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DISTRIBUTIONS, THEIR PRIMITIVES AND INTEGRALS WITH APPLICATIONS TO DISTRIBUTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper we will study integrability of distributions whose primitives are left regulated functions and locally or globally integrable in the Henstock-Kurzweil, Lebesgue or Riemann sense. Corresponding spaces of distributions and their primitives are defined and their properties are studied. Basic properties of primitive integrals are derived and applications to systems of first order nonlinear distributional differential equations and to an mth order distributional differential equation are presented. The domain of solutions can be unbounded, as shown by concrete examples.

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1. INTRODUCTION

One way of defining an integral is via its primitive. The primitive is a function whose derivative is in some sense equal to the integrand. For example, if f and F are functions on a real interval I and F is absolutely continuous, such that $F'(x) = f(x)$ for almost all $x \in I$, then the Lebesgue integral of f is $\int_a^b f(x)\,dx = F(b) - F(a)$ for all a, $b \in I$. If function F has a pointwise derivative at each point in I, except for a countable set, then the derivative is integrable in the Henstock-Kurzweil sense on each compact subinterval of I and $\int_a^b F'(x)\,dx = F(b) - F(a)$ for all a, $b \in I$. In this sense, the Henstock-Kurzweil integral inverts the pointwise derivative operator. There are also Henstock-Kurzweil integrable functions for which this fundamental theorem of calculus formula holds and yet these functions do not have a pointwise derivative on certain uncountable sets of measure zero. A function has a C-integral defined in [1] if and only if it is everywhere the pointwise derivative of its primitive. In this sense, the C-integral is the inverse of the pointwise derivative. It is well-known that the Riemann and Lebesgue integrals do not have this property. For details, see [16]. The Henstock-Kurzweil integral is equivalent to the Denjoy integral. We get the wide

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Denjoy integral if we use the approximate derivative. See, for example, [18] for the definition of the wide Denjoy integral. If we use the distributional derivative, then the primitives need not have any pointwise differentiation properties. The continuous and regulated primitive integrals defined in [22, 23] invert the distributional derivatives of continuous and regulated functions, respectively. See [6, 7, 8] for applications of these integrals to nonlinear distributional differential equations.

In this paper, we will study integrability and primitives of left regulated functions and distributions on a real interval I. We say that a distribution f is integrable if f is a distributional derivative of a function, called a primitive of f, that is left regulated, has a right limit at $\inf I$, and is Henstock-Kurzweil (HK) integrable, Lebesgue integrable or Riemann integrable locally on I, i.e., on each compact subinterval of I. We will show that every integrable distribution f also has a left continuous primitive $F : I \to \mathbb{R}$ that is right continuous at the possible left end point of I. Because any two such primitives of f differ by a constant, the difference $F(b) - F(a)$ for any two points of I is independent of the particular primitive F. This property allows us to define for all a, $b \in I$ the primitive integral of f from a to b by

$$\int_a^b f := F(b) - F(a). \tag{1.1}$$

There is a bijective mapping $\mathcal{F}$ between distributions f and those of their primitives F that have above mentioned one-sided continuity properties, their right limits vanish at $\inf I$, and they are locally integrable in the HK, Lebesgue or Riemann sense. In each of these three cases the spaces of primitives have the pointwise partial order $\le$, i.e., $F \le G$ if $F(x) \le G(x)$ for each $x \in I$. The bijection $\mathcal{F}$ can be used to define a partial order in the corresponding spaces of distributions by $f \preceq g$ if and only if $\mathcal{F}(f) \le \mathcal{F}(g)$. Moreover, if primitives are globally integrable in the HK and Lebesgue cases and I is compact in the Riemann integrable case these spaces can be normed by the Alexiewicz norm $\|\cdot\|_A$ in the HK integrable case, by the L^1-norm $\|\cdot\|_1$ in the Lebesgue integrable case, and by the sup-norm $\|\cdot\|_\infty$ in the Riemann integrable case. The bijection $\mathcal{F}$ inherits norms to the corresponding spaces of distributions by $\|f\|_A = \|\mathcal{F}(f)\|_A$, $\|f\|_1 = \|\mathcal{F}(f)\|_1$ and $\|f\|_\infty = \|\mathcal{F}(f)\|_\infty$. We will show that with respect to these partial orderings and norms both the spaces of integrable distributions and their corresponding primitives form in the HK integrable case an ordered normed space, in the Lebesgue integrable case a normed Riesz space, and in the Riemann integrable case a Banach lattice and Banach algebra if I is compact. If I is not compact, it can be represented as an increasing denumerable union of compact intervals I_n. Thus the spaces of locally integrable primitives can be equipped with the linear metric defined by $d(F_1, F_2) = \sum_n \frac{\|F_1-F_2\|_n}{1+\|F_1-F_2\|_n}$, where $\|F\|_n$ denotes the norm of the restriction of F to I_n.

The Fundamental Theorem of Calculus is valid, so that $f = F'$, the primitive derivative of $F = \mathcal{F}(f)$. This allows us to convert distributional differential equations to integral equations in the spaces of primitive functions. In [9] this property is applied in the Riemann integrable case to derive existence results for the unique, smallest, greatest, minimal and/or maximal solutions of finite systems of first order nonlinear distributional Cauchy problems. Dependence of solutions on the data is also studied, as well as systems of distributional differential equations with impulses and higher order distributional Cauchy problems. In section 7 we generalize to the HK integrable case a uniqueness result and the existence and comparison results derived in [9] for the smallest and greatest solutions of distributional Cauchy systems and higher order distributional Cauchy problems. Results of [9] dealing with minimal and maximal solutions are extended to the Lebesgue integrable case. Another generalization is that the solution interval can be unbounded, as shown by concrete examples.

2. PROPERTIES OF LEFT REGULATED FUNCTIONS

A function $H : I \to \mathbb{R}$ is left regulated if it has a left limit $H(t-) = \lim_{s\to t-} H(s)$ at each point t of I ($H(t-) = H(t)$ if $t = \min I$). H is regulated, if it is left regulated and also right regulated, i.e., H has right limit $H(t+) = \lim_{s\to t+} H(s)$ at each point of I ($H(t+) = H(t)$ if $t = \max I$). The main difference between regulated functions and left regulated functions is that the latter ones may have discontinuities of the second kind, while regulated functions can have only discontinuities of the first kind. Hence, regulated functions on a closed interval are bounded while left regulated functions need not be bounded. A left regulated function H is left continuous if $H(t) = H(t-)$ for all points of I. G is said to be countably stepped on a subinterval $[a, b]$ of I if $(a, b]$ is equal to a countable disjoint union of intervals where H is constant on each interval, singletons being considered as closed intervals.

We say that a property holds locally for a function defined on I, if the function has that property on every compact subinterval of I.

The following lemma presents useful properties for left regulated functions.

Lemma 2.1. *Let $H : I \to \mathbb{R}$ be left regulated. Then*
(a) H has at most a countable number of discontinuities. (b) There is a sequence (H_n) of countably stepped functions on I that $|H_n(t) - H(t)| \le \frac{1}{n}$ for all $n \in \mathbb{N}$ and $t \in I$. (c) H is Lebesgue measurable.

Proof. (a) Given a compact subinterval $[a, b]$ of I and a positive integer n, define $G_n : [a, b] \to [a, b]$ by $G_n(a) = a$, and for $x \in (a, b]$,

$$(2.1)\quad G_n(x) = \inf\left\{y \in [a, x) \mid |H(s) - H(t)| \le \frac{1}{n} \text{ for all } s,\, t \in (y, x)\right\},\ x \in (a, b].$$

It is easy to verify that G_n is increasing, i.e., $G_n(x) \le G_n(y)$ whenever $a \le x \le y \le b$. Because H is left regulated, then $G_n(x) < x$ for each $x \in (a,b]$. By [11, Proposition 1.2.1] there is exactly one subset C_n of $[a,b]$ that is inversely well-ordered, i.e., each nonempty subset of C_n has the greatest number, and has the following property:

$$b = \max C_n, \quad \text{and } b > x \in C_n \text{ if and only if } \; x = \inf\{G_n[\{y \in C_n | y > x\}]\}. \tag{2.2}$$

Because $\inf G_n[C_n]$ exists, it is by [11, Proposition 1.2.1] a fixed point of G_n. Since a is the only fixed point of G_n, then $\inf G_n[C_n] = a$. This result and [11, Theorem 1.2.3] imply that $a = \min C_n$. Define

$$D_n := C_n \setminus \{a\}, \text{where } C_n \text{ is determined by (2.1), (2.2).} \tag{2.3}$$

Because $G_n(x) < x$ for each $x \in D_n$, it follows from dual of [11, Lemma 1.1.3] that $G_n(x) = \max\{y \in D_n | y < x\}$ for all $x \in D_n$. Thus $(a,b]$ is the disjoint union of half-open intervals $(G_n(x), x]$, $x \in D_n$. The definition of G_n and the choice of n imply that $|H(s) - H(t)| \le \frac{1}{n}$ for each $s \in (G_n(x), x)$. Thus all the discontinuity points of H in $[a,b]$ belong to the countable set $Z = \bigcup_{m=1}^\infty D_m \cup \{a\}$. This implies the conclusion of (a) because I can be represented as a denumerable union of its compact subintervals.

(b) Given a bounded subinterval $(a,b]$ of I and $n \in \mathbb{N}$, define $H_n : (a,b] \to \mathbb{R}$ by

$$H_n(t) = H(x-), \; t \in (G_n(x), x), \; H_n(x) = H(x), \quad x \in D_n. \tag{2.4}$$

H_n is countably stepped and $|H_n(t) - H(t)| \le \frac{1}{n}$ for all $t \in (a,b]$. This holds for each $n \in \mathbb{N}$, so that (H_n) converges to H uniformly on $(a,b]$. If $\sup I = \infty$ there is $\alpha \in I$ such that if $x, y \in (\alpha, \infty)$ then $|H(x) - H(y)| < 1/n$. Define $H_n(x) = \lim_{t\to\infty} H(t)$ for $x \in (\alpha, \infty)$. Now write $I \setminus (\alpha, \infty)$ as a disjoint union of intervals $(a,b]$. Define H_n as above on each such interval and define $H_n(\inf I) = H(\inf I)$.

(c) By (a) the set Z of discontinuity points of H is a null set, whence H is Lebesgue measurable. □

Applying results of Lemma 2.1 we obtain the following integrability criteria for a left regulated function.

