Dynamic Systems and Applications 22 (2013) 197-206

COMPOSING FUNCTIONS OF BOUNDED KORENBLUM VARIATION

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To the Memory of V. Lakshmikantham (1924–2012)

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ABSTRACT. We prove that the autonomous composition operator $Hf = h \circ f$ maps the space κBV of all functions of bounded Korenblum variation into itself if and only if the generating function h is locally Lipschitz on the real line.

AMS (MOS) Subject Classification. primary 47H30, secondary 26A45.

1. KORENBLUM VARIATION

More than 100 years ago, Camille Jordan [9] introduced the total variation

$$\mathrm{Var}(f) = \mathrm{Var}(f;[a,b]) := \sup\left\{\mathrm{Var}(f,P;[a,b]) : P \in \mathcal{P}([a,b])\right\} \tag{1.1}$$

of a function $f : [a,b] \to \mathbb{R}$, where

$$\mathrm{Var}(f,P) = \mathrm{Var}(f,P;[a,b]) := \sum_{j=1}^{m} |f(t_j) - f(t_{j-1})| \tag{1.2}$$

denotes the variation of f with respect to a partition $P = \{t_0, t_1, \dots, t_m\}$, and the supremum in (1.1) is taken over the family $\mathcal{P}([a,b])$ of all partitions of $[a,b]$. In the same paper [9], Jordan proved that every function of bounded total variation can be represented as difference of two monotonically increasing functions, which means that the corresponding space $BV([a,b])$ of *functions of bounded variation* is the linear hull of all monotone functions. This space plays a prominent role in real analysis, functional analysis, Fourier analysis, geometric measure theory, and even some parts of mathematical physics.

Subsequently, Jordan's concept of variation has been generalized in many directions. Wiener [36] distorted the measurement of intervals in the range using powers $|f(t_j) - f(t_{j-1})|^p$ in (1.2), while Young [37] used more general distortions of the form

Received November 7, 2012

$\phi(|f(t_j) - f(t_{j-1})|)$, with ϕ being a convex increasing function. The most general concept by Schramm [34] replaced such distortion functions ϕ by countable families Φ. All these extensions have the advantage to make it possible to define quite general Riemann-Stieltjes integrals. On the other hand, a flaw is the loss of an effective decomposition of a function from the corresponding function classes into, hopefully, simpler functions, such as in the Jordan decomposition for functions from the classical space $BV([a, b])$.

Another important generalization consists in replacing the difference $|f(t_j) - f(t_{j-1})|$ in (1.2) by certain higher order divided differences; the corresponding higher order variations have been introduced, as far as we know, by Popoviciu [21, 22] and subsequently studied in detail by Russell [23–33]. Such variations in fact admit natural decomposition theorems; for example, one can prove that every function of bounded second variation can be represented as difference of two convex functions.

In 1975, B. Korenblum [11] considered a completely new kind of variation, called *κ-variation*, introducing a function κ for distorting the expression $|t_j - t_{j-1}|$ in the partition itself, rather than the expression $|f(t_j) - f(t_{j-1})|$ in the range. Subsequently, this class of functions has been studied in some detail in [8] and [13] and applied to Fourier series in [12]. One advantage of this alternate approach is that a *function of bounded κ-variation* may be decomposed into the difference of two simpler functions called *κ-decreasing functions*, for the precise definition and a proof of this result see [8]. In what follows, we will mostly consider, without loss of generality, functions over $[a, b] = [0, 1]$.

Definition 1.1. A function $\kappa : [0, 1] \to [0, 1]$ is called *distortion function* if κ is continuous, increasing, concave, and satisfies $\kappa(0) = 0$, $\kappa(1) = 1$, and

$$\lim_{t\to 0+} \frac{\kappa(t)}{t} = \infty, \tag{1.3}$$

i.e., has infinite slope at the origin.

Note that from the estimate

$$\frac{\kappa(s + t) - \kappa(t)}{(s + t) - t} \le \frac{\kappa(s) - \kappa(0)}{s - 0}$$

it follows that a distortion function is always subadditive in the sense that

$$\kappa(s + t) \le \kappa(s) + \kappa(t) \qquad (0 \le s, t \le 1).$$

In addition, without loss of generality we may assume that

$$\kappa(t) \ge t \qquad (0 \le t \le 1). \tag{1.4}$$

The simplest example is of course $\kappa(t) = t^\alpha$ for $0 < \alpha < 1$, another interesting example is $\kappa(t) = t(1 - \log t)$.

