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STRICT STABILITY OF IMPULSIVE FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAYS

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ABSTRACT. This paper focuses on the strict stability for a class of impulsive functional differential equations with infinite delays by using Lyapunov functions and Razumikhin technique. Some new Razumikhin type theorems on stability are obtained, which show that impulses do contribute to the system's strict stability behavior. Also, we point out a technical error in [7]. Our results improve and generalize some results in the literature.

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1. INTRODUCTION

Impulsive differential equations have become important in recent years in some mathematical models of real processes and phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics. For recent research we refer the reader to [1–4, 7–9, 11, 13]. Recently, systems with impulses and time delay have been discussed in [1, 2, 9, 12, 14–16]. In fact, the system stability and convergence properties are strongly affected by time delays, which are often encountered in many industrial and natural processes due to measurement and computational delays, transmission and transport lags. In [2, 3, 8, 10], the authors considered the stability of impulsive differential equations with finite delays. In [4, 14], by using Lyapunov functions and Razumikhin technique, Li obtained some Razumikhin type theorems on stability for a class of impulsive functional differential equations with infinite delays. However very little is known on stability theory for impulsive functional differential systems, especially for infinite delay impulsive functional differential systems.

On the other hand, as we know, the asymptotic stability of the trivial solution of a differential system implies that the solutions near the trivial solution tend to zero,

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but it does not guarantee any information about the rate of decay of the solutions. In other words, these definitions of stability are one-sided estimates of solutions, so they are not strict. It is natural to expect that an estimation on the lower bound for the rate at which solutions approach to the trivial solution would be beneficial. Such concepts are called stability in tube-like domain or strict stability [5–7]. In [5], Lakshmikantham and Mohapatra obtained some results on strict stability for ordinary differential systems. Considering the effects of time delay, Lakshmikantham and Zhang [6] further studied the strict practical stability of delay differential equations. Recently, Zhang and Sun [7] investigated the strict stability of a class of differential systems with finite delays and impulsive perturbations by means of Lyapunov functions and Razumikhin technique. The results show that impulses do contribute to the system's strict stability behavior. Unfortunately some results in [7] are not correct.

Inspired by the above discussion, in this paper, we consider the strict stability of impulsive functional differential systems with infinite delays. Some new stability results are obtained by employing Lyapunov functions and Razumikhin technique. The results obtained improve and generalize [5–7]. The effects of delays and impulses which do contribute to the equation's stability properties will be shown in this paper.

This work is organized as follows. In Section 2, we introduce some notations and definitions. In Section 3, we establish some strict uniform stability criteria for impulsive infinite delays differential equations.

2. PRELIMINARIES

Let R denote the set of real numbers, R_+ the set of nonnegative real numbers and R^n the n-dimensional real space equipped with the Euclidean norm $\|\cdot\|$. For any $t \geq t_0 \geq 0 > \alpha \geq -\infty$, let $f(t, x(s))$ where $s \in [t+\alpha, t]$ or $f(t, x(\cdot))$ be a Volterra type functional. In the case when $\alpha = -\infty$, the interval $[t+\alpha, t]$ is understood to be $(-\infty, t]$.

We consider the impulsive functional differential equations

$$
\begin{cases} x'(t) = f(t, x(\cdot)), & t \geq t_0, \quad t \neq t_k, \\ \Delta x|_{t=t_k} = x(t_k) - x(t_k^-) = I_k(t_k, x(t_k^-)), & k = 1, 2\ldots, \end{cases} \tag{2.1}
$$

where the impulse times t_k satisfy $0 \leq t_0 < t_1 < \cdots < t_k < \cdots$, $\lim_{k\to+\infty} t_k = +\infty$ and x' denotes the right-hand derivative of x. Also $f \in C([t_{k-1}, t_k) \times \mathbb{C},\ R^n)$, $f(t, 0) = 0$, where $\mathbb{C}$ is an open set in $PC([\alpha, 0], R^n)$, where $PC([\alpha, 0], R^n) = \{\psi : [\alpha, 0] \to\ R^n$ is continuous everywhere except at finite number of points t, at which $\psi(t^+)$ and $\psi(t^-)$ exist and $\psi(t^+) = \psi(t)\}$. For each $k = 1, 2\ldots$, $I_k(t, x) \in C([t_0, \infty) \times R^n, R^n)$, $I(t_k, 0) = 0$, and for any $\rho > 0$, there exists a $\rho_1 > 0$ $(0 < \rho_1 < \rho)$ such that $x \in S(\rho_1)$ implies that $x + I(t_k, x) \in S(\rho)$, where $S(\rho) = \{x : \|x\| < \rho, x \in R^n\}$.

Let $PCB(t) = \{x_t \in \mathbb{C} : x_t \text{ is bounded}\}$. For $\psi \in PCB(t)$, $\|\psi\|_1$ is defined by $\|\psi\|_1 = \sup_{\alpha \le \theta \le 0} \|\psi(\theta)\|$ and $\|\psi\|_2$ by $\|\psi\|_2 = \inf_{\alpha \le \theta \le 0} \|\psi(\theta)\|$.

For any given $\sigma \ge t_0$, the initial condition for system (2.1) is given by

$$x_\sigma = \phi, \tag{2.2}$$

where $\phi \in PC([\alpha, 0], R^n)$.

For convenience, we also have the following classes for later use:

$K_1 = \{a \in C(R_+, R_+) |\ a(0) = 0 \text{ and } a(s) > 0 \text{ for } s > 0\}$;

$K_2 = \{a \in C(R_+, R_+) |\ a(0) = 0 \text{ and } a \text{ is monotone strictly increasing}\}$;

$PCB^1_\delta(\sigma) = \{\psi \in PCB(\sigma) : \|\psi\|_1 < \delta\}$;

$PCB^2_\zeta(\sigma) = \{\psi \in PCB(\sigma) : \|\psi\|_2 > \zeta\}$.

