Dynamic Systems and Applications 22 (2013) 183-196

THREE SOLUTIONS FOR DISCRETE ANISOTROPIC PERIODIC AND NEUMANN PROBLEMS

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Dedicated to Professor V. Lakshmikantham

ABSTRACT. Using critical point theory, we study the existence of at least three solutions for some periodic and Neumann boundary value problems involving the discrete $p(\cdot)$-Laplacian operator.

2010 AMS Subject Classification. 39A12; 39A23; 39A70; 65Q10

Keywords. Discrete $p(\cdot)$-Laplacian operator; Variational methods; Palais-Smale condition; Three solutions

1. INTRODUCTION

Let T be a positive integer, $p : \mathbb{Z}[0,T] \to (1,\infty)$, $r : \mathbb{Z}[1,T] \to (0,\infty)$ and the homeomorphism $h_{p(k)} : \mathbb{R} \to \mathbb{R}$ be defined by $h_{p(k)}(x) = |x|^{p(k)-2}x$, for all $x \in \mathbb{R}$ and $k \in \mathbb{Z}[0,T]$. Here and below, for $a, b \in \mathbb{N}$ with $a < b$, we use the notation $\mathbb{Z}[a,b] := \{a, a+1, \ldots, b\}$.

In this paper we deal with the existence of at least three solutions for the periodic problem

$$(\mathcal{P}_P) \qquad \begin{cases} -\Delta_{p(k-1)}x(k-1) + r(k)h_{p(k)}(x(k)) = \lambda f(k, x(k)), & (\forall)\ k \in \mathbb{Z}[1,T], \\ x(0) - x(T+1) = 0 = \Delta x(0) - \Delta x(T), \end{cases}$$

as well as for the Neumann problem

$$(\mathcal{P}_N) \qquad \begin{cases} -\Delta_{p(k-1)}x(k-1) + r(k)h_{p(k)}(x(k)) = \lambda f(k, x(k)), & (\forall)\ k \in \mathbb{Z}[1,T], \\ \Delta x(0) = 0 = \Delta x(T), \end{cases}$$

Received October 13, 2012

where λ is a positive parameter, $\Delta x(k) = x(k+1) - x(k)$ is the forward difference operator, $\Delta_{p(\cdot)}$ stands for the discrete $p(\cdot)$-Laplacian operator, i.e.,

$$\begin{aligned} \Delta_{p(k-1)} x(k-1) &:= \Delta(h_{p(k-1)}(\Delta x(k-1))) \\ &= h_{p(k)}(\Delta x(k)) - h_{p(k-1)}(\Delta x(k-1)) \end{aligned} \tag{1.1}$$

and $f : \mathbb{Z}[1,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Throughout the paper, we assume that the variable exponent p satisfies

$$p(0) = p(T) \tag{1.2}$$

whenever we refer to the periodic problem $(\mathcal{P}_P)$. From now on, we also employ the notations:

$$p^- = \min_{k \in \mathbb{Z}[0,T]} p(k), \quad p^+ = \max_{k \in \mathbb{Z}[0,T]} p(k) \ \text{ and } \ r_- = \min_{k \in \mathbb{Z}[1,T]} r(k).$$

In the last years, the critical point theory has been extensively used to obtain multiplicity of solutions for boundary value problems involving the discrete p-Laplacian operator (see e.g., [1], [4], [7]–[13], [18], [26] and the references therein).

Boundary value problems with discrete $p(\cdot)$-Laplacian were studied in recent time; we refer the reader to [3], [15], [16], [19], [21], [22]. The existence of at least three solutions for discrete anisotropic equations subjected to homogeneous Dirichlet boundary conditions was obtained in [14], [20]. Also, using some related variational arguments, the existence of infinitely many solutions for such equations is studied in [23].

In the recent work [2], the authors have obtained the existence of ground state and saddle point solutions for problems $(\mathcal{P}_P)$ and $(\mathcal{P}_N)$ with $\lambda = 1$; also, they give an alternative variational proof of the upper and lower solutions theorem for both of the problems. By mountain pass type arguments, in [25], the existence of at least two positive solutions for problems $(\mathcal{P}_P)$ and $(\mathcal{P}_N)$ is established, for sufficiently large values of the parameter λ.

The aim of this paper is to present suitable assumptions which guarantee the existence of at least three solutions for problems $(\mathcal{P}_P)$ and $(\mathcal{P}_N)$. Hence, the first results (see Theorems 3.1 and 3.4) ensured the existence of an open interval Λ_h, such that for every $\lambda \in \Lambda_h$, problems $(\mathcal{P}_P)$ and $(\mathcal{P}_N)$ admit at least three solutions whose norms are bounded with respect to λ. Next, under a suitable sign hypothesis on f and without assuming any asymptotic condition on the primitive F of f, we obtain the existence of at least three positive solutions for problems $(\mathcal{P}_P)$, $(\mathcal{P}_N)$ (see Theorems 3.6 and 3.9), for each λ belonging to a well-defined interval.