Lemma 2.2. *Let $H : I \to \mathbb{R}$ be left regulated. Then (a) H is locally Riemann integrable if and only if H is locally bounded. (b) H is locally Lebesgue integrable if and only if for each subinterval $[a,b]$ of I the function H_n, defined by* (2.4)*, is Lebesgue integrable for some $n \in \mathbb{N}$. (c) H is locally HK integrable if and only if for each subinterval $[a,b]$ of I the function H_n, defined by* (2.4)*, is HK integrable for some $n \in \mathbb{N}$.*

Proof. (a) Because H by Lemma 2.1 is continuous almost everywhere on I, then H is locally Riemann integrable if and only if H is locally bounded (see [17]).

(b) and (c) Let $[a, b]$ be a subinterval of I. By Lemma 2.1 (b) the function H_n defined by (2.4) is countably stepped, and hence Lebesgue measurable. If H is locally HK integrable, then H_n is bounded above and below by HK integrable functions $H \pm \frac{1}{n}$, whence H_n is HK integrable (see [27, Theorem 2.5.16]). Conversely, if H_n is HK integrable, then H is bounded above and below on $[a, b]$ by HK integrable functions $H_n \pm \frac{1}{n}$. Because H is also Lebesgue measurable by Lemma 2.2 (c), then H is HK integrable on $[a, b]$. The above reasoning holds also when HK integrability is replaced by Lebesgue integrability. □

The next result is also a consequence of Lemma 2.1.

Lemma 2.3. *Let $H : I \to \mathbb{R}$ be left regulated. Then for each compact subinterval $[a, b]$ of I, either H is integrable on $[a, b]$ in Riemann, Lebesgue or HK sense, or there exists the smallest number a_1 in $[a, b)$ such that H is locally integrable on $(a_1, b]$ in Riemann, Lebesgue or HK sense, respectively.*

Proof. Let $[a, b]$ be a compact subinterval of I, and let C_1 be the inversely well ordered subset of $[a, b]$ defined by (2.2) with $n = 1$. H is integrable on $[x, b]$ for every $x \in C_1$, if and only if H is integrable on $[a, b]$ because $a = \min C_1$. Hence, if H is not integrable on $[a, b]$, the set A_1 of those $x \in C_1$ for which H is not integrable on $[x, b]$ is nonempty. Because C_1 is inversely well ordered, then $a_1 = \max A_1$ exists. Since H has the left limit at b, then $a_1 < b$. If $c \in (a_1, b)$, then $c \in (G_1(x), x]$ for some $x \in C_1$, where G_1 is defined by (2.1). Because $a_1 < x$, then H is integrable on $[x, b]$, and since $|H(t) - H(x-)| \leq 1$ for all $t \in [c, x]$, then H is integrable on $[c, b]$. This shows that H is locally integrable on $[a_1, b]$. If $c < a_1$, then H is not integrable on $[c, b]$ because of the choice of a_1. □

By a CD *primitive* of a function H from an interval I of $\mathbb{R} \cup \{-\infty\}$ to $\mathbb{R}$ we mean a continuous function $F : I \to \mathbb{R}$ that is differentiable in the complement of a countable subset Z of I, and $F'(t) = H(t)$ for all $t \in I \setminus Z$. The following lemma, which is a consequence, e.g., of [2, Lemma 1.12], presents a sufficient condition for local HK integrability.

Lemma 2.4. *Let I be an interval in $\mathbb{R}$. If a function $H : I \to \mathbb{R}$ has a CD primitive F, then H is locally HK integrable on I, and $\int_a^b H(t)\,dt = F(b) - F(a)$ for all a, $b \in I$.*

The next result follows from [4, (8.6.4)].

Lemma 2.5. *Let $(H_n)_{n=1}^\infty$ be a sequence of functions from an interval $I \subseteq \mathbb{R}$ into $\mathbb{R}$. Suppose that*

(i) *there is a CD primitive F_n of H_n for each $n \in \mathbb{N}$;*
(ii) *there is a point $t_0 \in I$ such that $(F_n(t_0))$ converges in $\mathbb{R}$;*

(iii) *for every point $t \in I$ there is a neighbourhood $B(t)$ with respect to I such that in $B(t)$ the sequence (H_n) converges uniformly.*

Then for each $t \in I$, the sequence (F_n) converges uniformly in $B(t)$; and if we put $H(t) = \lim_{n\to\infty} H_n(t)$ and $F(t) = \lim_{n\to\infty} F_n(t)$, then F is a CD primitive of H.

Lemmas 2.4 and 2.5 are used in Example 6.1 to verify the HK integrability of a left regulated function that has a discontinuity of the second kind at every rational point.

As an application of Lemmas 2.1 and 2.5 we shall prove the following result.

Lemma 2.6. *Assume that $H : I \to \mathbb{R}$ is a left regulated and locally HK integrable function. Then H has a CD primitive on each compact subinterval of I.*

Proof. Let $[a,b]$ be a compact subinterval of I, and let functions $H_n : [a,b] \to \mathbb{R}$, $n \in \mathbb{N}$, be defined on $(a,b]$ by (2.4), and $H_n(a) = H(a)$, $n \in \mathbb{N}$. Every function H_n is HK integrable by Lemma 2.2(c). Define $F_n : [a,b] \to \mathbb{R}$ by

$$F_n(t) = -\int_t^b H_n(t)\,dt, \quad t \in [a,b]. \tag{2.5}$$

Because H_n is constant on every subinterval $(G_n(x), x)$, $x \in D_n$, then $F_n'(t) = H_n(t)$ whenever $t \in (a,b) \setminus D_n$. F_n is continuous, whence F_n is a CD primitive of the restriction of H_n to $[a,b]$. Moreover, $H_n(t) \to H(t)$ uniformly on $[a,b]$ by Lemma 2.1, and $F_n(b) = 0$ for all $n \in \mathbb{N}$. It then follows from Lemma 2.5 that the sequence (F_n) has a uniform limit that is a CD primitive of H on $[a,b]$. □

Now we are in position to prove The Fundamental Theorem of Calculus for left regulated functions.

Theorem 2.7. *Assume that $H : (a,b] \to \mathbb{R}$, $-\infty \le a < b < \infty$, is left regulated.*

(a) *H is locally HK integrable if and only if it has a CD primitive.*
(b) *H is locally Lebesgue integrable if and only if it has a locally absolutely continuous CD primitive.*
(c) *H is locally Riemann integrable if and only if it is locally bounded, in which case H has a locally Lipschitz continuous CD primitive.*

Proof. (a) Assume first that H has a CD primitive. It follows from Lemma 2.4 that g is locally HK integrable. To prove converse, choose a decreasing sequence $(c_n)_{n=1}^\infty$ from (a,b) so that it converges to a. The domain $(a,b]$ of H is the union of increasing sequence of compact intervals $[c_n,b]$, and H is HK integrable on these compact intervals. By Lemma 2.6 the restriction of H to $[c_n,b]$ has a CD primitive $F_n : [c_n,b] \to \mathbb{R}$, i.e., F_n is continuous, and there is a countable subset Z_n of $[c_n,b]$

such that $F_n'(t) = H(t)$ for all $t \in [c_n, b] \setminus Z_n$. Moreover, $F_n(b) = 0$ for each $n \in \mathbb{N}$. Defining

$$F(t) = \begin{cases} F_1(t) - F_1(b), & t \in (c_1, b], \\ F_{n+1}(t) - F_{n+1}(b), & t \in (c_{n+1}, c_n],\ n \in \mathbb{N}, \end{cases}$$

we obtain continuous mapping $F : I \to \mathbb{R}$, and $F'(t) = H(t)$ for all $t \in I \setminus \bigcup_{n=1}^{\infty} Z_n$ (cf. Remark after [4, (8.7.1)]). Consequently, F is a CD primitive of H.

(b) If H has a locally absolutely continuous CD primitive F, then H is locally Lebesgue integrable by the Fundamental Theorem of Calculus. Conversely, assume that H is locally Lebesgue integrable, and let $[c, b]$ be a compact subinterval of $(a, b]$. Then H is Lebesgue integrable on $[c, b]$, whence there is an absolutely continuous function $G : [c, b] \to \mathbb{R}$ and a null-set Z_1 such that $G'(t) = H(t)$ for all $t \in [c, b] \setminus Z_1$. H is also HK integrable on $[c, b]$. Thus H has by the proof of (a) a CD primitive, i.e., a continuous function $F : I \to \mathbb{R}$ and a countable subset Z_2 of $(a, b]$ such that $F'(t) = H(t)$ for each $t \in (a, b] \setminus Z_2$. Then $f = F - G$ is HK integrable on $[c, b]$, and $f'(t) = F'(t) - H'(t) = 0$ for all $t \in [c, d] \setminus (Z_1 \cup Z_2)$. Because $Z_1 \cup Z_2$ is a null-set, then $f(t) = F(t) - G(t) = C$, i.e., $F(t) = G(t) + C$, for all $t \in [c, b]$, whence F is absolutely continuous on $[c, b]$. Consequently, F is a locally absolutely continuous CD primitive of H.

(c) Let $[c, b]$ be a compact subinterval of $(a, b]$. It follows from Lemma 2.2 that H is Riemann integrable on $[c, b]$ if and only if H is bounded on $[c, b]$, in which case there is such a positive constant M that $|H(t)| \le M$ for each $t \in [c, b]$. Because H is also HK integrable, it has a CD primitive $F : I \to \mathbb{R}$ by the proof of (a). If $c \le x < y \le b$, then and $F(y) - F(x) = \int_x^y H(t)\,dt$ by Lemma 2.4. Thus, $|F(y) - F(x)| \le \int_x^y |H(t)|\,dt \le M(y - x)$. This holds for every compact subinterval $[c, b]$ of $(a, b]$, whence F is locally Lipschitz continuous. □

The definition of integrability and the primitive integral in the HK integrable case is based on the following result, where $\int$ denotes the Henstock-Kurzweil integral.