Building on the concept of distortion functions, Korenblum [11] introduced a new class of functions of bounded variation as follows.

Definition 1.2. Given a partition $P = \{t_0, t_1, \ldots, t_m\} \in \mathcal{P}([0,1])$, a distortion function $\kappa : [0,1] \to [0,1]$, and a function $f : [0,1] \to \mathbb{R}$, the nonnegative real number

$$\mathrm{Var}_\kappa(f, P) = \mathrm{Var}_\kappa(f, P; [0,1]) := \frac{\sum_{j=1}^{m} |f(t_j) - f(t_{j-1})|}{\sum_{j=1}^{m} \kappa(t_j - t_{j-1})} \tag{1.5}$$

is called the *κ-variation* of f on $[0,1]$ with respect to P, while the (possibly infinite) number

$$\mathrm{Var}_\kappa(f) = \mathrm{Var}_\kappa(f; [0,1]) := \sup \{\mathrm{Var}_\kappa(f, P; [0,1]) : P \in \mathcal{P}([0,1])\}, \tag{1.6}$$

where the supremum is taken over all partitions of $[0,1]$, is called the *total κ-variation* of f on $[0,1]$. In case $\mathrm{Var}_\kappa(f; [0,1]) < \infty$ we say that f has *finite κ-variation* on $[0,1]$ and write $f \in \kappa BV([0,1])$.

It is not hard to see that the set $\kappa BV([0,1])$ equipped with the norm

$$\|f\|_{\kappa BV} = |f(0)| + \mathrm{Var}_\kappa(f; [0,1]) \tag{1.7}$$

is a Banach space. Considering the special partition $P_t := \{0, t, 1\}$ for fixed t, we have

$$\mathrm{Var}_\kappa(f, P_t) = \frac{|f(1) - f(t)| + |f(t) - f(0)|}{\kappa(1-t) + \kappa(t)} \geq \frac{|f(t) - f(0)|}{2} \geq \frac{1}{2}(|f(t)| - |f(0)|)$$

which shows that every function $f \in \kappa BV([0,1])$ is bounded with

$$\|f\|_\infty \leq 2\|f\|_{\kappa BV}, \tag{1.8}$$

where $\|\cdot\|_\infty$ denotes the supremum norm.

Of course, Definition 1.2 may be formulated also for functions on an arbitrary interval $[a,b]$ by defining that f belongs to $\kappa BV([a,b])$ if the function $x \mapsto f((b-a)x + a)$ belongs to $\kappa BV([0,1])$. Equivalently, this means that we replace (1.5) by

$$\mathrm{Var}_\kappa(f, P) = \mathrm{Var}_\kappa(f, P; [a,b]) := \frac{\sum_{j=1}^{m} |f(t_j) - f(t_{j-1})|}{\sum_{j=1}^{m} \kappa\left(\frac{t_j - t_{j-1}}{b-a}\right)}.$$

From the subadditivity of κ it follows then that

$$\sum_{j=1}^{m} \kappa\left(\frac{t_j - t_{j-1}}{b-a}\right) \geq \kappa\left(\sum_{j=1}^{m} \frac{t_j - t_{j-1}}{b-a}\right) = \kappa(1) = 1.$$

For any distortion function κ, the inclusions

$$BV([0,1]) \subseteq \kappa BV([0,1]) \subseteq R([0,1]) \tag{1.9}$$

hold, where $R([a,b])$ denotes the set of *regular functions*, i.e., functions which have at most jump discontinuities. We point out that the slope condition (1.3) ensures that the first inclusion in (1.9) is strict. In fact, if the limit in (1.3) is finite, one may show that $\kappa BV([0,1]) = BV([0,1])$, see [13].

To find an example of a regular function which does not belong to $\kappa BV([0,1])$ for a given distortion function κ is easy. In our first example we show that also the first inclusion in (1.9) is strict by constructing a function $f \in \kappa BV([0,1]) \setminus BV([0,1])$.