The rest of the paper is organized as follows. The functional framework, the variational setting and the abstract three critical points theorems are presented in Section 2. In Section 3 we give our main results.

2. FUNCTIONAL FRAMEWORK AND ABSTRACT CRITICAL POINTS THEOREMS

To establish the main results we shall use a variational approach. With this aim, to treat the periodic problem $(\mathcal{P}_P)$, we introduce the space

$$X_P := \{x : \mathbb{Z}[0, T+1] \to \mathbb{R} \mid x(0) = x(T+1)\},$$

while in the case of Neumann problem $(\mathcal{P}_N)$, we shall use

$$X_N := \{x : \mathbb{Z}[0, T+1] \to \mathbb{R}\}.$$

For convenience in notations we generically denote by X one of the spaces X_P or X_N. The space X will be endowed with the Luxemburg type norm

$$\|x\|_{\eta,p(\cdot)} = \inf\left\{\nu > 0 \;:\; \sum_{k=1}^{T+1} \frac{1}{p(k-1)} \left|\frac{\Delta x(k-1)}{\nu}\right|^{p(k-1)} + \eta \sum_{k=1}^{T} \frac{1}{p(k)} \left|\frac{x(k)}{\nu}\right|^{p(k)} \le 1\right\},$$

for some $\eta > 0$. It is easy to check that for all $x \in X$ and any $\eta > 0$, one has

$$\|x\|_{\eta,p(\cdot)}^{p^-} \le \sum_{k=1}^{T+1} \frac{|\Delta x(k-1)|^{p(k-1)}}{p(k-1)} + \eta \sum_{k=1}^{T} \frac{|x(k)|^{p(k)}}{p(k)} \le \|x\|_{\eta,p(\cdot)}^{p^+}, \text{ if } \|x\|_{\eta,p(\cdot)} > 1. \tag{2.1}$$

We shall use the functional $\varphi_X : X \to \mathbb{R}$ given by

$$\varphi_X(x) = \sum_{k=1}^{T+1} \frac{|\Delta x(k-1)|^{p(k-1)}}{p(k-1)} + \sum_{k=1}^{T} \frac{r(k)}{p(k)} |x(k)|^{p(k)}, \quad (\forall)\ x \in X. \tag{2.2}$$

Standard arguments show that $\varphi_X \in C^1(X, \mathbb{R})$ and

$$\langle \varphi_X'(x), y\rangle = \sum_{k=1}^{T+1} h_{p(k-1)}(\Delta x(k-1)) \Delta y(k-1) + \sum_{k=1}^{T} r(k) h_{p(k)}(x(k)) y(k), \tag{2.3}$$

for all $x \in X$. Also, we define

$$\mathcal{F}_X(x) = \sum_{k=1}^{T} F(k, x(k)), \quad (\forall)\ x \in X, \tag{2.4}$$

where $F : \mathbb{Z}[1, T] \times \mathbb{R} \to \mathbb{R}$ is the primitive of f, i.e.,

$$F(k, t) = \int_0^t f(k, \tau) d\tau, \quad (\forall)\ k \in \mathbb{Z}[1, T],\ (\forall)\ t \in \mathbb{R}.$$

It is easy to see that $\mathcal{F}_X \in C^1(X, \mathbb{R})$ and

$$\langle \mathcal{F}_X'(x), y\rangle = \sum_{k=1}^{T} f(k, x(k)) y(k), \quad (\forall)\ x, y \in X. \tag{2.5}$$

The energy functional corresponding to problem $(\mathcal{P}_P)$ (resp. $(\mathcal{P}_N)$) is

$$\Phi_X(x) = \varphi_X(x) - \lambda \mathcal{F}_X(x), \quad (\forall)\ x \in X,$$

with $X = X_P$ (resp. $X = X_N$). From (2.3) and (2.5), one has

$$\begin{aligned}\langle \Phi'_X(x), y\rangle &= \sum_{k=1}^{T+1} h_{p(k-1)}(\Delta x(k-1))\Delta y(k-1) \\ &\quad + \sum_{k=1}^{T} r(k)h_{p(k)}(x(k))y(k) - \lambda \sum_{k=1}^{T} f(k,x(k))y(k), \quad (\forall)\ x,y \in X.\end{aligned}$$

The search of solutions of problem $(\mathcal{P}_P)$ reduces to finding critical points of the energy functional Φ_{X_P} by the following

Proposition 2.1 (see [2, Proposition 2.1])**.** *Assume that hypothesis* (1.2) *holds true. A function $x \in X_P$ is solution of problem $(\mathcal{P}_P)$ if and only if it is a critical point of Φ_{X_P}.*

Also, we have

Proposition 2.2 (see [2, Proposition 2.3])**.** *A function $x \in X_N$ is solution of problem $(\mathcal{P}_N)$ if and only if it is a critical point of Φ_{X_N}.*

Next, we recall for reader's convenience, two theorems which will be employed in our proofs. The first was obtained in [5] as a consequence of a three critical points theorem of B. Ricceri [24], by using some results on a suitable minimax inequality (also see [6]). The second one was established in [7] (also see [10]) and it is a finite dimensional variant of Theorem 3.3 in [8].