We assume that the solution for the initial problem (2.1)–(2.2) is unique and is written in the form $x(t, \sigma, \phi)$, see [1, 16]. Since $f(t, 0) = 0$, $I_k(t_k, 0) = 0$, $k = 1, 2, \ldots$, then $x = 0$ is a solution of (2.1)–(2.2), which is called the zero solution. In this paper, we always assume that the solution $x(t, \sigma, \phi)$ of (2.1)–(2.2) can be continued to ∞ from the right of σ.

We introduce some definitions as follows:

Definition 2.1. The function $V : [\alpha, \infty) \times \mathbb{C} \to R_+$ belongs to class v_0 if

- (A_1) V is continuous on each of the sets $[t_{k-1}, t_k) \times \mathbb{C}$ and $\lim_{(t,\varphi) \to (t_k^-, \psi)} V(t, \varphi) = V(t_k^-, \psi)$ exists;
- (A_2) $V(t, x)$ is locally Lipschitzian in x and $V(t, 0) \equiv 0$.

Definition 2.2. Let $V \in v_0$, for any $(t, \psi) \in [t_{k-1}, t_k) \times \mathbb{C}$, the upper right-hand Dini derivative of $V(t, x)$ along the solution of (2.1)–(2.2) is defined by

$$D^+ V(t, \psi(0)) = \limsup_{h \to 0^+} \{V(t + h, \psi(0) + hf(t, \psi)) - V(t, \psi(0))\}/h.$$

Definition 2.3. Assume $x(t) = x(t, \sigma, \phi)$ be the solution of (2.1)–(2.2) through (σ, ϕ). Then the trivial solution of (2.1)–(2.2) is said to be

(1) strictly stable, if for any $\sigma \ge t_0$ and $\varepsilon_1 > 0$, there exists a $\delta_1 = \delta_1(\varepsilon_1, \sigma) > 0$ such that $\phi \in PCB^1_{\delta_1}(\sigma)$ implies that $\|x(t, \sigma, \phi)\| < \varepsilon_1, t \ge \sigma$, and for every $\delta_2 \in (0, \delta_1]$, there exists an $\varepsilon_2 \in (0, \delta_2)$ such that $\phi \in PCB^2_{\delta_2}(\sigma)$ implies $\|x(t, \sigma, \phi)\| > \varepsilon_2, t \ge \sigma$;

(2) strictly uniformly stable, if δ_1, δ_2 and ε_2 in (1) are independent of σ;

(3) strictly attractive, if given $\sigma \ge t_0$ and $\delta_1 > 0$, $\varepsilon_1 > 0$, for any $\delta_2 \le \delta_1$, there exists $\varepsilon_2 < \varepsilon_1$, $T_1 = T_1(\sigma, \varepsilon_1)$ and $T_2 = T_2(\sigma, \varepsilon_2)$ such that $\phi \in PCB^1_{\delta_1}(\sigma) \cap PCB^2_{\delta_2}(\sigma)$ implies $\varepsilon_2 < \|x(t)\| < \varepsilon_1$, $\sigma + T_1 \le t \le \sigma + T_2$;

(4) strictly uniformly attractive, if T_1 and T_2 in (3) are independent of σ;

(5) strictly asymptotically stable, if (3) holds, and the trivial solution of (1) is stable;

(6) strictly uniformly asymptotically stable, if (4) holds, and the trivial solution of (1) is uniformly stable.

It is very important to note that (1) and (3), or (2) and (4) cannot hold at the same time. When $\|x(t)\| \to 0, t \to \infty$, or $\liminf \|x(t)\| = 0, \limsup \|x(t)\| \neq 0$, the trivial solution of system (2.1)–(2.2) cannot be strictly stable.

3. MAIN RESULTS

In this section, we shall develop Lyapunov-Razumikhin methods and establish some theorems which provide sufficient conditions for strict uniform stability of the trivial solution of (2.1)–(2.2).

Theorem 3.1. *Assume that there exist functions* $w_{ij} \in K_1, g, h \in K_2, c_i, p_i \in C(R_+, R_+)$, $V_i \in v_0, i, j = 1, 2$ *such that the following conditions hold:*

(i) $w_{i1}(\|x\|) \le V_i(t, x) \le w_{i2}(\|x\|)$, $i = 1, 2$, $(t, x) \in [\alpha, \infty) \times S(\rho)$;

(ii) *For any* $\sigma \ge t_0$ *and* $\psi \in PC([\alpha, 0], S(\rho))$, *if* $V_1(t, \psi(0)) \ge g(V_1(t + \theta, \psi(\theta)))$, $\alpha \le \theta \le 0$, $t \neq t_k$, *then*

$$D^+V_1(t, \psi(0)) \le p_1(t)c_1(V_1(t, \psi(0))).$$

Also, for all $(t_k, \psi) \in R_+ \times PC([\alpha, 0], S(\rho_1))$,

$$V_1(t_k, \psi(0) + I_k(t_k, \psi)) \le g(V_1(t_k^-, \psi(0))),$$

where $g(s) < s$ *for any* $s > 0$;

(iii) *There exist constants* $M_1, M_2 > 0$ *such that the following inequalities hold:*

$$\sup_{t \ge 0} \int_t^{t+\tau} p_1(s)ds = M_1 < \infty, \quad \inf_{s>0} \int_{g(s)}^{s} \frac{dt}{c_1(t)} = M_2 > M_1,$$

where $\tau = \max_{k\ge1}\{t_k - t_{k-1}\} < \infty$;