Example 1.3. Let $\kappa : [0,1] \to [0,1]$ be defined by $\kappa(t) = t^\alpha$ for some $\alpha \in (0,1)$. Put

$$\gamma := \sum_{k=1}^{\infty} \frac{1}{k^{1/\alpha}}, \qquad t_n := \frac{1}{\gamma} \sum_{k=1}^{n} \frac{1}{k^{1/\alpha}} \quad (n = 1, 2, 3, \ldots).$$

We define a function $f : [0,1] \to \mathbb{R}$ by

$$f(x) := \begin{cases} 0 & \text{for} \quad x = 0 \text{ or } x = 1, \\ (x - t_n)^\alpha & \text{for} \quad t_n \le x < t_{n+1}. \end{cases}$$

Considering partitions containing the points $t_0, t_1, \ldots, t_n$, one easily sees that

$$\mathrm{Var}(f; [0,1]) = 2 \sum_{k=1}^{\infty} \kappa \left(\frac{1}{k^{1/\alpha}} \right) = 2 \sum_{k=1}^{\infty} \frac{1}{k} = \infty,$$

and so $f \notin BV([0,1])$. However, a straightforward computation shows that $f(y) - f(x) \le \kappa(y - x)$, and so $f \in \kappa BV([0,1])$ with $\|f\|_{\kappa BV} = \mathrm{Var}_\kappa(f; [0,1]) \le 1$.

2. MAIN RESULT

Now we consider the (autonomous) composition operator

$$Hf(x) := h(f(x)) \qquad (0 \le x \le 1) \tag{2.1}$$

generated by some function $h : \mathbb{R} \to \mathbb{R}$. In what follows, the local Lipschitz condition

$$|h(u) - h(v)| \le k(r)|u - v| \qquad (|u|, |v| \le r) \tag{2.2}$$

for h will play a crucial role, because it is both necessary and sufficient for the inclusion $H(X) \subseteq X$ for many familiar function spaces X. This was shown for the space $BV([a,b])$ by Josephy [10], for the space $Lip([a,b])$ of Lipschitz continuous functions by Babaev [5], for the space $Lip_\alpha([a,b])$ of Hölder continuous functions $(0 < \alpha < 1)$ by Mukhtarov [19], for the space $AC([a,b])$ of absolutely continuous functions by the second author [16], for the space $HBV([a,b])$ of functions of bounded harmonic variation by Chaika and Waterman [6], for the space $\Lambda BV([a,b])$ of functions of bounded Λ-variation by Pierce and Waterman [20], for the space $RBV_p([a,b])$ of functions of bounded p-variation in Riesz's sense by the second author and Rivas [18], for the space $RBV_\varphi([a,b])$ of functions of bounded φ-variation in Riesz's sense by the second author [17], and for the space $WBV_\varphi([a,b])$ of functions of bounded φ-variation in Wiener's sense by the Ciemnoczołowski and Orlicz [7]. On the other hand, there are functions spaces in which condition (2.2) is either too strong or too weak to ensure that the corresponding operator (2.1) maps this space into itself. For example, it is completely obvious that the operator (2.1) maps the space $C([a,b])$ into itself if and only if h is continuous on $\mathbb{R}$, and the space $C^1([a,b])$ into itself if and only if h is continuously differentiable on $\mathbb{R}$. As was shown in [4], a more sophisticated argument based on Sierpiński's decomposition theorem [35] implies the following result in the space $R([a,b])$ of regular functions:

Theorem 2.1. *The composition operator* (2.1) *maps the space* $R([a,b])$ *into itself if and only if the corresponding function* h *is continuous on* $\mathbb{R}$*. Moreover, in this case the operator* (2.1) *is automatically bounded in the supremum norm.*

All these examples show that the problem of determining the precise class of all "admissible" functions h such that the corresponding operator H maps a certain function class into itself, is in general nontrivial and may have an unexpected solution. A unified approach to this problem for several function spaces is contained in the following theorem from [1] which we recall for further reference.

Theorem 2.2. *The composition operator* (2.1) *maps the space* $Lip([a,b])$ *into the space* $BV([a,b])$ *if and only if the corresponding function* h *satisfies* (2.2)*. Moreover, in this case the operator* (2.1) *is automatically bounded in the corresponding norms.*

Of course, the sufficiency of (2.2) for the inclusion $H(Lip) \subseteq BV$ is evident. Since

$$Lip([a,b]) \subseteq RBV_p([a,b]) \subseteq AC([a,b]) \subseteq BV([a,b]), \tag{2.3}$$

Theorem 2.2 contains the results from [5], [10], [16], [18] as special cases. Note that Theorem 2.2 covers not only the spaces occurring in (2.3), but any other "intermediate" space between Lip and BV.

Now we are interested in the problem of finding a condition on h, both necessary and sufficient, under which the operator (2.1) maps the space κBV into itself. Unfortunately, since this space is *larger* than BV, by (1.9), Theorem 2.2 does not help. In the recent paper [3] the authors study the Lipschitz continuity of the operator (2.1) in the norm (1.7), *assuming* that this operator maps κBV into itself. As far as we know, conditions for the inclusion $H(\kappa BV) \subseteq \kappa BV$ have not been given in the literature.