Lemma 2.8. *Suppose that $H : I \to \mathbb{R}$ is left regulated, that $\phi_n \to \phi$ in $\mathcal{D}$, and that $[a, b] \subseteq I$. Then $\int_a^b H(t)\phi_n(t)\,dt \to \int_a^b H(t)\phi(t)\,dt$ if and only if H is HK integrable on $[a, b]$.*

Proof. Since H is left regulated, it is Lebesgue measurable by Lemma 2.1. Since the sequence $(\phi_n - \phi)$ converges to 0 in $\mathcal{D}$, the sequence $(\phi_n' - \phi')$ converges to 0 uniformly on $[a, b]$. Thus the sequence (ϕ_n) is of uniform bounded variation, and converges uniformly to ϕ. The conclusion follows from [20, Corollary 3.2], since the hypotheses of it are valid by the above proof when $f = H$, $(g_n) = (\phi_n)$ and $g = \phi$. □

3. THE LD PRIMITIVE INTEGRAL AND BASIC PROPERTIES

In this section we will study integrability and the integral of distributions on a real interval I having locally or globally HK integrable primitives. We will first fix some notation for distributions. Let I be a real interval. The space $\mathcal{D}$ of test functions are formed by functions of $C_0^\infty(I)$, that is, the smooth functions which, together with all their derivatives, have compact support in I (cf. [5, 26]). The support of a function ϕ is the closure of the set on which ϕ does not vanish. Denote this as $\mathrm{supp}(\phi)$. There is a notion of continuity in $\mathcal{D}$. If (ϕ_n) is a sequence in $\mathcal{D}$, then $\phi_n \to \phi$ in $\mathcal{D}$ if there is a compact subset K in I such that for all $n \in \mathbb{N}$, $\mathrm{supp}(\phi_n) \subseteq K$, and for each integer $m \geq 0$, $\phi_n^{(m)} \to \phi^{(m)}$ uniformly on K as $n \to \infty$. The distributions on I are the continuous linear functionals on $\mathcal{D}$, denoted $\mathcal{D}'$. If $T \in \mathcal{D}'$, then $T : \mathcal{D} \to \mathbb{R}$ and we write $\langle T, \phi\rangle \in \mathbb{R}$ for $\phi \in \mathcal{D}$. If $\phi_n \to \phi$ in $\mathcal{D}$, then $\langle T, \phi_n\rangle \to \langle T, \phi\rangle$ in $\mathbb{R}$. And, for all a_1, $a_2 \in \mathbb{R}$ and all $\phi_1, \phi_2 \in D$, $\langle T, a_1\phi_1 + a_2\phi_2\rangle = a_1 \langle T, \phi_1\rangle + a_2 \langle T, \phi_2\rangle$. The differentiation formula $\langle T', \phi\rangle = -\langle T, \phi'\rangle$ ensures that distributions have derivatives which are distributions. Results on distributions can be found in [5].

3.1. LDP **integrability and the** LD **primitive integral.** We will first describe the spaces of primitives for the LD primitive integral on I. Denote

(3.1)

$$\mathcal{D}^{lr}(I) = \{H : I \to \mathbb{R} \mid H \text{ is left regulated, locally } HK \text{ integrable, and has a right limit at } \inf I\},$$

$$\mathcal{D}^{lc}(I) = \{G \in \mathcal{D}^{lr}(I) \mid G \text{ is left continuous, and } G(a) = G(a+) \text{ if } a = \min I \text{ exists}\},$$

$$\mathcal{D}_0^{lc}(I) = \{F \in \mathcal{D}^{lc}(I) | F(\inf I+) = 0\}.$$

The same notations are also used when local HK integrability is replaced by global integrability.

Let $H \in \mathcal{D}^{lr}(I)$. We will prove in Theorem 3.1 that H uniquely determines a distribution, also denoted by H, on I by

$$\langle H, \phi\rangle = \int_a^b H(t)\phi(t)\,dt, \quad \phi \in \mathcal{D}, \text{ where } \mathrm{supp}(\phi) \subseteq [a, b] \subseteq I, \tag{3.2}$$

where $\int$ denotes the Henstock-Kurzweil integral. Define functions $G \in \mathcal{D}^{lc}(I)$ and $F \in \mathcal{D}_0^{lc}(I)$ by

(3.3)

$$G(x) = \begin{cases} H(x-), & x > \inf I, \\ H(x+), & x = \min I \text{ if } \min I \text{ exists}, \end{cases} \qquad F(x) = G(x) - G(\inf I+),\ x \in I.$$

Replacing H in (3.2) by the so defined functions G and F we get distributions G and F. Because H has by Lemma 2.1 only a countable number of discontinuity points, then all the distributions F, G and H are equal. All these three distributions have the same derivative which is itself a distribution. This is known as the distributional derivative or weak derivative. We will usually denote the distributional derivative of a distribution F by F' and the possible pointwise derivative of F by $F'(t)$.

As a consequence of Lemma 2.8 we obtain

Theorem 3.1. *Every function of $\mathcal{D}^{lr}(I)$ determines a unique distribution on I by* (3.2).

Proof. Let $H \in \mathcal{D}^{lr}(I)$. Since H is locally HK integrable and left regulated, then (3.2) defines by Lemma 2.8 a continuous function $H : \mathcal{D} \to \mathbb{R}$ by $\phi \mapsto \int_a^b H(s)\phi(t)\, dt$, where $\mathrm{supp}(\phi) \subseteq [a, b] \subseteq I$. H is linear because of linearity properties of Henstock-Kurzweil integrals. Thus $H \in \mathcal{D}'$. □

A distribution f on I is called LDP integrable on I if it is the distributional derivative of some primitive $H \in \mathcal{D}^{lr}(I)$, that is, for all $\phi \in \mathcal{D}$ we have

$$\langle f, \phi\rangle = \langle H', \phi\rangle = -\langle H, \phi'\rangle = -\int_a^b H(t)\phi'(t)\, dt, \text{ where } \mathrm{supp}(\phi') \subseteq [a, b] \subseteq I. \tag{3.4}$$

The last integral is a Henstock-Kurzweil integral. Denote

$$\mathcal{A}_D(I) = \{f \in \mathcal{D}' \mid f \text{ is } LDP \text{ integrable on } I\}. \tag{3.5}$$

Linearity of the distributional derivative shows that $\mathcal{A}_D(I)$ is a linear subspace of $\mathcal{D}'$. If f is LDP integrable with a primitive $H \in \mathcal{D}^{lr}(I)$, then (3.3) determines a primitive F of f in $\mathcal{D}_0^{lc}(I)$.

If $f \in \mathcal{A}_D(I)$, F is a primitive of f in $\mathcal{D}^{lc}(I)$ and a, $b \in I$, define the LD primitive integral of f from a to b by (1.1).

3.2. **Basic properties of the *LD* primitive integral.** We now present some of the basic properties of the LD primitive integral. Linear combinations are defined by $\langle a_1f_1 + a_2f_2, \phi\rangle = \langle a_1F_1' + a_2F_2', \phi\rangle$ for $\phi \in \mathcal{D}$; $a_1, a_2 \in \mathbb{R}$; $f_1, f_2 \in \mathcal{A}_D(I)$ with primitives $F_1, F_2 \in \mathcal{D}_0^{lc}(I)$.

Theorem 3.2 (Basic properties of the integral)**.** (a) *The LD primitive integral is unique.*

(b) *Addivity over intervals. If $f \in \mathcal{A}_D(I)$, then for all $a \le c \le b$ we have $\int_a^c f + \int_c^b f = \int_a^b f$.*
(c) *Linearity. If $f_1, f_2 \in \mathcal{A}_D(I)$ and $a_1, a_2 \in \mathbb{R}$, then $a_1f_1 + a_2f_2 \in \mathcal{A}_D(I)$ and $\int_a^b(a_1f_1 + a_2f_2) = a_1\int_a^b f_1 + a_2\int_a^b f_2, \quad a, b \in I$.*
(d) *Reverse limits of integration. Let a, $b \in I$. Then $\int_b^a f = -\int_a^b f$.*

Proof. (a) To prove the LD primitive integral is unique we need to prove primitives in $\mathcal{D}_0^{lc}(I)$ are unique. Suppose $F, G \in \mathcal{D}_0^{lc}(I)$ and $F' = G'$. Then $(F - G)' = 0$ and the only solutions of this distributional differential equation are the constant distributions [5, Section 2.4]. The only constant distribution in $\mathcal{D}_0^{lc}(I)$ is the zero function.

(b) Note that $[F(b) - F(c)] + [F(c) - F(a)] = F(b) - F(a)$.

(c) Since $a_1 f_1 + a_2 f_2 = (a_1 F_1 + a_2 F_2)'$, where $F_i \in \mathcal{D}_0^{lc}(I)$, $F_i' = f_i$, we have

$$\begin{cases} \int_a^b (a_1 f_1 + a_2 f_2) = (a_1 F_1 + a_2 F_2)(b) - (a_1 F_1 + a_2 F_2)(a) \\ = a_1 F_1(b) + a_2 F_2(b) - a_1 F_1(a) - a_2 F_2(a) = a_1 \int_a^b f_1 + a_2 \int_a^b f_2. \end{cases}$$

(d) $\int_b^a f = F(a) - F(b) = -[F(b) - F(a)] = -\int_a^b f$. □

Theorem 3.3 (Fundamental theorem of calculus)**.** *Let $f \in \mathcal{A}_D(I)$, $G \in \mathcal{D}^{lc}(I)$, $a \in I$ and $c \in \mathbb{R}$. Then $G' = f$ and $G(a) = c$ if and only if $G(x) = c + \int_a^x f$ for every $x \in I$.*

Proof. If $G' = f$ and $G(a) = c$, it follows from (3.3) and (1.1) that

$$\begin{aligned} G(x) &= c + G(x) - G(a) = c + (G(x) - G(\inf I+)) - (G(a) - G(\inf I+)) \\ &= c + F(x) - F(a) = c + \int_a^x f. \end{aligned}$$

Conversely, let $G(x) = c + \int_a^x f$. Then $G(a) = c + \int_a^a f = c$ by (1.1). Since $F_a(x) = \int_a^x f$ is a primitive of f in $\mathcal{D}^{lc}(I)$, then $F_a' = f$. Thus $G' = (c + F_a)' = 0 + F_a' = f$. □

The result of Theorem 3.3 can be used to convert distributional initial value problems into integral equations.

As a consequence of the definitions of Theorem 3.2 and the definitions of $\mathcal{A}_D(I)$ and $\mathcal{D}_0^{lc}(I)$ we obtain

Corollary 3.4. *The mapping $\mathcal{F}$, defined by*

$$\mathcal{F}(f) = F,\ f \in \mathcal{A}_D(I),\ \text{where } F \text{ is the primitive of } f \text{ in } \mathcal{D}_0^{lc}(I), \tag{3.6}$$

is a linear isomorphism from $\mathcal{A}_D(I)$ to $\mathcal{D}_0^{lc}(I)$.

As with the Henstock-Kurzweil integral, there are no improper integrals.