(iv) *For any* $\sigma \ge t_0$ *and* $\psi \in PC([\alpha, 0], S(\rho))$, *if* $V_2(t, \psi(0)) \le h^2(V_2(t + \theta, \psi(\theta)))$, $\alpha \le \theta \le 0$, $t \neq t_k$, *then*

$$D^+V_2(t, \psi(0)) \ge p_2(t)c_2(V_2(t, \psi(0))).$$

Also, for all $(t_k, \psi) \in R_+ \times PC([\alpha, 0], S(\rho_1))$,

$$V_2(t_k, \psi(0) + I_k(t_k, \psi)) \ge h^{-1}(V_2(t_k^-, \psi(0))),$$

where $h(s) > s$ *for any* $s > 0$;

(v) *There exist constants* $J_1, J_2 > 0$ *such that the following inequalities hold:*

$$\inf_{t \ge 0} \int_t^{t+\mu} p_2(s)ds = J_1 < \infty, \quad \sup_{s>0} \int_{s}^{h^2(s)} \frac{dt}{c_2(t)} = J_2 < J_1,$$

where $\mu = \min_{k\ge1}\{t_k - t_{k-1}\} > 0$.

Then the trivial solution of (2.1)–(2.2) *is strictly uniformly stable.*

Proof. Condition (i) implies that $w_{i1}(s) \le w_{i2}(s)$ for $s \in [0,\rho]$. Let W_{i1} and W_{i2} be continuous, strictly increasing functions satisfying $W_{i1}(s) \le w_{i1}(s) \le w_{i2}(s) \le W_{i2}(s)$ for all $s \in [0,\rho]$. Thus, for all $(t,x) \in [\alpha,\infty) \times S(\rho)$, we have

$$W_{i1}(\|x\|) \le V_i(t,x) \le W_{i2}(\|x\|).$$

For any $\varepsilon_1 > 0 (< \rho_1)$, one may choose a $\delta_1 = \delta_1(\varepsilon_1) > 0$ such that $W_{12}(\delta_1) \le g(W_{11}(\varepsilon_1))$. Let $x(t) = x(t,\sigma,\phi)$ be a solution of (2.1)–(2.2) through (σ,ϕ), $\sigma \ge t_0$. Suppose that $\sigma \in [t_{l-1}, t_l)$, $l \in Z_+$. For any $\phi \in PCB^1_{\delta_1}(\sigma)$, we shall prove that $\|x(t)\| < \varepsilon_1$, $t \ge \sigma$. For convenience, let $V_i(t) = V_i(t,x(t))$.

First, for $\sigma + \alpha \le t \le \sigma$, we have

$$W_{11}(\|x\|) \le V_1(t) < W_{12}(\delta_1) \le g(W_{11}(\varepsilon_1)) < W_{11}(\varepsilon_1), \tag{3.1}$$

which implies that $\|x(t)\| < \varepsilon_1 < \rho_1$, $t \in [\sigma+\alpha,\sigma]$. Next we claim that

$$V_1(t) < W_{11}(\varepsilon_1),\ t \in [\sigma, t_l). \tag{3.2}$$

Suppose that this assertion is false. Then there exists some $t \in [\sigma, t_l)$ such that $V_1(t) \ge W_{11}(\varepsilon_1)$. Since $V_1(\sigma) < W_{11}(\varepsilon_1)$, we can define $\hat{t} = \inf\{t \in [\sigma,t_l) \mid V_1(t) \ge W_{11}(\varepsilon_1)\}$. Thus, $\hat{t} \in (\sigma, t_l)$, $V_1(\hat{t}) = W_{11}(\varepsilon_1)$ and $V_1(t) < W_{11}(\varepsilon_1)$, $t \in [\sigma,\hat{t})$. Also, in view of (3.1) we obtain

$$V_1(t) < W_{11}(\varepsilon_1), \quad t \in [\sigma+\alpha, \hat{t}). \tag{3.3}$$

On the other hand, note that $V_1(\hat{t}) = W_{11}(\varepsilon_1) > g(W_{11}(\varepsilon_1))$ and $V_1(\sigma) < g(W_{11}(\varepsilon_1))$ in view of (3.1), we can define $t^* = \sup\{t \in [\sigma,\hat{t}] \mid V_1(t) \le g(W_{11}(\varepsilon_1))\}$. Then it is obvious that $t^* \in [\sigma,\hat{t})$, $V_1(t^*) = g(W_{11}(\varepsilon_1))$ and $V_1(t) > g(W_{11}(\varepsilon_1))$ for $t \in (t^*, \hat{t}]$. Therefore, combining with (3.3), we have for $t \in (t^*, \hat{t})$

$$V_1(t) > g(W_{11}(\varepsilon_1)) > g(V_1(t+\theta)), \quad \alpha \le \theta \le 0.$$

By assumption (ii), (iii), we have

$$\int_{V_1(t^*)}^{V_1(\hat{t})} \frac{ds}{c_1(s)} = \int_{g(W_{11}(\varepsilon_1))}^{W_{11}(\varepsilon_1)} \frac{ds}{c_1(s)} \ge M_2 > M_1.$$

However, we note that

$$\int_{V_1(t^*)}^{V_1(\hat{t})} \frac{ds}{c_1(s)} \le \int_{t^*}^{\hat{t}} p_1(s)ds < \int_{t^*}^{t^*+\tau} p_1(s)ds \le M_1,$$

which is a contradiction. Thus (3.2) holds.