Theorem 2.3 ([5, Theorem 2.1])**.** *Let $(Y, \|\cdot\|)$ be a separable and reflexive real Banach space, and let $\psi, J : Y \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals. Assume that there exists $y_0 \in Y$ such that $\psi(y_0) = J(y_0) = 0$ and $\psi(y) \geq 0$ for every $y \in Y$ and that there exist $y_1 \in Y$, $\omega > 0$ such that*

(i$_1$) *$\omega < \psi(y_1)$;*

(i$_2$) $\sup\limits_{\psi(y)<\omega} J(y) < \omega \dfrac{J(y_1)}{\psi(y_1)}$.

Further, put

$$\overline{a} = \frac{h\omega}{\omega \frac{J(y_1)}{\psi(y_1)} - \sup_{\psi(y)<\omega} J(y)},$$

with $h > 1$, assume that the functional $\psi - \lambda J$ is sequentially weakly lower semicontinuous, satisfies the Palais-Smale (in short, (PS)) condition and

(i$_3$) $\lim\limits_{\|y\|\to+\infty} (\psi(y) - \lambda J(y)) = +\infty$, *for every $\lambda \in [0, \overline{a}]$.*

Then, there exists an open interval $\Lambda \subseteq [0, \overline{a}]$ and a positive real number μ such that, for each $\lambda \in \Lambda$, the equation $\psi'(y) - \lambda J'(y) = 0$ admits at least three solutions in Y, whose norms are less than μ.

Theorem 2.4 ([7, Theorem 2.1], [10, Theorem 31]). *Let Y be a finite dimensional real Banach space and $\psi, J : Y \to \mathbb{R}$ be two functionals of class C^1 on Y, with ψ coercive. Moreover, assume that*

(i$_4$) *ψ is convex and $\inf_Y \psi = \psi(0) = J(0) = 0$;*

(i$_5$) *for each $\lambda > 0$ and every u_1, u_2 which are local minima for the functional $\psi - \lambda J$ such that $J(u_1) \geq 0$ and $J(u_2) \geq 0$, one has*

$$\inf_{\xi \in [0,1]} J(\xi u_1 + (1-\xi)u_2) \geq 0.$$

Further, assume that there are two positive constants ω_1, ω_2 and $v \in Y$, with $\omega_1 < \psi(v) < \omega_2/2$ such that

(i$_6$) $\dfrac{\sup_{y \in \psi^{-1}(-\infty,\omega_1)} J(y)}{\omega_1} \leq \dfrac{J(v)}{2\psi(v)}$ *and* $\dfrac{\sup_{y \in \psi^{-1}(-\infty,\omega_2)} J(y)}{\omega_2} \leq \dfrac{J(v)}{4\psi(v)}$.

Then, for each

$$\lambda \in \left(\frac{2\psi(v)}{J(v)}, \min\left\{ \frac{\omega_1}{\sup_{y \in \psi^{-1}(-\infty,\omega_1)} J(y)}, \frac{\omega_2/2}{\sup_{y \in \psi^{-1}(-\infty,\omega_2)} J(y)} \right\} \right),$$

the functional $\psi - \lambda J$ admits at least three distinct critical points y_1, y_2, y_3 such that $y_1 \in \psi^{-1}(-\infty, \omega_1)$, $y_2 \in \psi^{-1}(\omega_1, \omega_2/2)$ and $y_3 \in \psi^{-1}(-\infty, \omega_2)$.

3. MAIN RESULTS

Under suitable assumptions, first we obtain the existence of an open interval Λ_h, depending on $h > 1$, such that problems $(\mathcal{P}_P)$ and $(\mathcal{P}_N)$ admit at least three solutions for every $\lambda \in \Lambda_h$. Moreover, an upper bound for Λ_h is established.