Theorem 3.5 (Hake theorem)**.** *Suppose $f \in \mathcal{D}'$, and that $f \in \mathcal{A}_D([x, y])$ for every proper subinterval $[x, y]$ of $[a, b] \subseteq I$. If for some $c \in (a, b)$, $\lim_{x \to a+} \int_x^c f$ exists, then $f \in \mathcal{A}_D[a, c]$, and $\int_a^c f = \lim_{x \to a+} \int_x^c f$, and if $\lim_{y \to b-} \int_c^y f$ exists, then $f \in \mathcal{A}_D[c, b]$, and $\int_c^b f = \lim_{y \to b-} \int_c^y f$.*

3.3. **Order and norm properties.** In $\mathcal{D}_0^{lc}(I)$, there is the partial order: $F \le G$ if and only if $F(x) \le G(x)$ for all $x \in I$. Since the mapping $\mathcal{F}$, defined by (3.6) is a linear isomorphism from $\mathcal{A}_D(I)$ to $\mathcal{D}_0^{lc}(I)$, we can define a partial order $\preceq$ in $\mathcal{A}_D(I)$ as follows. For $f, g \in \mathcal{A}_D(I)$, define $f \preceq g$ if $\mathcal{F}(f) \le \mathcal{F}(g)$ in $\mathcal{D}_0^{lc}(I)$. In particular, if $\min I$ exists, then

$$f \preceq g \text{ if and only if } \int_{\min I}^{x} f \le \int_{\min I}^{x} g \text{ for each } x \in I. \tag{3.7}$$

Thus $f \preceq g$ if and only if $F \le G$, where F and G are the respective primitives in $\mathcal{D}_0^{lc}(I)$. If $\preceq$ is a binary operation on set E, then it is a partial order if for all $x, y, z \in E$ it is reflexive ($x \preceq x$), antisymmetric ($x \preceq y$ and $y \preceq x$ imply $x = y$) and transitive ($x \preceq y$ and $y \preceq z$ imply $x \preceq z$). If E is a vector space and $\preceq$ is a partial order on E, then E is an ordered vector space if for all $x, y, z \in S$

(1) $x \preceq y$ implies $x + z \preceq y + z$.

(2) $x \preceq y$ implies $kx \preceq ky$ for all $k \in \mathbb{R}$ with $k \ge 0$.

If $x \preceq y$, we write $y \succeq x$.

Theorem 3.6 (Ordered vector space)**.** *(a) Both $\mathcal{D}^{lc}(I)$ and $\mathcal{D}_0^{lc}(I)$ are ordered vector spaces. (b) $\mathcal{A}_D(I)$ is order isomorphic to $\mathcal{D}_0^{lc}(I)$.*

Proof. (a) The following properties follow immediately from the definition. If $F \le G$ in $\mathcal{D}^{lc}(I)$, then for all $H \in \mathcal{D}^{lc}(I)$ we have $F + H \le G + H$. If $F \le G$ and $k \ge 0$ then $kF \le kG$. Hence, $\mathcal{D}^{lc}(I)$ is an ordered vector space, and so is $\mathcal{D}_0^{lc}(I)$ as a linear subspace of $\mathcal{D}^{lc}(I)$.

(b) If $f, g \in \mathcal{A}_D(I)$ and $f \preceq g$, then $\mathcal{F}(f) \le \mathcal{F}(g)$. Let $h \in \mathcal{A}_D(I)$. Then, $\mathcal{F}(f)+\mathcal{F}(h) \le \mathcal{F}(g)+\mathcal{F}(h)$. But then $(\mathcal{F}(f)+\mathcal{F}(h))' = \mathcal{F}(f)'+\mathcal{F}(h)' = f+h \preceq g+h$. If $k \in \mathbb{R}$ and $k \ge 0$, then $(k\mathcal{F}(f))' = k\mathcal{F}(f)' = kf$ so $kf \preceq kg$. Then $\mathcal{A}_D(I)$ is an ordered vector space that is order isomorphic to $\mathcal{D}_0^{lc}(I)$. □

A vector space E equipped with a partial order $\preceq$ and a norm $\|\cdot\|$ is said to be an ordered normed space if the order cone $E_+ = \{x \in E | x \succeq 0\}$ is a closed subset of E in its norm topology.

Assume next that the functions of $\mathcal{D}^{lc}(I)$ and $\mathcal{D}_0^{lc}(I)$ are HK integrable. We will show that $\mathcal{D}^{lc}(I)$ and $\mathcal{D}_0^{lc}(I)$, ordered pointwise, are ordered normed spaces with respect to the Alexiewicz norm:

$$\|F\|_A = \sup_{[a,b]\subseteq I} \left| \int_a^b F(t)\,dt \right|, \quad F \in \mathcal{D}^{lc}(I).$$

Using the isomorphism $\mathcal{F}$ we define a norm in $\mathcal{A}_D(I)$ by $\|f\|_A = \|\mathcal{F}(f)\|_A$. Equivalently,

$$\|f\|_A = \sup_{[a,b]\subseteq I} \left| \int_a^b F(t)\,dt \right|, \quad \text{where } F = \mathcal{F}(f). \tag{3.8}$$

Theorem 3.7 (Ordered normed space)**.** *Assume that the functions of $\mathcal{D}^{lc}(I)$ and $\mathcal{D}_0^{lc}(I)$ are HK integrable. (a) $\mathcal{D}^{lc}(I)$ and $\mathcal{D}_0^{lc}(I)$ are ordered normed spaces with respect to the Alexiewicz norm and pointwise order. (b) $\mathcal{A}_D(I)$ is an ordered normed space with respect to partial order and norm defined by* (3.7) *and* (3.8)*. (c) $\mathcal{A}_D(I)$ and $\mathcal{D}_0^{lc}(I)$ are isometrically isomorphic. The integral provides a linear isometry.*

Proof. (a) To prove that the Alexiewicz norm is a norm in $\mathcal{D}^{lc}(I)$ and in $\mathcal{D}_0^{lc}(I)$, first note they are linear subspaces of the Denjoy space $D(I)$ of all HK integrable functions from I to $\mathbb{R}$. And, if $F \in \mathcal{D}^{lc}(I)$ such that $\|F\|_A = 0$, then then $F(x) = 0$ for almost all $x \in I$. But F is left continuous in $I \setminus \{\inf I\}$ and right continuous at the possible $\min I$. So if there were $b \in I$ such that $F(b) \neq 0$ then there is an interval $(a, b]$ if $b > \min I$ or an interval $[b, c)$ if $b = \min I$ in which F does not vanish, which is a contradiction. Thus $F(x) = 0$ for all $x \in I$. Positivity, homogeneity and the triangle inequality are inherited from $D(I)$.

(b) Because HK integrability and strong HK integrability (called HL integrability in [2]) are equivalent for real functions by [19, Proposition 3.6.6], it follows from [2, Lemma 9.29] that the cone $D(I)_+ = \{G \in D(I) | G(x) \geq 0$ for a.e. $x \in I\}$ is closed with respect to the Alexiewicz norm. Hence, if $\|F_n - F\|_A \to 0$ in $\mathcal{D}^{lc}(I)$ and $F_n \in \mathcal{D}^{lc}(I)_+ = \{G \in \mathcal{D}^{lc}(I) | G(x) \geq 0$ for all $x \in I\}$, then $F(x) \geq 0$ for a.e. $x \in I$. Because F is left continuous in $I \setminus \{\min I\}$ and right continuous at $\min I$ one can show (cf. the proof of (a)) that F cannot have negative values, whence $F \in \mathcal{D}^{lc}(I)_+$. This proves that $\mathcal{D}^{lc}(I)_+$ is closed in the Alexiewicz norm topology of $\mathcal{D}^{lc}(I)$, so $\mathcal{D}^{lc}(I)$ is an ordered normed space. The proof that $\mathcal{D}_0^{lc}(I)$ is an ordered normed space is similar.

(c) The conclusion is a direct consequence of (a) because $\mathcal{F}$ is an order isomorphism from $\mathcal{A}_D(I)$ to $\mathcal{D}_0^{lc}(I)$, and $\|f\|_A = \|\mathcal{F}(f)\|_A$ for all $f \in \mathcal{A}_D(I)$. □

3.4. **Integration by parts, dual space.** It is well-known that if $f \in D([a, b])$ then the pointwise product fg is also Henstock-Kurzweil integrable if g is of bounded variation. Let $F(x) = \int_a^x f$. The integration by parts formula is then defined in terms of a Riemann-Stieltjes integral via $\int_a^b f(x)g(x)\,dx = F(b)g(b) - \int_a^b F(x)\,dg(x)$. See [16]. The functions of (essential) bounded variation also form the dual space of $D([a, b])$. We will see analogues of these results for the LD primitive integral.

Let $\mathcal{BV}([a, b])$ be the functions of bounded variation on $[a, b]$.

Definition 3.8. Let $c \in [a, b]$. Define

(3.9)
$$\mathcal{IBV}_c([a, b]) = \{g : [a, b] \to \mathbb{R} \mid g(x) = \int_c^x h(t)\,dt \text{ for some } h \in \mathcal{BV}([a, b]) \cap L^1([a, b])\}.$$

The L^1 condition is redundant if $[a, b]$ is a compact interval. Note that functions in $\mathcal{IBV}_c([a, b])$ vanish at c, are Lipschitz continuous on $[a, b]$ and are hence in $\mathcal{BV}([a, b])$.

Definition 3.9. Let $f \in \mathcal{A}_D([a,b])$ with primitive $F \in \mathcal{D}_0^{lc}([a,b])$. Let $g \in \mathcal{IBV}_c([a,b])$ for some $c \in [a,b]$. Define the integration by parts formula $\int_a^b fg = F(b)g(b) - \int_a^b F(t)g'(t)\,dt$.

There is no way of proving the integration by parts definition, although it clearly holds if $f \in D([a,b])$. However, we can use a sequential approach to justify it since the C^1 functions are dense in $D([a,b])$. Functions in $\mathcal{IBV}_a$ and $\mathcal{IBV}_c$ differ by a constant so we just need consider $\mathcal{IBV}_a$.