Hence, $W_1(\|x\|) \le V_1(t) < W_{11}(\varepsilon_1)$, $t \in [\sigma, t_l)$ implies that $\|x(t_l^-)\| < \varepsilon_1 < \rho_1$. Thus, $x(t_l) \in S(\rho)$. From condition (ii), we have

$$V_1(t_l) \le g(V_1(t_l^-)) \le g(W_{11}(\varepsilon_1)) < W_{11}(\varepsilon_1).$$

Next we claim that

$$V_1(t) < W_{11}(\varepsilon_1), \quad t \in [t_l, t_{l+1}).$$

Suppose on the contrary that there exists some $t \in [t_l, t_{l+1})$ such that $V_1(t) \geq W_{11}(\varepsilon_1)$. Then applying exactly the same argument as in the proof of (3.2) yields our desired contradiction.

By induction we have in general that for $t \in [t_{l+k}, t_{l+k+1})$, $k > 0$,

$$V_1(t) < W_{11}(\varepsilon_1).$$

Therefore, in view of condition (i) we obtain that $\|x(t)\| < \varepsilon_1$, $t \geq \sigma$.

Now, for any $\delta_2 \in (0, \delta_1]$, choose a $\delta_3 \in (0, \delta_2)$ such that $W_{21}^{-1}(h(W_{21}(\delta_3))) \leq \delta_2$, and choose $\varepsilon_2 \in (0, \delta_3)$ such that $\varepsilon_2 < W_{22}^{-1}(W_{21}(\delta_3))$. Next we claim that $\phi \in PCB^2_{\delta_2}(\sigma)$ implies that $\|x\| > \varepsilon_2$, $t \geq \sigma$. First, for $\sigma + \alpha \leq t \leq \sigma$, we have

$$V_2(t) \geq W_{21}(\|\phi\|) \geq W_{21}(\delta_2) \geq h(W_{21}(\delta_3)) > W_{21}(\delta_3) > W_{22}(\varepsilon_2), \tag{3.4}$$

which implies that $\|x(t)\| > \varepsilon_2$, $t \in [\sigma + \alpha, \sigma]$. Next we claim that

$$V_2(t) \geq W_{21}(\delta_3), \quad t \in [\sigma, t_l). \tag{3.5}$$

Suppose that this assertion is not true. Then there exists some $t \in [\sigma, t_l)$ such that $V_2(t) < W_{21}(\delta_3)$. Since $V_2(\sigma) > W_{21}(\delta_3)$. we can define $\hat{t} = \inf\{t \in [\sigma, t_l) \mid V_2(t) \leq W_{21}(\delta_3)\}$. Thus, $\hat{t} \in (\sigma, t_l)$, $V_2(\hat{t}) = W_{21}(\delta_3)$, and $V_2(t) > W_{21}(\delta_3)$, $t \in [\sigma, \hat{t})$. Also, combining with (3.4), we obtain

$$V_2(t) \geq W_{21}(\delta_3), \quad t \in [\sigma + \alpha, \hat{t}]. \tag{3.6}$$

On the other hand, considering $V_2(\hat{t}) = W_{21}(\delta_3) < h(W_{21}(\delta_3))$ and $V_2(\sigma) \geq h(W_{21}(\delta_3))$ in view of (3.4), we can define $t^* = \sup\{t \in [\sigma, \hat{t}] \mid V_2(t) \geq h(W_{21}(\delta_3))\}$. Thus, $t^* \in [\sigma, \hat{t})$, $V_2(t^*) = h(W_{21}(\delta_3))$, and $V_2(t) < h(W_{21}(\delta_3))$ for $t \in (t^*, \hat{t}]$. Therefore, combining with (3.6), we have for $t \in [t^*, \hat{t}]$

$$V_2(t) \leq h(W_{21}(\delta_3)) \leq h(V_2(t + \theta)) < h^2(V_2(t + \theta)), \quad \alpha \leq \theta \leq 0.$$

By assumption (iv), we get the inequality $D^+V_2(t, \psi(0)) \geq p_2(t)c_2(V_2(t, \psi(0))) \geq 0$ holds. Thus function $V_2(t)$ is monotone increasing for $t \in [t^*, \hat{t}]$. In particular, we get $V_2(t^*) \leq V_2(\hat{t})$. However, this contradicts the fact that $V_2(\hat{t}) = W_{21}(\delta_3) < h(W_{21}(\delta_3)) = V_2(t^*)$. Thus (3.5) holds.

Next we claim that $V_2(t_l^-) \geq h^2(W_{21}(\delta_3))$. Suppose that this assertion is false, then $V_2(t_l^-) < h^2(W_{21}(\delta_3))$. Thus either $V_2(t) < h^2(W_{21}(\delta_3))$ for all $t \in [t_{l-1}, t_l)$, or there exists some $t \in [t_{l-1}, t_l)$ for which $V(t) \geq h^2(W_{21}(\delta_3))$. In the first case, $V_2(t) < h^2(W_{21}(\delta_3)) \leq h^2(V_2(t + \theta))$, $\alpha \leq \theta \leq 0$, $t \in [t_{l-1}, t_l)$. Also, we obtain $V_2(t_l^-) < h^2(V_2(t_{l-1}))$. Therefore, by virtue of condition (iv), (v), we have

$$\int_{V_2(t_{l-1})}^{V_2(t_l^-)} \frac{ds}{c_2(s)} \leq \int_{V_2(t_{l-1})}^{h^2(V_2(t_{l-1}))} \frac{ds}{c_2(s)} \leq J_2 < J_1.$$

However, we note

$$\int_{V_2(t_{l-1})}^{V_2(t_l^-)} \frac{ds}{c_2(s)} \geq \int_{t_{l-1}}^{t_l} p_2(s)ds \geq \int_{t_{l-1}}^{t_{l-1}+\mu} p_2(s)ds \geq J_1.$$