This is our main objective now. First of all, we remark that the fact that h belongs locally to $\kappa BV(\mathbb{R})$ does not imply that $H(\kappa BV) \subseteq \kappa BV$:

Example 2.3. Let $A \subset [0,1]$ be an uncountable Cantor set of positive measure, and define $f : [0,1] \to \mathbb{R}$ by $f(x) := \operatorname{dist}(x, A) = \inf\{|x-a| : a \in A\}$. Clearly, f is Lipschitz continuous, and therefore belongs to $\kappa BV([0,1])$ for any distortion function κ. Moreover, the function $h := \chi_{\{0\}}$ certainly belongs to $\kappa BV([a,b])$ for any interval $[a,b] \subset \mathbb{R}$. However, the function $Hf = h \circ f = \chi_A$ is not regular, since it is discontinuous on the uncountable set A, and so it does not belong to $\kappa BV([0,1])$. In fact, since all functions in κBV are regular, by (1.9), they cannot have more than countably many points of discontinuity.

Note that the inclusion (1.9) and Theorems 2.1 and 2.2 imply that the local Lipschitz condition (2.2) for h is sufficient, while continuity of h is necessary for the inclusion $H(\kappa BV) \subseteq \kappa BV$. In the following theorem which is the main result of this paper we show that (2.2) is actually the "right" condition:

Theorem 2.4. *The composition operator* (2.1) *maps the space* $\kappa BV([a,b])$ *into itself if and only if the corresponding function* h *satisfies* (2.2). *Moreover, in this case the operator* (2.1) *is automatically bounded in the norm* (1.7).

Proof. We take $[a,b] = [0,b]$. The sufficiency of (2.2) for the inclusion $H(\kappa BV) \subseteq \kappa BV$ is obvious. Choose a positive sequence $(\varepsilon_k)_k$ satisfying

$$\sum_{k=1}^{\infty} \varepsilon_k \le \frac{b}{5}, \tag{2.4}$$

and suppose that h does not satisfy a local Lipschitz condition, which means that there is some $r > 0$ for which (2.2) does not hold for any constant $k(r) > 0$. Then there exist sequences $(u_k)_k$ and $(v_k)_k$ in $[-r,r]$ such that $u_k < v_k$,

$$\delta_k := \kappa^{-1}(v_k - u_k) < \varepsilon_k, \tag{2.5}$$

and

$$|h(v_k) - h(u_k)| > \frac{1}{\varepsilon_k}|v_k - u_k| \tag{2.6}$$

for $k = 1, 2, \ldots$. Passing, if necessary, to a subsequence, we can assume without loss of generality that $(u_k)_k$ is increasing and converges to some point $u_\infty \in [-r,r]$ such that

$$\kappa(u_{k+1} - u_k) \le \varepsilon_k, \quad \kappa(u_k - u_\infty) \le \varepsilon_k \qquad (k = 1, 2, \ldots). \tag{2.7}$$

Moreover, from (1.4) it follows that

$$\kappa(\delta_k) \ge \delta_k, \quad \kappa(u_{k+1} - u_k) \ge u_{k+1} - u_k \qquad (k = 1, 2, \ldots). \tag{2.8}$$

Denoting by $\operatorname{ent}(\xi)$ the integer part of ξ, let

$$n_k := \operatorname{ent}\left(\frac{\varepsilon_k}{\kappa(\delta_k)} + 1\right);$$

in particular, we have then

$$\varepsilon_k \leq \kappa(\delta_k) n_k < 2\varepsilon_k \qquad (k = 1, 2, \dots). \tag{2.9}$$

Finally, consider the recursively defined sequence $(t_k)_k$ of points

$$t_1 := 0, \quad t_{k+1} := t_k + u_{k+1} - u_k + 2n_k\delta_k \qquad (k = 1, 2, \dots). \tag{2.10}$$