Theorem 3.10. *Let $f \in \mathcal{A}_D([a,b])$ with primitive $F \in \mathcal{D}_0^{lc}([a,b])$; let $g \in \mathcal{IBV}_a([a,b])$. (a) Then $|\int_a^b fg| \le |F(b)||g(b)| + \|F\|_A(\|g'\|_\infty + Vg')$. (b) Suppose $(F_n) \subset C([a,b]) \cap C^1((a,b))$ such that $F_n(a) = 0$, $F_n(b) = F(b)$ and $\|F_n' - f\|_A \to 0$. Let $H(x) = F(x)g(x) - \int_a^x F(t)g'(t)\,dt$ and let $H_n(x) = \int_a^x F_n'(t)g(t)\,dt$. Then $|H(b) - H_n(b)| \le \|F - F_n\|_A Vg' \to 0$ as $n \to \infty$.*

Proof. (a) The inequality $\left|\int_a^b F\phi\right| \le \|F\|_A(\inf|\phi| + V\phi)$ holds for all $F \in D([a,b])$ and $\phi \in \mathcal{BV}([a,b])$. See [3] and [21, Lemma 24].

(b) By the multiplier result above, the product $Fg \in \mathcal{D}_0^{lc}([a,b])$. Since g' is almost everywhere equal to a function of bounded variation we also have $Fg' \in D([a,b])$. The function $x \mapsto \int_a^x F(t)g'(t)\,dt$ is continuous on $[a,b]$ and vanishes at a. Hence, H exists on $[a,b]$. There exist functions F_n satisfying the endpoint conditions since F has limits at $a+$ and $b-$. Each function H_n is continuous on $[a,b]$. The inequality follows from (a). □

For compact intervals there is convergence in the Alexiewicz norm to the integration by parts formula.

Theorem 3.11. *Let $[a,b]$ be a compact interval. Let $f \in \mathcal{A}_D([a,b])$ with primitive $F \in \mathcal{D}_0^{lc}([a,b])$. Let $g \in \mathcal{IBV}_a([a,b])$. Suppose $(F_n) \subset C([a,b]) \cap C^1([a,b])$ such that $\|F_n' - f\|_A \to 0$. Let $H(x) = F(x)g(x) - \int_a^x F(t)g'(t)\,dt$ and let $H_n(x) = \int_a^x F_n'(t)g(t)\,dt$. Then $H, H_n \in \mathcal{D}_0^{lc}([a,b])$ and $\|H - H_n\|_A \to 0$.*

Proof. The proof of Theorem 3.10(b) shows that $H, H_n \in \mathcal{D}_0^{lc}([a,b])$. Now,

$$\|H' - H_n'\|_A = \|H - H_n\|_A \le 2 \sup_{\beta\in[a,b]} \left|\int_a^\beta [H(x) - H_n(x)]\,dx\right|. \tag{3.10}$$

Let $\beta \in [a,b]$. Use (a) of Theorem 3.10. Then

$$\left\{\begin{array}{l} \left|\int_a^\beta [H(x) - H_n(x)]\,dx\right| \le \left|\int_a^\beta [F(x) - F_n(x)]\,g(x)\,dx\right| \\ + \left|\int_a^\beta [F(t) - F_n(t)]\,g'(t)\int_a^\beta \chi_{[t,\beta]}(x)\,dx\,dt\right| \\ \le \|F - F_n\|_A\left[(\|g\|_\infty + Vg) + (b-a)(\|g'\|_\infty + Vg')\right]. \end{array}\right. \tag{3.11}$$

The order of the iterated integrals can be interchanged by [3, Theorem 57]. It now follows that $\|H - H_n\|_A \to 0$. □

Notice that integration by parts defines a product $\mathcal{A}_D([a,b]) \times \mathcal{IBV}_a([a,b]) \to \mathcal{A}_D([a,b])$ given by $fg = H'$. A similar type of definition was used in [23]. Various properties of this product (or bimodule) were proved in Theorem 18 of that paper. Since $\mathcal{IBV}_a([a,b])$ is closed under pointwise products, the same results hold here with similar proofs.

Note that if J is a subinterval of $[a,b]$ then $\int_J f$ is not defined since $\chi_J \notin \mathcal{IBV}_c$.

Now we consider the dual space of $\mathcal{A}_D([a,b])$. First, denote the functions of essential bounded variation by $\mathcal{EBV}([a,b])$. A function $f \in \mathcal{EBV}([a,b])$ if $f = g$ a.e. for some $g \in \mathcal{BV}([a,b])$. See [14]. And, $\mathcal{EBV}([a,b])$ consists of equivalence classes of functions agreeing a.e. Each equivalence class of $\mathcal{EBV}([a,b])$ contains a unique function g of bounded variation that is left continuous on $(a,b]$ and right continuous at a such that g differs from each function in the equivalence class on a set of measure zero. If f is in the equivalence class then $\|f\|_{\mathcal{EBV}} = \inf_h Vh + \|f\|_\infty = Vg + \|g\|_\infty$. The infimum is taken over all $h \in \mathcal{BV}([a,b])$ such that $f = h$ a.e. Note that here $\|f\|_\infty$ is the essential supremum of f and $\|g\|_\infty$ reduces to the supremum of g.

Theorem 3.12. *(a) $\mathcal{A}_D([a,b])$ is not complete. (b) $\mathcal{D}_0^{lc}([a,b])$ is dense in $D([a,b])$. (c) The dual space of $\mathcal{A}_D([a,b])$ is isometrically isomorphic to $\mathcal{IBV}_b([a,b])$ and to $\mathcal{BV}([a,b])$.*

Proof. (a) Define $F_n = \sum_{m=2}^n (-1)^m \chi_{(-1/m,-1/(m+1)]}$. Then $F_n \in \mathcal{D}_0^{lc}([-1,0])$. Let $F = \sum_{m=2}^\infty (-1)^m \chi_{(-1/m,-1/(m+1)]}$. Then $F \in \mathcal{D}([-1,0])$ since $F \in L^1([-1,0])$. Note that $\|F - F_n\|_A = 1/(n+1) - 1/(n+2) \to 0$ as $n \to \infty$. So $F_n \to F$ in the Alexiewicz norm. But $\lim_{x\to 0^-} F(x)$ does not exist so F is not left regulated on $[-1,0]$. Since $\mathcal{D}_0^{lc}([a,b])$ and $\mathcal{A}_D([a,b])$ are isomorphic and isometric (Theorem 3.7) it follows that although $\mathcal{A}_D([a,b])$ is a normed linear space it is not a Banach space.

(b) Let $f \in D([a,b])$ with primitive $F(x) = \int_a^x f(t)\,dt$. Let $\epsilon > 0$. Find $g \in \mathcal{D}_0^{lc}([a,b])$ such that $\|f-g\|_A < \epsilon$. Let $G(x) = \int_a^x g$. Then $\|f-g\|_A = \sup_{I\subset[a,b]} |\int_I (f-g)| \le 2\|F-G\|_\infty$. Since F has limits at $a+$ and $b-$ there are $\alpha, \beta \in \mathbb{R}$ with $a < \alpha < \beta < b$ such that if $x \in [a,\alpha]$ then $|F(x)| < \epsilon/3$ and if $x \in [\beta, b]$ then $|F(x) - F(b)| < \epsilon/3$. Since F is continuous on $[a,b]$, by the Weierstrass approximation theorem, there is a polynomial P such that $\|(F-P)\chi_{[\alpha,\beta]}\|_\infty < \epsilon/3$. Define

$$G(x) = \begin{cases} 0, & a \le x \le \alpha/2 \\ 2P(\alpha)(x/\alpha - 1/2), & \alpha/2 \le x \le \alpha \\ P(x), & \alpha \le x \le \beta \\ F(\beta), & \beta \le x \le b. \end{cases} \tag{3.12}$$

For $\alpha/2 \le x \le \alpha$ we have

$$|F(x) - G(x)| \le |F(x)| + |P(\alpha)| \le |F(x)| + |F(\alpha) - P(\alpha)| + |F(\alpha)| < \epsilon. \tag{3.13}$$

Hence, $\|F-G\|_\infty < \epsilon$ and G is continuous on $[a,b]$ with a continuous derivative except perhaps at $\alpha/2$, α or β. Let $g(\alpha/2)=0$, $g(\alpha)=G'(\alpha-)$ and $g(\beta)=G'(\beta-)$. For other values of x let $g(x)=G'(x)$. Then $g \in \mathcal{D}_0^{lc}([a,b])$ and $\|f-g\|_A \leq 2\|F-G\|_\infty < 2\epsilon$.

(c) The dual spaces of $\mathcal{A}_D([a,b])$ and $\mathcal{D}_0^{lc}([a,b])$ are isometrically isomorphic. It is an elementary result of functional analysis that if Y is a Banach space and X is a dense subspace then $X^* = Y^*$. For example, [13, p. 194]. By (b) then, the dual space of $\mathcal{A}_D([a,b])$ is isometrically isomorphic to the dual of $D([a,b])$. But this is known to be $\mathcal{EBV}([a,b])$ [14]. Now, the (Lebesgue) integral provides a linear isometry and isomorphism between $\mathcal{IBV}_b([a,b])$ and $\mathcal{EBV}([a,b])$. If $g \in \mathcal{IBV}_b([a,b])$ and $g(x) = \int_a^x h(t)\,dt$ for $h \in \mathcal{EBV}([a,b])$ then $\|g\|_{\mathcal{IBV}} = \|g'\|_{\mathcal{EBV}} = \|h\|_{\mathcal{EBV}}$. The proof of (c) now follows. □

If T is a continuous linear functional on $D([a,b])$ then there is a unique $g \in \mathcal{EBV}([a,b])$ such that $\langle T, f\rangle = \int_a^b f(t)g(t)\,dt$ for all $f \in D([a,b])$. See [14]. Of course g can be changed on a set of measure zero without affecting the value of the integral. The continuous linear functionals on $\mathcal{A}_D([a,b])$ can now be characterised. If $S \in \mathcal{A}_D^*([a,b])$ then there is a unique function $h \in \mathcal{IBV}_b([a,b])$ such that if $f \in \mathcal{A}_D([a,b])$ then $\langle S, f\rangle = \int_a^b fh = -\int_a^b F(t)h'(t)\,dt$, where $F \in \mathcal{D}_0^{lc}([a,b])$ is the primitive of f. Observe that if $(f_n) \subset \mathcal{A}_D([a,b])$ such that $\|f_n\|_A \to 0$ then Theorem 3.10(a) shows $\int_a^b f_n h \to 0$, i.e., continuity of the linear functional.

4. THE LL PRIMITIVE INTEGRAL AND BASIC PROPERTIES

In this section we will study integrability of distributions whose primitives are left regulated and Lebesgue integrable, and define an integral for such distributions. Properties of the integral, integrable distributions and their primitives are studied.