This is a contradiction. In the second case, let $t^* = \sup\{t \in [\sigma, t_l) | V_2(t) \geq h^2(W_{21}(\delta_3))\}$. Then $V_2(t^*) = h^2(W_{21}(\delta_3)), V(t) < h^2(W_{21}(\delta_3)), t \in (t^*, t_l)$. Thus, $V_2(t) \leq h^2(W_{21}(\delta_3)) \leq h^2(V_2(t+\theta))$, $\alpha \leq \theta \leq 0$, $t \in [t^*, t_l)$. By assumption (iv), we get the inequality $D^+V_2(t, \psi(0)) \geq p_2(t)c_2(V_2(t, \psi(0))) \geq 0$ holds. Then the function $V_2(t)$ is monotone increasing for $t \in [t^*, \hat{t}]$, which implies that $V_2(t^*) \leq V_2(t_l^-)$. But this contradicts the fact that $V_2(t_l^-) < h^2(W_{21}(\delta_3)) = V_2(t^*)$. Thus, we have shown that $V_2(t_l^-) \geq h^2(W_{21}(\delta_3))$.

From condition (iv) and the inequality $V_2(t_l^-) \geq h^2(W_{21}(\delta_3))$, we have

$$V_2(t_l) \geq h^{-1}(V_2(t_l^-)) \geq h(W_{21}(\delta_3)) > W_{21}(\delta_3).$$

Next

$$V_2(t) \geq W_{21}(\delta_3), \ t \in [t_l, t_{l+1})$$

by the same argument that was employed in the proof of (3.5). By induction we have that for $t \in [t_{l+k}, t_{l+k+1})$, $k = 1, 2, \ldots$

$$V_2(t) \geq W_{21}(\delta_3),$$

i.e.,

$$V_2(t) \geq W_{21}(\delta_3) \geq W_{22}(\varepsilon_2), \quad t \geq \sigma,$$

which together with condition (i), we obtain $\|x\| > \varepsilon_2$, $t \geq \sigma$. Therefore, we finally obtain that $\varepsilon_2 < \|x\| < \varepsilon_1$ for $\phi \in PCB^1_{\delta_1}(\sigma) \cap PCB^2_{\delta_2}(\sigma)$, $t \geq \sigma$. The proof of Theorem 3.1 is complete. □

Corollary 3.2. *Assume that there exist functions $w_i \in K_1$, $g, h \in K_2$, $c_i, p_i \in C(R_+, R_+)$, $i = 1, 2$, $V \in v_0$ such that the following conditions hold:*

(i) *$w_1(\|x\|) \leq V(t, x) \leq w_2(\|x\|)$, $i = 1, 2$, $(t, x) \in [\alpha, \infty) \times S(\rho)$;*

(ii) *For any $\sigma \geq t_0$ and $\psi \in PC([\alpha, 0], S(\rho))$, if $g(V(t+\theta, \psi(\theta))) \leq V(t, \psi(0)) \leq h^2(V(t+\theta, \psi(\theta)))$, $\alpha \leq \theta \leq 0$, $t \neq t_k$, then*

$$p_2(t)c_2(V(t, \psi(0))) \leq D^+V(t, \psi(0)) \leq p_1(t)c_1(V(t, \psi(0))).$$

Also, for all $(t_k, \psi) \in R_+ \times PC([\alpha, 0], S(\rho_1))$,

$$h^{-1}(V(t_k^-, \psi(0))) \leq V(t_k, \psi(0) + I_k(t_k, \psi)) \leq g(V(t_k^-, \psi(0))),$$

where $g(s) < s < h(s)$ for any $s > 0$;

(iii) *There exist constants* $M_i > 0$, $i = 1, \dots, 4$ *such that the following inequalities hold:*

$$\sup_{t\geq 0} \int_t^{t+\tau} p_1(s)ds = M_1 < \infty, \quad \inf_{s>0} \int_{g(s)}^{s} \frac{dt}{c_1(t)} = M_2 > M_1,$$

$$\inf_{t\geq 0} \int_t^{t+\mu} p_2(s)ds = M_3 < \infty, \quad \sup_{s>0} \int_{s}^{h^2(s)} \frac{dt}{c_2(t)} = M_4 < M_3,$$

where $\mu = \min_{k\geq 1}\{t_k - t_{k-1}\} > 0$, $\tau = \max_{k\geq 1}\{t_k - t_{k-1}\} < \infty$.

Then the trivial solution of (2.1)–(2.2) *is strictly uniformly stable.*

Remark 3.3. In [7], the authors obtained some sufficient conditions for guaranteeing the strict stability of impulsive functional differential systems with finite delays. However, there is a technical error in Theorem 2 of [7]. That is, condition (v) contradicts condition (vi) in Theorem 2 of [7]. In fact, from condition (v), i.e., $p \in C(R_+, R_+)$ and $0 < \psi_2(u) < u$, we have

$$\int_u^{\psi_2(u)} \frac{ds}{p(s)} < 0,$$

which contradicts

$$\int_u^{\psi_2(u)} \frac{ds}{p(s)} \geq B > 0$$

in condition (vi).