For each positive constant c, we shall use the notations:

$$\Gamma_{\min}(c) := \frac{\sum_{k=1}^{T} F(k,c)}{\min\{c^{p^-}, c^{p^+}\}} \quad \text{and} \quad \Gamma_{\max}(c) := \frac{\sum_{k=1}^{T} F(k,c)}{\max\{c^{p^-}, c^{p^+}\}}.$$

Theorem 3.1. *Assume that there exist positive constants c, d with $c < d$ such that*

$$\sum_{k=1}^{T} \sup_{|t|<c} F(k,t) < \frac{p^- \min\{c^{p^-}, c^{p^+}\} r_-}{p^+ \sum_{k=1}^{T} r(k)} \Gamma_{\max}(d) \tag{3.1}$$

and

$$\limsup_{|t| \to \infty} \frac{F(k,t)}{|t|^{p(k)}} \leq 0, \quad (\forall)\ k \in \mathbb{Z}[1,T]. \tag{3.2}$$

Also, we set

$$\tilde{a} = \left(\frac{p^- \Gamma_{\max}(d)}{\sum_{k=1}^{T} r(k)} - \frac{p^+ \sum_{k=1}^{T} \sup_{|t|<c} F(k,t)}{r_- \min\{c^{p^-}, c^{p^+}\}} \right)^{-1}. \tag{3.3}$$

If (1.2) *holds true, then for every $h > 1$, there exists an open interval $\Lambda_h \subseteq [0, h\tilde{a}]$ and a positive real number μ such that, for all $\lambda \in \Lambda_h$, problem $(\mathcal{P}_P)$ admits at least three solutions in X_P, whose norms are less than μ.*

Proof. We apply Theorem 2.3 with $Y = X_P$, $\psi = \varphi_{X_P}$ (see (2.2)) and $J = \mathcal{F}_{X_P}$ (see (2.4)). Clearly, the regularity assumptions required on $\varphi_{X_P}, \mathcal{F}_{X_P}$ and X_P are satisfied. Also, $\varphi_{X_P}(0) = \mathcal{F}_{X_P}(0) = 0$ and $\varphi_{X_P}(x) \geq 0$, for all $x \in X_P$.

We denote

$$\omega = \frac{r_- \min\{c^{p^-}, c^{p^+}\}}{p^+} > 0$$

and since $c < d$, one has

$$\varphi_{X_P}(d) = \sum_{k=1}^{T} \frac{r(k)}{p(k)} d^{p(k)} \geq \frac{\min\{d^{p^-}, d^{p^+}\}}{p^+} \sum_{k=1}^{T} r(k) > \frac{r_- \min\{c^{p^-}, c^{p^+}\}}{p^+} = \omega, \tag{3.4}$$

that is, condition (i_1), with $y_1(k) \equiv d \in X_P$, for all $k \in \mathbb{Z}[0, T+1]$.

Also, it is easy to see that (3.1) implies

$$\sum_{k=1}^{T} F(k, d) > 0. \tag{3.5}$$

If $\varphi_{X_P}(x) < \omega$, then

$$\sum_{k=1}^{T} |x(k)|^{p(k)} < \min\{c^{p^-}, c^{p^+}\}$$

and hence, for each $k \in \mathbb{Z}[1, T]$, one obtains

$$|x(k)| < \min\{c^{p^-}, c^{p^+}\}^{\frac{1}{p(k)}}.$$

If $c \geq 1$, then $|x(k)| < c^{\frac{p^-}{p(k)}} < c$. Also, $|x(k)| < c^{\frac{p^+}{p(k)}} < c$, provided that $c \in (0, 1)$. Therefore, $\max_{k \in \mathbb{Z}[1,T]} |x(k)| < c$ and from (3.1) and (3.5), we infer

$$\begin{aligned}
\sup_{\varphi_{X_P}(x) < \omega} \mathcal{F}_{X_P}(x) &\leq \sup_{\max_{k \in \mathbb{Z}[1,T]} |x(k)| < c} \mathcal{F}_{X_P}(x) \leq \sum_{k=1}^{T} \sup_{|t| < c} F(k, t) \\
&< \frac{p^- \min\{c^{p^-}, c^{p^+}\} r_-}{p^+ \sum_{k=1}^{T} r(k)} \, \Gamma_{\max}(d) = \frac{\omega p^- \mathcal{F}_{X_P}(d)}{\max\{d^{p^-}, d^{p^+}\} \sum_{k=1}^{T} r(k)} \\
&\leq \omega \frac{\mathcal{F}_{X_P}(d)}{\varphi_{X_P}(d)}.
\end{aligned}$$

Thus, condition (i_2) is satisfied. Moreover, we have

$$\begin{aligned}
\overline{a} &= \frac{h\omega}{\omega \frac{\mathcal{F}_{X_P}(d)}{\varphi_{X_P}(d)} - \sup_{\varphi_{X_P}(x) < \omega} \mathcal{F}_{X_P}(x)} \\
&\leq \frac{h\omega}{\frac{p^- r_- \min\{c^{p^-}, c^{p^+}\} \Gamma_{\max}(d)}{p^+ \sum_{k=1}^{T} r(k)} - \sum_{k=1}^{T} \sup_{|t| < c} F(k, t)} = h\tilde{a},
\end{aligned}$$

with $\tilde{a}$ given in (3.3). Now, we consider $g(k, t) = \lambda f(k, t) - r(k) h_{p(k)}(t)$, for all $k \in \mathbb{Z}[1, T]$ and $t \in \mathbb{R}$. Then,

$$G(k, t) = \lambda F(k, t) - r(k) \frac{|t|^{p(k)}}{p(k)}$$

and in view of (3.2), we obtain the following Hammerstein type condition

$$\limsup_{|t|\to\infty} \frac{G(k,t)}{|t|^{p(k)}} \le -\frac{r(k)}{p(k)} < 0, \quad (\forall)\ k \in \mathbb{Z}[1,T]. \tag{3.6}$$