4.1. LLP integrability and the LL primitive integral.

We denote

(4.1)
$$\mathcal{L}^{lr}(I) = \{H : I \to \mathbb{R} \mid H \text{ is left regulated, locally Lebesgue integrable,}$$
$$\text{and has the right limit at } \inf I\},$$
$$\mathcal{L}^{lc}(I) = \{G \in \mathcal{L}^{lr}(I) \mid G \text{ is left continuous, and } G(a) = G(a+) \text{ if } a = \min I \text{ exists}\},$$
$$\mathcal{L}_0^{lc}(I) = \{F \in \mathcal{L}^{lc}(I) \mid F(\inf I+) = 0\}.$$

We use the same notations also in the case when local Lebesgue integrability is replaced by Lebesgue integrability. When Lebesgue integrability is needed we mention it. Properties of the Lebesgue integral ensure that to each $H \in \mathcal{L}^{lr}(I)$ there corresponds a unique distribution on I, denoted also by H, and defined by (3.2), where $\int$ denotes the Lebesgue integral.

A distribution f on I is called LLP integrable on I if it is the distributional derivative of some primitive $H \in \mathcal{L}^{lr}(I)$. Denote

$$\mathcal{A}_L(I) = \{f \in \mathcal{D}' | f \text{ is } LLP \text{ integrable on } I\}. \tag{4.2}$$

$\mathcal{A}_L(I)$ is a linear subspace of $\mathcal{A}_D(I)$. If f is LLP integrable with a primitive $H \in \mathcal{L}^{lr}(I)$, then (3.3) determines primitives $G \in \mathcal{L}^{lc}(I)$ and $F \in \mathcal{L}_0^{lc}(I)$ of f.

If $f \in \mathcal{A}_L(I)$, F is a primitive of f in $\mathcal{L}^{lc}(I)$ and $a, b \in I$, define the LL primitive integral of f from a to b by (1.1). The so obtained integral is unique, additive over intervals, linear, and changes its sign if the integration limits are reversed. Proofs of these properties are same as the proofs presented in Theorem 3.2 for corresponding properties for the LD primitive integral. The fundamental theorem of calculus holds, i.e., the result of Theorem 3.3 holds when $f \in \mathcal{A}_L(I)$ and $G \in \mathcal{L}^{lc}(I)$, and the proof is same. Thus the mapping $\mathcal{F}$ defined by

$$\mathcal{F}(f) = F, \ f \in \mathcal{A}_L(I), \text{ where } F \text{ is the primitive of } f \text{ in } \mathcal{L}_0^{lc}(I), \tag{4.3}$$

is a linear isomorphism from $\mathcal{A}_L(I)$ to $\mathcal{L}_0^{lc}(I)$. The Hake theorem, i.e., Theorem 3.5 holds as well for LL primitive integral.

4.2. **Order and norm properties.** Also in $\mathcal{L}_0^{lc}(I)$, there is the pointwise partial order: $F \le G$ if and only if $F(x) \le G(x)$ for all $x \in I$. This allows us to define a partial order $\preceq$ in $\mathcal{A}_L(I)$ by $f \preceq g$ in $\mathcal{A}_L(I)$ if and only if $\mathcal{F}(f) \le \mathcal{F}(g)$ in $\mathcal{L}_0^{lc}(I)$. In particular (3.7) holds, and $f \preceq g$ in $\mathcal{A}_L(I)$ if and only if $F \le G$, where F and G are the respective primitives in $\mathcal{L}_0^{lc}(I)$. The proofs that $\mathcal{L}^{lc}(I)$ and $\mathcal{L}_0^{lc}(I)$ are ordered vector spaces, and that $\mathcal{A}_L(I)$ is order isomorphic to $\mathcal{L}_0^{lc}(I)$, are the same as the proofs presented in Theorem 3.6 for corresponding properties for $\mathcal{D}^{lc}(I)$, $\mathcal{D}_0^{lc}(I)$ and $\mathcal{A}_D(I)$.

Next we will show that both $\mathcal{L}_0^{lc}(I)$ and $\mathcal{A}_L(I)$ are lattice-ordered vector spaces. Recall that an ordered vector space E is lattice-ordered if the partial order $\preceq$ of E satisfies the following condition.
(3) $x \vee y$ and $x \wedge y$ are in E. The join is $x \vee y = \sup\{x, y\} = w$ such that $x \preceq w$, $y \preceq w$ and if $x \preceq \tilde{w}$ and $y \preceq \tilde{w}$ then $w \preceq \tilde{w}$. The meet is $x \wedge y = \inf\{x, y\} = w$ such that $w \preceq x$, $w \preceq y$ and if $\tilde{w} \preceq x$ and $\tilde{w} \preceq y$ then $\tilde{w} \preceq w$.

The definitions $(F \vee G)(x) = \sup(F, G)(x) = \max(F(x), G(x))$, and $(F \wedge G)(x) = \inf(F, G)(x) = \min(F(x), G(x))$ define lattice operations in $\mathcal{L}^{lc}(I)$ and in $\mathcal{L}_0^{lc}(I)$.

Theorem 4.1. *(a) $\mathcal{L}_0^{lc}(I)$ and $\mathcal{L}^{lc}(I)$ are lattice-ordered. (b) $\mathcal{A}_L(I)$ is lattice-ordered.*

Proof. (a) Let F, $G \in \mathcal{L}_0^{lc}(I)$. Define $\Phi = (F \vee G)$ and $\Psi = (F \wedge G)$. We need to prove Φ, $\Psi \in \mathcal{L}_0^{lc}(I)$. Let $\inf I < c \le \max I$ and prove Φ is left continuous at c. Suppose $F(c) > G(c)$. Given $\epsilon > 0$ there is $\delta > 0$ such that $|F(x) - F(c)| < \epsilon$, $|G(x) - G(c)| < \epsilon$ and $F(x) > G(x)$ whenever $x \in (c - \delta, c)$. For such x, $|\Phi(x) - \Phi(c)| = |F(x) - F(c)| < \epsilon$. If $F(c) = G(c)$, then $|\Phi(x) - \Phi(c)| \le \max(|F(x) - F(c)|, |G(x) - G(c)|) < \epsilon$. Therefore,

Φ is left continuous on $I\setminus\{\inf I\}$. Similarly, Φ has the right limit at $\inf I$, and is right continuous at the possible left end point of I, so that $\Phi\in\mathcal{L}_0^{lc}(I)$. Similarly with the infimum. Hence, Φ, $\Psi\in\mathcal{L}_0^{lc}(I)$. The proof that $\mathcal{L}^{lc}(I)$ is lattice-ordered is similar.

(b) First, we show that $\mathcal{A}_L(I)$ is closed under the operations $f\vee g$ and $f\wedge g$. For f, $g\in\mathcal{A}_L(I)$, we have $f\vee g=\sup(f,g)$. There is h such that $f\preceq h$, $g\preceq h$, and if $f\preceq\tilde h$, $g\preceq\tilde h$, then $h\preceq\tilde h$. This last statement is equivalent to $\mathcal{F}(f)\le\mathcal{F}(g)$, $\mathcal{F}(g)\le\mathcal{F}(h)$, and if $\mathcal{F}(f)\le\mathcal{F}(\tilde h)$, $\mathcal{F}(g)\le\mathcal{F}(\tilde h)$, then $\mathcal{F}(h)\le\mathcal{F}(\tilde h)$. But then $\mathcal{F}(h)=\max(\mathcal{F}(f),\mathcal{F}(g))$ and $h=\mathcal{F}(h)'$ so $f\vee g=(\mathcal{F}(f)\vee\mathcal{F}(g))'\in\mathcal{A}_L(I)$. Similarly, $f\wedge g=(\mathcal{F}(f)\wedge\mathcal{F}(g))'\in\mathcal{A}_L(I)$. □

Let E be a lattice-ordered normed space. Define $|x|=x\vee(-x)$, $x^+=x\vee 0$ and $x^-=(-x)\vee 0$. Then $x=x^+-x^-$ and $|x|=x^++x^-$. E is called a normed Riesz space if the norm $\|\cdot\|$ of E is a Riesz-norm, i.e., it satisfies the following condition.
(4) $|x|\preceq|y|$ implies $\|x\|\le\|y\|$.

Assume next that the functions of $\mathcal{L}^{lc}(I)$ and $\mathcal{L}_0^{lc}(I)$ are Lebesgue integrable. We will see that these spaces can be normed by the L^1-norm:

$$\|F\|_1=\int_I|F(t)|\,dt. \tag{4.4}$$

Using the isomorphism $\mathcal{F}$ we define a 1-norm in $\mathcal{A}_L(I)$ by $\|f\|_1=\|\mathcal{F}(f)\|_1$, i.e.,

$$\|f\|_1=\int_I|F(t)|\,dt,\qquad\text{where}\quad F=\mathcal{F}(f). \tag{4.5}$$

The lattice operations show that the LL primitive is absolute: if f is integrable so is $|f|$.

Theorem 4.2. *Assume that the functions of $\mathcal{L}^{lc}(I)$ and $\mathcal{L}_0^{lc}(I)$ are Lebesgue integrable. (a) $\mathcal{L}_0^{lc}(I)$ and $\mathcal{L}^{lc}(I)$ are normed Riesz spaces with respect to pointwise ordering and L^1-norm. (b) $\mathcal{A}_L(I)$ is a normed Riesz space with respect to partial order and norm defined by* (3.7) *and* (4.5). *(c) If $f\in\mathcal{A}_L(I)$ then $|f|$, f^+ and f^- are in $\mathcal{A}_L(I)$; $\int_I f=\int_I f^+-\int_I f^-/\ |\int_I f|\ge|\int_I|f||$.*

Proof. (a) To prove that the L^1-norm is a norm in $\mathcal{L}^{lc}(I)$ and in $\mathcal{L}_0^{lc}(I)$, first note they are linear subspaces of the space $L^1(I)$ of all Lebesgue integrable functions from I to $\mathbb{R}$. And, if $F\in\mathcal{L}^{lc}(I)$ such that $\|F\|_1=0$, then then $F(x)=0$ for almost all $x\in I$. Since F is left continuous in $I\setminus\{\inf I\}$, and right continuous at the possible minimum of I, then $F(x)=0$ for all $x\in I$ (see the proof of Theorem 3.7). Positivity, homogeneity and the triangle inequality are inherited from $L^1(I)$.