Theorem 3.4. *Assume that there exist functions* $w_{ij} \in K_1$, $g, h \in K_2$, $c_i, p_i \in C(R_+, R_+)$, $V_i(t,x) \in v_0$, $i, j = 1, 2$ *such that the following conditions hold:*

(i) $w_{i1}(\|x\|) \leq V_i(t,x) \leq w_{i2}(\|x\|)$, $i = 1, 2$, $(t,x) \in [\alpha, \infty) \times S(\rho)$;

(ii) *For any* $\sigma \geq t_0$ *and* $\psi \in PC([\alpha, 0], S(\rho))$, *if* $g^2(V_1(t, \psi(0))) \geq V_1(t+\theta, \psi(\theta))$, $\alpha \leq \theta \leq 0$, $t \neq t_k$, *then*

$$D^+V_1(t, \psi(0)) \leq -p_1(t)c_1(V_1(t, \psi(0))).$$

Also, for all $(t_k, \psi) \in R_+ \times PC([\alpha, 0], S(\rho_1))$,

$$V_1(t_k, \psi(0) + I_k(t_k, \psi)) \leq g(V_1(t_k^-, \psi(0))),$$

where $g(s) > s$ *for any* $s > 0$;

(iii) *There exist constants* $M_1, M_2 > 0$ *such that the following inequalities hold:*

$$\inf_{t\geq 0} \int_t^{t+\mu} p_1(s)ds = M_1 > 0, \quad \sup_{s>0} \int_{s}^{g^2(s)} \frac{dt}{c_1(t)} = M_2 < M_1,$$

where $\mu = \min_{k\geq 1}\{t_k - t_{k-1}\} < \infty$;

(iv) *For any* $\sigma \geq t_0$ *and* $\psi \in PC([\alpha, 0], S(\rho))$, *if* $h(V_2(t, \psi(0))) \leq V_2(t+\theta, \psi(\theta))$, $\alpha \leq \theta \leq 0$, $t \neq t_k$, *then*

$$D^+V_2(t, \psi(0)) \geq -p_2(t)c_2(V_2(t, \psi(0))).$$

Also, for all $(t_k, \psi) \in R_+ \times PC([\alpha, 0], S(\rho_1))$,

$$V_2(t_k, \psi(0) + I_k(t_k, \psi)) \geq h^{-1}(V_2(t_k^-, \psi(0))),$$

where $h(s) < s$ *for any* $s > 0$*;*

(v) *There exist constants* $J_1, J_2 > 0$ *such that the following inequalities hold:*

$$\sup_{t\geq 0} \int_t^{t+\tau} p_2(s)ds = J_1 < \infty, \quad \inf_{s>0} \int_{h(s)}^{s} \frac{dt}{c_2(t)} = J_2 > J_1,$$

where $\tau = \max_{k\geq 1}\{t_k - t_{k-1}\} > 0$.

Then the trivial solution of (2.1)–(2.2) *is strictly uniformly stable.*

Proof. As in Theorem 3.1, let W_{i1} and W_{i2} be continuous, strictly increasing functions satisfying $W_{i1}(s) \leq w_{i1}(s) \leq w_{i2}(s) \leq W_{i2}(s)$ for all $s \in [0, \rho]$, $i = 1, 2$. Thus, we have

$$W_{i1}(\|x\|) \leq V_i(t, x) \leq W_{i2}(\|x\|), \quad (t, x) \in [\alpha, \infty) \times S(\rho).$$

Consider any $\varepsilon_1 > 0$ and assume without loss of generality that $\varepsilon_1 < \rho_1$. Choose a $\delta_1 = \delta_1(\varepsilon_1) > 0$ such that $g(W_{12}(\delta_1)) < W_{11}(\varepsilon_1)$. Let $x(t) = x(t, \sigma, \phi)$ be a solution of (2.1)–(2.2) through $(\sigma, \phi), \sigma \geq t_0$. Let $\phi \in PCB^1_{\delta_1}(\sigma)$, we shall prove that $\|x(t)\| < \varepsilon_1$, $t \geq \sigma$. For convenience, let $V_i(t) = V_i(t, x(t))$. Suppose that $\sigma \in [t_{l-1}, t_l)$, $l \in Z_+$. Then for $\sigma + \alpha \leq t \leq \sigma$, we have

$$W_{11}(\|x\|) \leq V_1(t) < g(V_1(t)) < g(W_{12}(\delta_1)) < W_{11}(\varepsilon_1). \tag{3.7}$$

Thus, we have $\|x(t)\| < \varepsilon_1 < \rho_1$, $t \in [\sigma + \alpha, \sigma]$. Next we claim that

$$V_1(t) < W_{11}(\varepsilon_1), \quad t \in [\sigma, t_l). \tag{3.8}$$

Suppose that on the contrary there exists some $t \in [\sigma, t_l)$ such that $V_1(t) \geq W_{11}(\varepsilon_1)$. Let $\hat{t} = \inf\{t \in [\sigma, t_l) \mid V_1(t) \geq W_{11}(\varepsilon_1)\}$. Since $V_1(\sigma) < W_{11}(\varepsilon_1)$, we have $\hat{t} \in (\sigma, t_l), V_1(\hat{t}) = W_{11}(\varepsilon_1)$ and $V_1(t) < W_{11}(\varepsilon_1)$, $t \in [\sigma, \hat{t})$. Hence, we get $V_1(t) < W_{11}(\varepsilon_1)$, $t \in [\sigma + \alpha, \hat{t})$. Also, since $g(V(\hat{t})) = g(W_{11}(\varepsilon_1)) > W_{11}(\varepsilon_1)$, and $g(V_1(\sigma)) < W_{11}(\varepsilon)$ in view of (3.7), we can define $t^* = \sup\{t \in [\sigma, \hat{t}] \mid g(V_1(t)) \leq W_{11}(\varepsilon_1)\}$. Then $t^* \in [\sigma, \hat{t})$, $g(V_1(t^*)) = W_{11}(\varepsilon_1)$ and $g(V_1(t)) > W_{11}(\varepsilon_1)$, $t \in (t^*, \hat{t}]$. Hence, we obtain

$$g^2(V_1(t)) \geq g(V_1(t)) \geq W_{11}(\varepsilon_1) > V_1(t + \theta, \psi(\theta)), \quad \alpha \leq \theta \leq 0, \quad t \in [t^*, \hat{t}].$$