Next, we use the same arguments as in the proof of Theorem 3.1 in [17]. From (3.6), there are constants $\sigma > 0$ and $\rho > 0$ such that

$$G(k,t) \le -\frac{\sigma}{p(k)}|t|^{p(k)}, \quad (\forall)\ k \in \mathbb{Z}[1,T],\ (\forall)\ t \in \mathbb{R} \text{ with } |t| > \rho.$$

On the other hand, by the continuity of G, there is a constant $M_\rho > 0$ such that

$$|G(k,t)| \le M_\rho, \quad (\forall)\ k \in \mathbb{Z}[1,T],\ (\forall)\ t \in \mathbb{R} \text{ with } |t| \le \rho.$$

Hence, we infer

$$G(k,t) \le M_\rho + \frac{\sigma}{p(k)}\rho^{p(k)} - \frac{\sigma}{p(k)}|t|^{p(k)}, \quad (\forall)\ k \in \mathbb{Z}[1,T],\ (\forall)\ t \in \mathbb{R}.$$

To prove the coercivity of $\varphi_{X_P} - \lambda\mathcal{F}_{X_P}$ (i.e., Φ_{X_P}), from the above inequality we have

$$\begin{aligned}
\Phi_{X_P}(x) &= \sum_{k=1}^{T+1} \frac{|\Delta x(k-1)|^{p(k-1)}}{p(k-1)} - \sum_{k=1}^{T} G(k,x(k)) \\
&\ge \sum_{k=1}^{T+1} \frac{|\Delta x(k-1)|^{p(k-1)}}{p(k-1)} - \frac{\sigma}{p^-}(\rho^{p^-} + \rho^{p^+})T - M_\rho T + \sigma \sum_{k=1}^{T} \frac{|x(k)|^{p(k)}}{p(k)} \\
&= \sum_{k=1}^{T+1} \frac{|\Delta x(k-1)|^{p(k-1)}}{p(k-1)} + \sigma \sum_{k=1}^{T} \frac{|x(k)|^{p(k)}}{p(k)} - C_1, \quad (\forall)\ x \in X_P,
\end{aligned}$$

where $C_1 = M_\rho T + \frac{\sigma}{p^-}(\rho^{p^-} + \rho^{p^+})T$. This, together with (2.1), yields

$$\Phi_{X_P}(x) \ge \|x\|_{\sigma,p(\cdot)}^{p^-} - C_1, \quad (\forall)\ x \in X_P,\ \|x\|_{\sigma,p(\cdot)} > 1.$$

Consequently, Φ_{X_P} is coercive, for every $\lambda > 0$. So, condition (i_3) is fulfilled. Also, it is easy to see that from coercivity, Φ_{X_P} satisfies (PS) condition.

Thus, from Theorem 2.3, for every $h > 1$, there exists an open interval $\Lambda_h \subseteq [0, h\tilde{a}]$ and a positive real number μ such that, for all $\lambda \in \Lambda_h$, equation

$$\varphi'_{X_P}(x) - \lambda\mathcal{F}'_{X_P}(x) = 0$$

admits at least three solutions in X_P and by virtue of (1.2) and Proposition 2.1, we have that problem $(\mathcal{P}_P)$ admits at least three solutions in X_P, whose norms are less than μ and the proof is complete. □

Remark 3.2. (i) We note that according to the proof of Theorem 2.3 (also see [5, Proposition 1.3]), μ entering in Theorem 3.1 satisfies

$$\sup_{\varphi_{X_P}(x)<\omega} \mathcal{F}_{X_P}(x) + \frac{\omega\frac{\mathcal{F}_{X_P}(d)}{\varphi_{X_P}(d)} - \sup_{\varphi_{X_P}(x)<\omega}\mathcal{F}_{X_P}(x)}{h} < \mu < \omega\frac{\mathcal{F}_{X_P}(d)}{\varphi_{X_P}(d)},$$

for every $h > 1$.