Observe that this sequence is strictly increasing and convergent with

$$\begin{aligned} t_k \to t_\infty = \sum_{k=1}^{\infty}(t_{k+1} - t_k) &= 2\sum_{k=1}^{\infty} n_k\delta_k + \sum_{k=1}^{\infty}(u_{k+1} - u_k) \\ &\leq 2\sum_{k=1}^{\infty} n_k\kappa(\delta_k) + \sum_{k=1}^{\infty}\kappa(u_{k+1} - u_k) \leq 5\sum_{k=1}^{\infty}\varepsilon_k \leq b, \end{aligned}$$

where we have used (2.4), (2.7), (2.8) and (2.9). Now we are ready to define a function $f : [0, b] \to \mathbb{R}$ such that $f \in \kappa BV([0, b])$, but $h \circ f \notin \kappa BV([0, b])$, thus proving the assertion. Consider the zigzag function defined by

$$f(x) := \begin{cases} u_k & \text{if } x = t_k + 2j\delta_k \text{ for } j \in \{0, \dots, n_k\}, \\ v_k & \text{if } x = t_k + (2j-1)\delta_k \text{ for } j \in \{1, \dots, n_k\}, \\ u_\infty & \text{if } t_\infty \leq x \leq b, \\ \text{linear} & \text{otherwise.} \end{cases}$$

By means of the sequence $(t_k)_k$ defined in (2.10), we define partitions

$$P_k := \{t_k, t_k + \delta_k, t_k + 2\delta_k, \dots, t_k + 2n_k\delta_k\} \in \mathcal{P}([t_k, t_k + 2n_k\delta_k]) \qquad (k = 1, 2, \dots)$$

and put for $m \in \mathbb{N}$

$$\Pi_m := \bigcup_{k=1}^{m} P_k \cup \{t_\infty, b\}.$$

Since the partitions $\Pi_m \in \mathcal{P}([0, b])$ are extremal for f, we get from (2.5) and (2.8)

$$\operatorname{Var}_\kappa(f, \Pi_m) \leq \frac{\displaystyle\sum_{k=1}^{m} n_k(v_k - u_k) + (u_{k+1} - u_k) + (u_k - u_\infty)}{\displaystyle\sum_{k=1}^{m} n_k\kappa(\delta_k) + \kappa(u_{k+1} - u_k) + \kappa(t_k + m\delta_m - t_\infty)} \leq 1,$$

hence $f \in BV_\kappa([0, b])$. On the other hand, the κ-variation of $Hf = h \circ f$ on Π_m may be estimated from below by

$$\operatorname{Var}_\kappa(Hf, \Pi_m) \geq \frac{\displaystyle\sum_{k=1}^{m} 2n_k|h(u_k) - h(v_k)|}{\displaystyle\sum_{k=1}^{m}[2n_k\kappa(\delta_k) + \kappa(u_{k+1} - u_k)]} \geq \frac{\displaystyle\sum_{k=1}^{m}\frac{2n_k}{\varepsilon_k}\kappa(\delta_k)}{\displaystyle 5\sum_{k=1}^{m}\varepsilon_k} \geq \frac{m}{b},$$

where we have used (2.4), (2.6), (2.7) and (2.9). We conclude that $Hf \notin \kappa BV([0,b])$, and the proof of the first assertion is complete.

It remains to show that, under the hypothesis (2.2), the operator H is bounded. But from $\|f\|_{\kappa BV} \leq r$ it follows that $\|Hf\|_{\kappa BV} \leq 2k(r)$, by (1.8), and so H is bounded in the norm (1.7). □

We make some comments on Theorem 2.4. First, merely from the fact that the operator H maps the space κBV into itself we get its boundedness for free; this phenomenon is typical for the operator (2.1) in many function spaces. On the other hand, we do not know whether or not condition (2.2) also implies the *continuity* of H in the norm (1.7); this is an open problem even in the simpler space BV.

3. FINAL REMARKS

We point out that the situation becomes much more complicated in the case of the *non-autonomous* composition operator

$$Hf(x) := h(x, f(x)) \qquad (0 \leq x \leq 1) \tag{3.1}$$

generated by some function $h : [0,1] \times \mathbb{R} \to \mathbb{R}$. Here it seems natural to impose a local Lipschitz condition on $h(x,\cdot)$ on $\mathbb{R}$, uniformly with respect to $x \in [0,1]$, together with the requirement $h(\cdot,u) \in \kappa BV([0,1])$, uniformly with respect to $u \in \mathbb{R}$, in order to ensure that the corresponding operator (3.1) maps κBV into itself. In fact, this condition was stated in [14] for the space BV without proof, and afterwards used as "obvious" by several authors (e.g., in Chapter 6 of the monograph [2]). However, tempting as this sufficient condition appears at first glance, on reflection it becomes less convincing. Only quite recently it was shown by Maćkowicz [15] by means of a counterexample that this is false. So even the harmless looking problem of finding a sufficient condition for the inclusion $H(BV) \subseteq BV$ for the operator (3.1) which is possibly not "too far" from being necessary, is open.

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