Thus, by assumption (ii), the inequality $D^+V_1(t, \psi(0)) \leq -p_1(t)c_1(V_1(t, \psi(0))) \leq 0$ holds. Then function $V_1(t)$ is monotone nonincreasing for $t \in [t^*, \hat{t}]$, which implies that $V_1(t^*) \geq V_1(\hat{t})$. Thus, $g(W_{11}(\varepsilon_1)) = g(V_1(\hat{t})) \leq g(V_1(t^*)) = W_{11}(\varepsilon_1)$, which is a contradiction with $g(s) > s$. Thus (3.8) holds, which implies $x(t_l^-) \in S(\rho_1)$, $x(t_l) \in S(\rho)$.

Next we claim that $V_1(t_l^-) \leq g^{-2}(W_{11}(\varepsilon_1))$. Suppose that this assertion is false. Then $V_1(t_l^-) > g^{-2}(W_{11}(\varepsilon_1))$. Thus either $V_1(t) > g^{-2}(W_{11}(\varepsilon_1))$ for all $t \in [t_{l-1}, t_l)$, or there exists some $t \in [t_{l-1}, t_l)$ for which $V_1(t) \leq g^{-2}(W_{11}(\varepsilon_1))$. In the first case,

$g^2(V_1(t)) > W_{11}(\varepsilon_1) > V_1(t+\theta,\psi(\theta))$, $\alpha \le \theta \le 0$ in view of (3.8). In particular, we obtain $g^2(V_1(t_l^-)) > V_1(t_{l-1})$. Hence, by virtue of (ii), (iii), we have

$$\int_{V_1(t_l^-)}^{V_1(t_{l-1})} \frac{ds}{c(s)} \le \int_{V_1(t_l^-)}^{g^2(V(t_l^-))} \frac{ds}{c_1(s)} \le M_2 < M_1.$$

However,

$$\int_{V_1(t_l^-)}^{V_1(t_{l-1})} \frac{ds}{c_1(s)} \ge \int_{t_{l-1}}^{t_l} p_1(s)ds \ge \int_{t_{l-1}}^{t_{l-1}+\mu} p_1(s)ds \ge M_1.$$

This is a contradiction. In the second case, let $t^* = \sup\{t \in [\sigma, t_l) \mid V_1(t) \le g^{-2}(W_{11}(\varepsilon_1))\}$. Then $V_1(t^*) = g^{-2}(W_{11}(\varepsilon_1))$, $V_1(t) > g^{-2}(W_{11}(\varepsilon_1))$, $t \in (t^*, t_l)$, which implies $g^2(V_1(t)) \ge W_{11}(\varepsilon_1) > V(t+\theta,\psi(\theta))$, $\alpha \le \theta \le 0$, $t \in [t^*, t_l)$. Hence, the function $V_1(t)$ is monotone nonincreasing for $t \in [t^*, \hat{t}]$, which implies that $V_1(t^*) \ge V_1(t_l^-)$. Thus $g^{-2}(W_{11}(\varepsilon_1)) = V_1(t^*) \ge V(t_l^-) > g^{-2}(W_{11}(\varepsilon_1))$, which is a contradiction. Thus, we have proven that $V_1(t_l^-) \le g^{-2}(W_{11}(\varepsilon_1))$.

Furthermore, we obtain

$$V_1(t_l) \le g(V_1(t_l^-)) \le g^{-1}(W_{11}(\varepsilon_1)) < W_{11}(\varepsilon_1). \tag{3.9}$$

We have

$$V_1(t) < W_{11}(\varepsilon_1), \quad t \in [t_l, t_{l+1})$$

by the same argument that was employed in the proof of (3.8). By the induction, we have that for $t \in [t_{l+k}, t_{l+k+1})$, $k = 1, 2, \ldots$

$$V_1(t) < W_{11}(\varepsilon_1),$$

i.e.,

$$V_1(t) < W_{11}(\varepsilon_1), \quad t \ge \sigma,$$

which together with condition (i), we obtain $\|x\| < \varepsilon_1$, $t \ge \sigma$.

Now, for any $\delta_2 \in (0, \delta_1]$, choose a $\delta_3 \in (0, \delta_2)$ such that $W_{21}^{-1}(h^{-1}(W_{21}(\delta_3))) \le \delta_2$, and choose $\varepsilon_2 \in (0, \delta_3)$ such that $\varepsilon_2 < W_{22}^{-1}(W_{21}(\delta_3))$. Next we claim that $\phi \in PCB_{\delta_2}^2(\sigma)$ implies that $\|x\| > \varepsilon_2$, $t \ge \sigma$. First, for $\sigma + \alpha \le t \le \sigma$, we have

$$V_2(t) \ge W_{21}(\|\phi\|) \ge W_{21}(\delta_2) \ge h^{-1}(W_{21}(\delta_3)) > W_{21}(\delta_3) > W_{22}(\varepsilon_2) \tag{3.10}$$

which implies that $\|x(t)\| > \varepsilon_2$, $t \in [\sigma + \alpha, \sigma]$. Next we claim that

$$V_2(t) \ge W_{21}(\delta_3), \quad t \in [\sigma, t_l). \tag{3.11}$$

Suppose that this assertion is not true. Then there exists some $t \in [\sigma, t_l)$ such that $V_2(t) < W_{21}(\delta_3)$. Since $V_2(\sigma) > W_{21}(\delta_3)$, we can define $\hat{t} = \inf\{t \in [\sigma, t_l) \mid V_2(t) \le W_{21}(\delta_3)\}$. Thus, $\hat{t} \in (\sigma, t_l)$, $V_2(\hat{t}) = W_{21}(\delta_3)$, and $V_2(t) > W_{21}(\delta_3)$, $t \in [\sigma, \hat{t})$. Also, combining with (3.10), we obtain

$$V_2(t) \ge W_{21}(\delta_3), \quad t \in [\sigma + \alpha, \hat{t}]. \tag{3.12}$$