The cone $L^1(I)_+=\{G\in L^1(I)|G(x)\ge 0$ for a.e. $x\in I\}$ is closed with respect to the L^1-norm. Hence, if $\|F_n-F\|_1\to 0$ in $\mathcal{L}^{lc}(I)$ and $F_n\in\mathcal{L}^{lc}(I)_+=\{G\in\mathcal{L}^{lc}(I)|G(x)\ge 0$ for all $x\in I\}$, then $F(x)\ge 0$ for a.e. $x\in I$. One-sided continuity properties of F ensure that F cannot have negative values, whence $F\in\mathcal{L}^{lc}(I)_+$. This proves that $\mathcal{L}^{lc}(I)_+$ is closed in the L^1-norm topology of $\mathcal{L}^{lc}(I)$, so $\mathcal{L}^{lc}(I)$ is an ordered normed space. The proof that $\mathcal{L}_0^{lc}(I)$ is an ordered normed space is similar.

If $|F| \le |G|$, then $\|F\|_1 \le \|G\|_1$. Hence, the L^1-norm is a Riesz norm. Thus $\mathcal{L}^{lc}(I)$ and $\mathcal{L}_0^{lc}(I)$ are normed Riesz spaces.

(b) Because $\mathcal{F}$, defined by (4.3), is an order isomorphism from $\mathcal{A}_L(I)$ to $\mathcal{L}_0^{lc}(I)$, and $\|f\|_1 = \|\mathcal{F}(f)\|_1$ for all $f \in \mathcal{A}_L(I)$, then $\mathcal{A}_L(I)$ is an ordered normed space because $\mathcal{L}_0^{lc}(I)$ is. And, if $|f| \preceq |g|$ then $|\mathcal{F}(f)|' \preceq |\mathcal{F}(g)|'$ so $|\mathcal{F}(f)| \le |\mathcal{F}(g)|$, that is, $|\mathcal{F}(f)(x)| \le |\mathcal{F}(g)(x)|$ for all $x \in I$. Then $\|f\| = \|\mathcal{F}(f)\|_1 \le \|\mathcal{F}(g)\|_1 = \|g\|$. Thus the norm defined by (4.5) is a Riesz norm. This concludes the proof that $\mathcal{A}_L(I)$ is a normed Riesz space.

(c) Theorem 4.1(b) establishes the first part. If $F \in \mathcal{L}_0^{lc}(I)$ is the primitive of f then $|\int_a^b f| = |F(b) - F(a)| \ge ||F(b)| - |F(a)|| = |\int_a^b |F|'| = |\int_a^b |f||$. □

4.3. **Integration by parts, dual space.** The multipliers for $L^1([a,b])$ are the essentially bounded functions, $L^\infty([a,b])$. If $f \in L^1([a,b])$ and $g \in L^\infty([a,b])$ then $fg \in L^1([a,b])$. The dual space of $L^1([a,b])$ is also $L^\infty([a,b])$. If $g(x) = \int_a^x h(t)\,dt$ for some $h \in L^\infty([a,b]) \cap L^1([a,b])$ then $\int_a^b f(t)g(t)\,dt = F(b)g(b) - \int_a^b F(t)h(t)\,dt$, where $F(x) = \int_a^x f(t)\,dt$. There are analogues for the LL primitive integral.

Definition 4.3. Let $c \in [a,b]$. Define

(4.6)
$$\Lambda_c([a,b]) = \{g : [a,b] \to \mathbb{R} \mid g(x) = \int_c^x h(t)\,dt \quad \text{for some } h \in L^\infty([a,b]) \cap L^1([a,b])\}.$$

The L^1 condition is redundant when $[a,b]$ is compact. Note that functions in $\Lambda_c([a,b])$ are Lipschitz continuous and vanish at c.

Definition 4.4. Let $f \in \mathcal{A}_L([a,b])$ with primitive $F \in \mathcal{L}_0^{lc}([a,b])$. Let $g \in \Lambda_c([a,b])$ for some $c \in [a,b]$. Define the integration by parts formula $\int_a^b fg = F(b)g(b) - \int_a^b F(t)g'(t)\,dt$.

As in Section 3.4, density of C^1 functions in L^1 and a sequential approach justifies the definition.

Theorem 4.5. *Let $f \in \mathcal{A}_L([a,b])$ with primitive $F \in \mathcal{L}_0^{lc}([a,b])$. Let $g \in \Lambda_a([a,b])$. (a) Then $|\int_a^b fg| \le |F(b)||g(b)| + \|F\|_1\|g'\|_\infty$. (b) Suppose $(F_n) \subset C([a,b]) \cap C^1((a,b))$ such that $F_n(a) = 0$, $F_n(b) = F(b)$ and $\|F_n' - f\|_1 \to 0$. Let $H(x) = F(x)g(x) - \int_a^x F(t)g'(t)\,dt$; let $H_n(x) = \int_a^x F_n'(t)g(t)\,dt$. Then $|H(b) - H_n(b)| \le \|F - F_n\|_1\|g'\|_\infty \to 0$ as $n \to \infty$.*

Proof. (a) This is the Hölder inequality.

(b) See the proof of Theorem 3.10. □

Proposition 4.1. *Let $[a,b]$ be a compact interval. Suppose $(F_n)\subset C([a,b])\cap C^1([a,b])$ such that $\|F_n'-f\|_1\to 0$. Let $H(x)=F(x)g(x)-\int_a^x F(t)g'(t)\,dt$ and let $H_n(x)=\int_a^x F_n'(t)g(t)\,dt$. Then $H, H_n\in\mathcal{L}_0^{lc}([a,b])$ and $\|H-H_n\|_1\to 0$.*

Proof. The proof is similar to that for Proposition 3.11. Now, using the Hölder inequality and the Fubini–Tonelli theorem,

(4.7)
$$\begin{cases} \|H'-H_n'\|_1\le\int_a^b|F(x)-F_n(x)||g(x)|\,dx+\int_a^b|F(t)-F_n(t)||g'(t)|\int_a^t\,dx\,dt\\ \le\|F-F_n\|_1\left(\|g\|_\infty+(b-a)\|g'\|_\infty\right).\end{cases}$$

This completes the proof. □

Note that if J is a subinterval of $[a,b]$ then $\int_J f$ is not defined since χ_J is not Lipschitz continuous.

Due to the isometric isomorphism between $\mathcal{A}_L([a,b])$ and $L^1([a,b])$, the dual space of $\mathcal{A}_L([a,b])$ is isometrically isomorphic to $L^\infty([a,b])$.

Theorem 4.6. *(a) $\mathcal{A}_L([a,b])$ is not complete. (b) $\mathcal{L}_0^{lc}([a,b])$ is dense in $L^1([a,b])$. (c) The dual space of $\mathcal{A}_L([a,b])$ is isometrically isomorphic to $\Lambda_b([a,b])$ and to $L^\infty([a,b])$.*

Proof. (a) Use the example in Theorem 3.12(a). Now, $\|F-F_n\|_1=1/(n+1)$.

(b) (c) These are essentially the same as in Theorem 3.12. □

If $S\in\mathcal{A}_L^*([a,b])$ then there is a unique function $g\in\Lambda_b([a,b])$ such that if $f\in\mathcal{A}_L([a,b])$ then $\langle S,f\rangle=\int_a^b fg=F(b)g(b)-\int_a^b F(t)g'(t)\,dt$, where $F\in\mathcal{L}_0^{lc}([a,b])$ is the primitive of f.

5. THE *LR* PRIMITIVE INTEGRAL AND BASIC PROPERTIES

In this section we will study integrability and an integral of distributions whose primitives are left regulated and Riemann integrable. Properties for the integral, integrable distributions and their primitives are derived.

5.1. ***LRP* integrability and the *LR* primitive integral.** We denote

(5.1)
$$\begin{cases} \mathcal{R}^{lr}(I)=\{H:I\to\mathbb{R}\mid H \text{ is left regulated, locally Riemann integrable,}\\ \text{and has a right limit at }\inf I\},\\ \mathcal{R}^{lc}(I)=\{G\in\mathcal{R}^{lr}(I)\mid G\text{ is left continuous, } G(a)=G(a+)\text{ if } a=\min I\text{ exists}\},\\ \mathcal{R}_0^{lc}(I)=\{F\in\mathcal{R}^{lc}(I)\mid F(\inf I+)=0\}.\end{cases}$$

It is well-known that to each $H\in\mathcal{R}^{lr}(I)$ there corresponds a unique distribution on I, denoted also by H, and defined by (3.2), where $\int$ denotes the Riemann integral.

A distribution f on I is called LRP integrable on I if it is the distributional derivative of some primitive $H \in \mathcal{R}^{lr}(I)$. Denote

$$\mathcal{A}_R(I) = \{f \in \mathcal{D}' \mid f \text{ is } LRP \text{ integrable on } I\}. \tag{5.2}$$

$\mathcal{A}_R(I)$ is a linear subspace of $\mathcal{A}_L(I)$. If f is LRP integrable with a primitive $H \in \mathcal{R}^{lr}(I)$, then (3.3) determines primitives $G \in \mathcal{R}^{lc}(I)$ and $F \in \mathcal{R}_0^{lc}(I)$ of f.

If $f \in \mathcal{A}_R(I)$, F is a primitive of f in $\mathcal{R}^{lc}(I)$ and a, $b \in I$, define the LL primitive integral of f from a to b by (1.1). The so obtained integral is unique, additive over intervals, linear, and changes its sign if the integration limits are reversed. Proofs of these properties are the same as the proofs presented in Theorem 3.2 for corresponding properties for the LD primitive integral. The fundamental theorem of calculus holds, i.e., the result of Theorem 3.3 holds when $f \in \mathcal{A}_R(I)$ and $G \in \mathcal{R}^{lc}(I)$, and the proof is the same. The mapping $\mathcal{F}$, defined by

$$\mathcal{F}(f) = F, \ f \in \mathcal{A}_R(I), \text{ where } F \text{ is the primitive of } f \text{ in } \mathcal{R}_0^{lc}(I), \tag{5.3}$$

is a linear isomorphism from $\mathcal{A}_R(I)$ to $\mathcal{R}_0^{lc}(I)$. The Hake theorem, i.e., Theorem 3.5 holds as well for LL primitive integral.

