suppose the following conditions hold:*

(B1) $uf(t,u) \geq 0$ *for all* $t \in [0,1]$*;*

(B2) $\liminf_{u\to 0^+} \min_{t\in[0,1]} \dfrac{f(t,u)}{u} > \lambda_1$*;*

(B3) $\limsup_{u\to +\infty} \max_{t\in[0,1]} \dfrac{f(t,u)}{u} < \lambda_1$*,*

where λ_1 *is the first eigenvalue of the operator* T *defined by* (2.2)*. Then BVP* (1.1)–(1.3) *has a positive solution and a negative solution.*

Proof of Theorem 3.2. By (B1), we obtain $A(P) \subset P$. The proof for existence of a positive solution when (B2)–(B3) are satisfied is nearly identical with Theorem 4.2 in [14] and will be omitted. Thereafter, we show that the existence of a negative solution.

Define

$$f_2(t,u) = -f(t,-u), \quad \text{for all } u \in \mathbb{R}, \quad t \in [0,1],$$

and

$$(A_2\varphi)(t) = \int_0^1 G(t,s)q(s)f_2(s,\varphi(s))ds.$$

Obviously, $A_2(P) \subset P$ and f_2 satisfies (B2) and (B3). Hence, A_2 has a positive fixed point ϕ.

For $\phi \in P\backslash\{\theta\}$ and fixed $t \in [0,1]$,

$$f_2(t,\phi) = -f(t,-\phi),$$

one has

$$(A_2\phi)(t) = -\int_0^1 G(t,s)q(s)f(s,-\phi(s))ds = \phi. \tag{3.11}$$

Notice (3.1) and (3.11), $-\phi$ is a negative fixed point of A. The proof is complete. □

Theorem 3.3. *Let* β *is a real nonnegative constant. Assume that* $f(t,u)$ *satisfies* $f(t,0) \neq 0,\ t \in (0,1)$ *and* $f(t,u) \geq -\beta$*, and*

$$\limsup_{|u|\to +\infty} \max_{t\in[0,1]} \frac{f(t,u)}{|u|} < \frac{1}{\int_0^1 G(1,s)q(s)ds}. \tag{3.12}$$

Then BVP (1.1)–(1.3) *has at least a nontrivial solution.*

Proof of Theorem 3.3. By (3.12), there exist $\gamma > 0$ and Q such that

$$0 < Q < \frac{1}{\int_0^1 G(1,s)q(s)ds},$$

and

$$-\beta \le f(t,u) \le Q|u| + \gamma, \quad \text{for } t \in [0,1],\ u \in \mathbb{R}.$$

Let

$$B_R = \{u \in E \mid \left| u - \gamma \int_0^1 G(t,s)q(s)ds \right| \le R\}.$$

Since B_R is a bounded, convex, closed set, then we have

$$\|u\| \le R + \gamma \int_0^1 G(1,s)q(s)ds,$$

and

$$\begin{aligned} &\left| Au(t) - \gamma \int_0^1 G(t,s)q(s)ds \right| \\ &\le \int_0^1 G(t,s)q(s)\,|f(s,u(s)) - \gamma|\,ds \\ &\le \max\left\{ (\beta+\gamma) \int_0^1 G(1,s)q(s)ds,\ Q\|u\| \int_0^1 G(1,s)q(s)ds \right\}, \end{aligned}$$

as long as

$$R \ge \frac{Q\gamma(\int_0^1 G(1,s)q(s)ds)^2}{1 - Q\int_0^1 G(1,s)q(s)ds}.$$

So, we obtain $A(B_R) \subset B_R$. By Schauder fixed point theorem, A has at least one fixed point in B_R and then BVP (1.1)–(1.3) has at least one solution. This completes the proof. □

Remark 3.2. Assumption (3.12) implies

$$\limsup_{|u|\to\infty} \max_{t\in[0,1]} \frac{f(t,u)}{|u|} < \lambda_1. \tag{3.13}$$

Indeed, by (2.2), one has

$$\|T\| \le \int_0^1 G(1,s)q(s)ds, \tag{3.14}$$

then by definition of spectral radius of linear operator, we have

$$\lambda_1 = (r(T))^{-1} \ge \|T\|^{-1} \ge \frac{1}{\int_0^1 G(1,s)q(s)ds}.$$

Finally, we give some examples to illustrate the results obtained in this paper.

4. EXAMPLES

Example 4.1. In BVP (1.1)–(1.3), let $f(t,u) = \dfrac{1-u}{1+u^2} + \ln\dfrac{1+t}{2}$ and $q(t) = t^{r-1}(1-t)^{s-1}$, $0 < r, s < 1$. Obviously, q is singular at $t = 0$ and $t = 1$. Let $b(t) = 1 - \ln\frac{1+t}{2}$. By the direct calculation, we can easily see that assumptions (A1)–(A3) hold. So by Theorem 3.1, we know that BVP (1.1)–(1.3) has at least a nontrivial solution.

Example 4.2. In BVP (1.1)–(1.3), let

$$f(u) = \begin{cases} \sqrt[3]{u}, & u \le 1 \\ \ln u + 1, & u > 1. \end{cases}$$

and $q(t) = \dfrac{1}{t\sqrt{1-t}}$. It is easy to show that conditions (B1)–(B3) are satisfied. By Theorem 3.2, BVP (1.1)–(1.3) has a positive solution and a negative solution.

Example 4.3. In BVP (1.1)–(1.3), let

$$f(u) = \begin{cases} \dfrac{\sin^4 u}{u^2} - 1, & u \ne 0 \\ -1, & u = 0. \end{cases}$$

and $q(t) = (1-t)^2$, it is easy to see that f satisfies all the assumptions in Theorem 3.3, then BVP (1.1)–(1.3) has at least a nontrivial solution.

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