On the other hand, considering $h(V_2(\hat{t})) = h(W_{21}(\delta_3)) < W_{21}(\delta_3)$ and $h(V_2(\sigma)) > W_{21}(\delta_3)$ in view of (3.10), we can define $t^* = \sup\{t \in [\sigma, \hat{t}] \mid h(V_2(t)) \geq W_{21}(\delta_3)\}$. Thus, $t^* \in [\sigma, \hat{t})$, $h(V_2(t^*)) = W_{21}(\delta_3)$, and $h(V_2(t)) < W_{21}(\delta_3)$ for $t \in (t^*, \hat{t}]$. Consequently, combining with (3.12), we have for $t \in [t^*, \hat{t}]$,

$$h(V_2(t)) \leq W_{21}(\delta_3) \leq V_2(t+\theta), \quad \alpha \leq \theta \leq 0.$$

By assumption (iv), we get the inequality $D^+V_2(t, \psi(0)) \geq -p_2(t)c_2(V_2(t, \psi(0)))$ holds. Hence, we note that

$$\int_{V_2(t^*)}^{V_2(\hat{t})} \frac{ds}{c_2(s)} = \int_{V_2(t^*)}^{h(V_2(t^*))} \frac{ds}{c_2(s)} = -\int_{h(V_2(t^*))}^{V_2(t^*)} \frac{ds}{c_2(s)} \leq -J_2 < -J_1.$$

However, we also have

$$\int_{V_2(t^*)}^{V_2(\hat{t})} \frac{ds}{c_2(s)} \geq -\int_{t^*}^{\hat{t}} p_2(s)ds \geq -\int_{t^*}^{t^*+\tau} p_2(s)ds \geq -J_1,$$

which is a contradiction. Thus (3.11) holds.

From condition (iv) and (3.11), we have

$$V_2(t_l) \geq h^{-1}(V_2(t_l^-)) \geq h^{-1}(W_{21}(\delta_3)) > W_{21}(\delta_3).$$

Next

$$V_2(t) \geq W_{21}(\delta_3), \quad t \in [t_l, t_{l+1})$$

by the same argument that was employed in the proof of (3.11). By induction we have that for $t \in [t_{l+k}, t_{l+k+1})$, $k = 1, 2, \ldots$

$$V_2(t) \geq W_{21}(\delta_3),$$

i.e.,

$$V_2(t) \geq W_{21}(\delta_3) \geq W_{22}(\varepsilon_2), \quad t \geq \sigma,$$

which together with condition (i), we obtain $\|x\| > \varepsilon_2$, $t \geq \sigma$. Therefore, we finally obtain that $\varepsilon_2 < \|x\| < \varepsilon_1$ for $\phi \in PCB^1_{\delta_1}(\sigma) \cap PCB^2_{\delta_2}(\sigma)$, $t \geq \sigma$. The proof of Theorem 3.4 is complete. □

Corollary 3.5. *Assume that there exist functions $w_i \in K_1$, $g, h \in K_2$, $c_i, p_i \in C(R_+, R_+)$, $V(t,x) \in v_0$, $i = 1, 2$ such that the following conditions hold:*

(i) $w_1(\|x\|) \leq V(t,x) \leq w_2(\|x\|)$, $(t,x) \in [\alpha, \infty) \times S(\rho)$;

(ii) *For any $\sigma \geq t_0$ and $\psi \in PC([\alpha, 0], S(\rho))$, if $h(V(t, \psi(0))) \leq V(t+\theta, \psi(\theta)) \leq g^2(V(t, \psi(0)))$, $\alpha \leq \theta \leq 0$, $t \neq t_k$, then*

$$-p_2(t)c_2(V(t, \psi(0))) \leq D^+V(t, \psi(0)) \leq -p_1(t)c_1(V(t, \psi(0))).$$

Also, for all $(t_k, \psi) \in R_+ \times PC([\alpha, 0], S(\rho_1))$,

$$h^{-1}(V(t_k^-, \psi(0))) \leq V(t_k, \psi(0) + I_k(t_k, \psi)) \leq g(V(t_k^-, \psi(0))),$$

where $h(s) < s < g(s)$ for any $s > 0$;

(iii) *There exist constants* $M_i > 0$, $i = 1, \ldots, 4$ *such that the following inequalities hold:*

$$\inf_{t\geq 0}\int_t^{t+\mu} p_1(s)ds = M_1 > 0, \quad \sup_{s>0}\int_s^{g^2(s)} \frac{dt}{c_1(t)} = M_2 < M_1,$$

$$\sup_{t\geq 0}\int_t^{t+\tau} p_2(s)ds = M_3 < \infty, \quad \inf_{s>0}\int_{h(s)}^{s} \frac{dt}{c_2(t)} = M_4 > M_3,$$

where $\tau = \max_{k\geq 1}\{t_k - t_{k-1}\} < \infty$, $\mu = \min_{k\geq 1}\{t_k - t_{k-1}\} > 0$.

Then the trivial solution of (2.1)–(2.2) *is strictly uniformly stable.*

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