(ii) It is worth to point out that applying [10, Theorem 30], under the assumptions (1.2), (3.1) and (3.2) from Theorem 3.1, we also obtain in a similar way as above that, for every

$$\lambda \in \left(\frac{\sum_{k=1}^{T} r(k)}{p^- \Gamma_{\max}(d)}, \ \frac{r_- \min\{c^{p^-}, c^{p^+}\}}{p^+ \sum_{k=1}^{T} \sup_{|t|<c} F(k,t)} \right),$$

problem $(\mathcal{P}_P)$ admits at least three solutions in X_P, such that at least one is in

$$\varphi_{X_P}^{-1}\left(-\infty, \ \frac{r_- \min\{c^{p^-}, c^{p^+}\}}{p^+} \right)$$

and another one in

$$\varphi_{X_P}^{-1}\left(\frac{r_- \min\{c^{p^-}, c^{p^+}\}}{p^+}, \ +\infty \right).$$

(iii) Theorem 3.2 proved in [4] for p=constant is an immediate consequence of Theorem 3.1.

Example 3.3. Let $p^- = 5$, $p^+ = 17$, $T = 15$, $\lambda > 0$, $r(k) \equiv 1$ and $f(k,t) = 2k(t^3 - t)$, for all $k \in \mathbb{Z}[1,15]$, $t \in \mathbb{R}$. We consider the problem

$$\left\{ \begin{array}{l} -\Delta_{p(k-1)} x(k-1) + h_{p(k)}(x(k)) = 2\lambda k (x(k)^3 - x(k)), \quad (\forall)\ k \in \mathbb{Z}[1,15], \\ x(0) - x(16) = 0 = \Delta x(0) - \Delta x(15). \end{array} \right. \tag{3.7}$$

We have

$$F(k,t) = k\left(\frac{t^4}{2} - t^2 \right), \quad (\forall)\ k \in \mathbb{Z}[1,15],\ (\forall)\ t \in \mathbb{R}.$$

If we choose $c = 1$ and $d = 2$, then it is easy to see that the conditions of Theorem 3.1 are satisfied. Hence, if $p(0) = p(15)$, then for every $h > 1$, there exists an open interval $\Lambda_h \subseteq [0, 2^{12}h/5]$ and a positive real number μ such that, for every $\lambda \in \Lambda_h$, problem (3.7) has at least three solutions in X_P, whose norms are less than μ.

Using exactly the same strategy as above we have the following

Theorem 3.4. *Assume that there exist positive constants c, d with $c < d$ such that* (3.1) *and* (3.2) *hold true and let $\tilde{a}$ be given by* (3.3). *Then, for every $h > 1$, there exists an open interval $\Lambda_h \subseteq [0, h\tilde{a}]$ and a positive real number μ such that, for all $\lambda \in \Lambda_h$, problem $(\mathcal{P}_N)$ admits at least three solutions in X_N, whose norms are less than μ.*

In order to obtain at least three positive solutions for the periodic problem $(\mathcal{P}_P)$, we shall need the following maximum principle.

Lemma 3.5. *If*

$$\left\{ \begin{array}{l} -\Delta_{p(k-1)} x(k-1) + r(k) h_{p(k)}(x(k)) \geq 0, \quad (\forall)\ k \in \mathbb{Z}[1,T], \\ x(0) - x(T+1) = 0 = \Delta x(0) - \Delta x(T), \end{array} \right. \tag{3.8}$$

then either $x > 0$ *in* $\mathbb{Z}[0, T+1]$ *or* $x \equiv 0$.

Proof. Let $j \in \mathbb{Z}[1, T]$ be such that $x(j) = \min_{k \in \mathbb{Z}[1,T]} x(k)$. Clearly,

$$\Delta x(j) \geq 0 \quad \text{and} \quad \Delta x(j-1) \leq 0. \tag{3.9}$$

From (3.8), (1.1) and (3.9), we obtain

$$r(j) h_{p(j)}(x(j)) \geq |\Delta x(j)|^{p(j)-2} \Delta x(j) - |\Delta x(j-1)|^{p(j-1)-2} \Delta x(j-1) \geq 0,$$

which implies that $x \geq 0$, for all $k \in \mathbb{Z}[0, T+1]$.

Moreover, assuming that $x(j) = 0$, from the previous inequality and nonnegativity of $x(j-1)$ and $x(j+1)$, we have

$$0 \leq |x(j+1)|^{p(j)-2} x(j+1) + |x(j-1)|^{p(j-1)-2} x(j-1) \leq 0$$

and so, $x(j+1) = x(j-1) = 0$. Thus, repeating these arguments, the conclusion follows at once. □

Theorem 3.6. *Let* f *be a positive continuous function on* $\mathbb{Z}[1, T] \times [0, \infty)$. *Assume that there exist three positive constants* c_1, d *and* c_2, *with* $c_1 < d$, *such that*

$$\max\{d^{p^-}, d^{p^+}\} < \frac{p^-}{2p^+} \frac{r_-}{\sum_{k=1}^T r(k)} \min\{c_2^{p^-}, c_2^{p^+}\} \tag{3.10}$$

and

$$\max\{\Gamma_{\min}(c_1),\ 2\Gamma_{\min}(c_2)\} < \frac{p^-}{2p^+} \frac{r_-}{\sum_{k=1}^T r(k)}\ \Gamma_{\max}(d). \tag{3.11}$$

If (1.2) *holds true, then for each*

$$\lambda \in \left(\frac{2\sum_{k=1}^T r(k)}{p^- \Gamma_{\max}(d)}, \frac{r_-}{p^+ \max\{\Gamma_{\min}(c_1),\ 2\Gamma_{\min}(c_2)\}} \right), \tag{3.12}$$

problem $(\mathcal{P}_P)$ *admits at least three distinct positive solutions* x_1, x_2, x_3 *in* X_P, *such that*

$$x_i(k) < c_2, \quad (\forall)\ k \in \mathbb{Z}[0, T+1],\ i = 1, 2, 3.$$

Proof. Without loss of generality, we may assume that $f(k,t) = f(k,0)$, for all $(k,t) \in \mathbb{Z}[1, T] \times (-\infty, 0)$. We shall apply Theorem 2.4 with $Y = X_P$, $\psi = \varphi_{X_P}$ and $J = \mathcal{F}_{X_P}$. From (2.1), we have

$$\varphi_{X_P}(x) \geq \|x\|_{r_-, p(\cdot)}^{p^-}, \quad (\forall)\ x \in X_P,\ \|x\|_{r_-, p(\cdot)} > 1,$$

which imply that φ_{X_P} is coercive. Clearly, (i_4) is fulfilled.

Let u_1, u_2 be two local minima of Φ_{X_P}. They are two solutions for problem $(\mathcal{P}_P)$ and owing to Lemma 3.5, one has $\xi u_1(k) + (1-\xi) u_2(k) \geq 0$, for all $k \in \mathbb{Z}[0, T+1]$ and all $\xi \in [0, 1]$. Hence,

$$\mathcal{F}_{X_P}(\xi u_1 + (1-\xi) u_2) \geq 0, \quad (\forall)\ \xi \in [0, 1]$$

and (i_5) is verified.

Setting

$$\omega_1 = \frac{r_- \min\{c_1^{p^-}, c_1^{p^+}\}}{p^+} \quad \text{and} \quad \omega_2 = \frac{r_- \min\{c_2^{p^-}, c_2^{p^+}\}}{p^+},$$

in the same way as in the proof of Theorem 3.1, if $\varphi_{X_P}(x) < \omega_1$ (resp. $\varphi_{X_P}(x) < \omega_2$), we have that $\max_{k\in\mathbb{Z}[1,T]} |x(k)| < c_1$ (resp. $\max_{k\in\mathbb{Z}[1,T]} |x(k)| < c_2$). Therefore, one obtains

$$\frac{\sup_{x\in\varphi_{X_P}^{-1}(-\infty,\omega_1)} \mathcal{F}_{X_P}(x)}{\omega_1} \leq \frac{\sup_{\max_{k\in\mathbb{Z}[1,T]} |x(k)|<c_1} \mathcal{F}_{X_P}(x)}{\omega_1} \leq$$

$$\frac{\sum_{k=1}^T \sup_{|t|<c_1} F(k,t)}{\omega_1} \leq \frac{\sum_{k=1}^T F(k,c_1)}{\omega_1} = \frac{p^+}{r_-}\, \Gamma_{\min}(c_1), \tag{3.13}$$

as well as

$$\frac{\sup_{x\in\varphi_{X_P}^{-1}(-\infty,\omega_2)} \mathcal{F}_{X_P}(x)}{\omega_2} \leq \frac{p^+}{r_-}\, \Gamma_{\min}(c_2). \tag{3.14}$$

On the other hand, since $c_1 < d$, we get $\varphi_{X_P}(d) > \omega_1$ (see (3.4)). Also, from (3.10), one has

$$\varphi_{X_P}(d) \leq \frac{\max\{d^{p^-}, d^{p^+}\}}{p^-} \sum_{k=1}^T r(k) < \frac{r_- \min\{c_2^{p^-}, c_2^{p^+}\}}{2p^+} = \frac{\omega_2}{2}.$$

So, we have $\omega_1 < \varphi_{X_P}(d) < \omega_2/2$. Now, using (3.11), (3.13), (3.14) and the fact that $\sum_{k=1}^T F(k,d) > 0$, we infer

$$\frac{\sup_{x\in\varphi_{X_P}^{-1}(-\infty,\omega_1)} \mathcal{F}_{X_P}(x)}{\omega_1} \leq \frac{p^+}{r_-}\, \Gamma_{\min}(c_1) < \frac{p^-}{2\sum_{k=1}^T r(k)}\, \Gamma_{\max}(d) \leq \frac{\mathcal{F}_{X_P}(d)}{2\varphi_{X_P}(d)},$$

respectively,

$$\frac{\sup_{x\in\varphi_{X_P}^{-1}(-\infty,\omega_2)} \mathcal{F}_{X_P}(x)}{\omega_2} \leq \frac{p^+}{r_-}\, \Gamma_{\min}(c_2) < \frac{p^-}{4\sum_{k=1}^T r(k)}\, \Gamma_{\max}(d) \leq \frac{\mathcal{F}_{X_P}(d)}{4\varphi_{X_P}(d)}$$

and (i_6) holds true, with $v(k) \equiv d \in X_P$, for all $k \in \mathbb{Z}[0, T+1]$. Further, again from (3.13) and (3.14), one has that

$$\lambda \in \left(\frac{2\varphi_{X_P}(d)}{\mathcal{F}_{X_P}(d)},\ \min\left\{ \frac{\omega_1}{\sup_{x\in\varphi_{X_P}^{-1}(-\infty,\omega_1)} \mathcal{F}_{X_P}(x)}; \frac{\omega_2/2}{\sup_{x\in\varphi_{X_P}^{-1}(-\infty,\omega_2)} \mathcal{F}_{X_P}(x)} \right\} \right).$$

Therefore, the functional Φ_{X_P} admits at least three critical points $x_i \in X_P$, $i = 1, 2, 3$, which on account of (1.2) and Proposition 2.1 are solutions of problem $(\mathcal{P}_P)$ and owing to Lemma 3.5, are positive functions.

Finally, for $i = 1, 2, 3$, since $\omega_1 < \omega_2$ and $\varphi_{X_P}(x_i) < \omega_2$, we get that

$$\max_{k\in\mathbb{Z}[1,T]} x_i(k) < c_2$$

and the end points inequality follows from the boundary conditions and the proof is complete. □

Remark 3.7. Since the range of solutions obtained in Theorem 3.6 is in $[0, c_2]$, the conclusion still remains true if we assume that f is a positive continuous function only in $\mathbb{Z}[1, T] \times [0, c_2]$. Also, it can be seen from the proof of Theorem 3.6 that if f is only nonnegative on $\mathbb{Z}[1, T] \times [0, c_2]$, then problem $(\mathcal{P}_P)$ has at least two positive solutions.

Example 3.8. Let $p^- = 7$, $p^+ = 12$, $T = 15$, $\lambda > 0$, $r(k) \equiv 1$ and

$$f(k, t) = \begin{cases} k, & 0 \leq t < 1, \\ kt^{18}, & 1 \leq t < 5, \\ k(10 - t)^{18}, & 5 \leq t < 9, \\ k, & t \geq 9, \end{cases}$$

for all $k \in \mathbb{Z}[1, 15]$, $t \in [0, \infty)$. By a simple computation we see that the conditions in Theorem 3.6 are satisfied if we choose $c_1 = 1$, $d = 2$ and $c_2 = 9$. Hence, with f defined above, if $p(0) = p(15)$, then for each $\lambda \in (19/3584, 19/1440)$, the problem

$$\begin{cases} -\Delta_{p(k-1)} x(k - 1) + h_{p(k)}(x(k)) = \lambda f(k, t), \quad (\forall)\ k \in \mathbb{Z}[1, 15], \\ x(0) - x(16) = 0 = \Delta x(0) - \Delta x(15) \end{cases}$$

has at least three distinct positive solutions $x_i \in X_P$, $i = 1, 2, 3$, such that, for each $k \in \mathbb{Z}[0, 16]$, one has $x_i < 9$, $i = 1, 2, 3$.

It is easy to check that Lemma 3.5 remains valid with

$$\begin{cases} -\Delta_{p(k-1)} x(k - 1) + r(k) h_{p(k)}(x(k)) \geq 0, \quad (\forall)\ k \in \mathbb{Z}[1, T], \\ \Delta x(0) = 0 = \Delta x(T), \end{cases}$$

instead of (3.8). Hence, for the Neumann problem $(\mathcal{P}_N)$, by no longer than "mutatis mutandis" arguments (also, see Remark 3.7), we have the following

Theorem 3.9. *Assume that there exist three positive constants c_1, d and c_2, with $c_1 < d$, such that* (3.10) *and* (3.11) *hold true. If f is a positive continuous function on $\mathbb{Z}[1, T] \times [0, c_2]$, then for each λ as in* (3.12)*, problem $(\mathcal{P}_N)$ admits at least three distinct positive solutions x_1, x_2, x_3 in X_N, such that*

$$x_i(k) < c_2, \quad (\forall)\ k \in \mathbb{Z}[0, T + 1],\ i = 1, 2, 3.$$

Acknowledgements. The support for P. Jebelean from the grant TE-PN-II-RU-TE-2011-3-0157 (CNCS-Romania) is gratefully acknowledged. Also, the work of C. Şerban was supported by the strategic grant POSDRU/CPP107/DMI1.5/S/78421, Project ID 78421 (2010), co-financed by the European Social Fund - Investing in People, within the Sectoral Operational Programme Human Resources Development 2007-2013.

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