5.2. **Order and norm properties.** Also in $\mathcal{R}_0^{lc}(I)$, there is the pointwise partial order: $F \le G$ if and only if $F(x) \le G(x)$ for all $x \in I$. We can thus define a partial order $\preceq$ in $\mathcal{A}_R(I)$ by $f \preceq g$ in $\mathcal{A}_R(I)$ if and only if $\mathcal{F}(f) \le \mathcal{F}(g)$ in $\mathcal{R}_0^{lc}(I)$. In particular (3.7) holds, and $f \preceq g$ in $\mathcal{A}_R(I)$ if and only if $F \le G$, where F and G are the respective primitives in $\mathcal{R}_0^{lc}(I)$. The proofs that $\mathcal{R}^{lc}(I)$ and $\mathcal{R}_0^{lc}(I)$ are ordered vector spaces, and that $\mathcal{A}_R(I)$ is order isomorphic to $\mathcal{R}_0^{lc}(I)$, are the same as the proofs presented in Theorem 3.6 for corresponding properties for $\mathcal{D}^{lc}(I)$, $\mathcal{D}_0^{lc}(I)$ and $\mathcal{A}_D(I)$.

Define the lattice operations for F, $G \in \mathcal{R}^{lc}(I)$ by $(F \vee G)(x) = \sup(F,G)(x) = \max(F(x),G(x))$ and $(F \wedge G)(x) = \inf(F,G)(x) = \min(F(x),G(x))$. The proof that both $\mathcal{R}_0^{lc}(I)$ and $\mathcal{R}^{lc}(I)$ are lattice-ordered, and that $\mathcal{A}_R(I)$ is lattice-ordered with respect to the partial order $\preceq$ defined above is same as that given in the proof of Theorem 4.1 to $\mathcal{L}_0^{lc}(I)$, $\mathcal{L}^{lc}(I)$ and $\mathcal{A}_L(I)$.

Assume next that I is compact. By Lemma 2.1 Riemann integrability of a left regulated function $F : I \to \mathbb{R}$ is equivalent to the boundedness of F. Thus we can define the sup-norm norm $\|\cdot\|_\infty$ in $\mathcal{R}^{lc}(I)$ and in $\mathcal{R}_0^{lc}(I)$:

$$\|F\|_\infty = \sup_{x\in I} |F(x)|. \tag{5.4}$$

Using the isomorphism $\mathcal{F}$ we define an Alexiewicz norm in $\mathcal{A}_R(I)$ by $\|f\| = \|\mathcal{F}(f)\|_\infty$. Equivalently,

$$\|f\| = \|F\|_\infty = \sup_{x\in I} |F(x)|, \quad \text{where } F(t) = \int_{\min I}^t f. \tag{5.5}$$

We will show that the spaces $\mathcal{R}^{lc}(I)$ and $\mathcal{R}_0^{lc}(I)$ and $\mathcal{A}_R(I)$ are Banach lattices, i.e., complete normed Riesz spaces.

Theorem 5.1. *Assume that I is compact. (a) $\mathcal{R}_0^{lc}(I)$ and $\mathcal{R}^{lc}(I)$ are Banach lattices with respect to pointwise ordering and sup-norm. (b) $\mathcal{A}_R(I)$ is a normed Riesz space with respect to partial order and norm defined by* (3.7) *and* (5.5)*. (c) If $f \in \mathcal{A}_R(I)$ then $|f|$, f^+ and f^- are in $\mathcal{A}_R(I)$; $\int_I f = \int_I f^+ - \int_I f^-$; $|\int_I f| \geq |\int_I |f||$.*

Proof. (a) Noticing that the spaces $\mathcal{R}^{lc}(I)$ and in $\mathcal{R}_0^{lc}(I)$ are also linear subspaces of $L^\infty(I)$, the proof that they are normed Riesz spaces is similar to that given for $\mathcal{L}^{lc}(I)$ and in $\mathcal{L}_0^{lc}(I)$ in Theorem 4.2 when $L^1(I)$ is replaced by $L^\infty(I)$. To show $\mathcal{R}^{lc}(I)$ is complete, suppose (F_n) is a Cauchy sequence in $\mathcal{R}^{lc}(I)$. Then (F_n) is a Cauchy sequence in $L^\infty(I)$ so there is $F \in L^\infty(I)$ such that $\|F - F_n\|_\infty \to 0$. To show F is left continuous in $I \setminus \{\min I\}$, suppose $c \in I \setminus \{\min I\}$. For $x \in (\min I, c)$ and $n \in \mathbb{N}$,

$$\begin{cases} |F(c) - F(x)| \leq |F(c) - F_n(c)| + |F_n(c) - F_n(x)| + |F_n(x) - F(x)| \\ \leq 2\|F - F_n\|_\infty + |F_n(c) - F_n(x)|. \end{cases}$$

Given $\epsilon > 0$, fix n large enough so that $\|F - F_n\|_\infty < \epsilon/3$. Then let $x \to c-$. Hence, F is left continuous on $I \setminus \{\min I\}$. We can see that F has a right limit at $c = \min I$ by taking x, $y > c$ and letting x, $y \to c+$ in $|F(x) - F(y)| \leq 2\|F - F_n\|_\infty + |F_n(x) - F_n(y)|$. Therefore, $F \in \mathcal{R}^{lc}(I)$ and the space $\mathcal{R}^{lc}(I)$ is complete. The space $\mathcal{R}_0^{lc}(I)$ is complete since it is a closed subspace of $\mathcal{R}^{lc}(I)$.

(b) Because $\mathcal{F}$ is an order isomorphism from $\mathcal{A}_R(I)$ to $\mathcal{R}_0^{lc}(I)$, and $\|f\| = \|\mathcal{F}(f)\|_\infty$ for all $f \in \mathcal{A}_R(I)$, the proof that $\mathcal{A}_L(I)$ is a normed Riesz space is the same as that given for $\mathcal{A}_L(I)$ in the proof of Theorem 4.2. To prove it is complete, suppose (f_n) is a Cauchy sequence in $\mathcal{A}_R(I)$. Then $\|\mathcal{F}(f_n) - \mathcal{F}(f_m)\|_\infty = \|f_n - f_m\|$ so $(\mathcal{F}(f_n))$ a Cauchy sequence in $\mathcal{R}_0^{lc}(I)$. There is $F \in \mathcal{R}_0^{lc}(I)$ such that $\|\mathcal{F}(f_n) - F\|_\infty \to 0$. And then $\|f_n - F'\| = \|\mathcal{F}(f_n) - F\|_\infty \to 0$. Since $F \in \mathcal{R}_0^{lc}(I)$, we have $F' \in \mathcal{A}_R(I)$ and $\mathcal{A}_R(I)$ is complete.

(c) The proof is the same as in Theorem 4.2. □

5.3. **Banach algebra.** In this subsection we assume that I is compact.

The spaces of Lebesgue and HK integrable functions are not closed under pointwise multiplication. For example, if $f(x) = x^{-2/3}$ then f is Lebesgue integrable on $[0, 1]$ but f^2 is not. However, each of the spaces $\mathcal{R}^{lr}(I)$, $\mathcal{R}^{lc}(I)$ and $\mathcal{R}_0^{lc}(I)$ is closed under pointwise multiplication. This makes them into commutative Banach algebras. The isomorphism between $\mathcal{R}_0^{lc}(I)$ and $\mathcal{A}_R(I)$ makes this latter space into a commutative Banach algebra.

A commutative algebra is a vector space V over scalar field $\mathbb{R}$ with a multiplication $V \times V \mapsto V$ such that for all $u, v, w \in V$ and all $a \in \mathbb{R}$, $u(vw) = (uv)w$ (associative), $uv = vu$ (commutative), $u(v + w) = uv + uw$ and $(u + v)w = uw + vw$

(distributive), $a(uv) = (au)v$. If $(V, \|\cdot\|_V)$ is a Banach space and $\|uv\|_V \leq \|u\|_V\|v\|_V$ then it is a Banach algebra.

The spaces $\mathcal{R}^{lr}(I)$, $\mathcal{R}^{lc}(I)$ have unit $e = 1$. A unit for $\mathcal{R}_0^{lc}(I)$ would need to equal 1 on $(\min I, \max I]$ and 0 at $\min I$. But such a function is not right continuous at $\min I$ so it is not in $\mathcal{R}_0^{lc}(I)$. However, $\mathcal{R}_0^{lc}(I)$ has an approximate identity. If $\min I > -\infty$ define the sequence of continuous functions

$$u_n(x) = \begin{cases} n(x - \min I), & \min I \leq x \leq \min I + 1/n \\ 1, & \min I + 1/n \leq x \leq \max I. \end{cases}$$

For each $F \in \mathcal{R}_0^{lc}(I)$ we have $\|F - u_nF\|_\infty \to 0$. $\mathcal{R}_0^{lc}(I)$ is then said to have an approximate identity. A similar construction was used in [24] when $I = [-\infty, \infty]$.

Theorem 5.2. *(a) $\mathcal{R}_0^{lc}(I)$ and $\mathcal{R}^{lc}(I)$ are commutative Banach algebras with respect to pointwise multiplication and sup-norm. (b) For $f, g \in \mathcal{A}_R(I)$, with respective primitives $F, G \in \mathcal{R}_0^{lc}(I)$, define a product by $fg = (FG)'$. Then $\mathcal{A}_R(I)$ is a commutative Banach algebra. It has no unit.*

Proof. (a) Limits and pointwise products commute. If two functions vanish at a point so does their product.

(b) This follows from the isomorphism between $\mathcal{R}_0^{lc}(I)$ and $\mathcal{A}_R(I)$. See [24, Lemma 1]. □

5.4. **Integration by parts.** The Riemann-Stieltjes integral $\int_a^b F\,dg$ exists for all $F \in C([a, b])$ and all $g \in \mathcal{BV}([a, b])$. It is defined with a globally fine partition. The integral equals $A \in \mathbb{R}$ if for all $\epsilon > 0$ there is $\delta > 0$ such that if $a = x_0 < x_1 < \cdots < x_n = b$ satisfies $\max_{1\leq i\leq n} |x_i - x_{i-1}| < \delta$ and ξ_i is any point in $[x_{i-1}, x_i]$ then $|\sum_{i=1}^n F(\xi_i)[g(x_i) - g(x_{i-1})] - A| < \epsilon$. For example, see [16]. For the LR primitive integral we use a similar type of Stieltjes integral with a locally fine countable partition of left open intervals. This yields an integration by parts formula for which the multipliers are right continuous functions of bounded variation. While a necessary condition for existence of the Riemann-Stieltjes integral is that at each point of $[a, b]$ one of F and g is continuous (see [12, 10.6]) for our version we will require that F is left continuous and g is right continuous.

The following definitions are necessary.

Definition 5.3. A left gauge is a mapping γ from $(a, b]$ to the intervals in $(a, b]$ such that for each $y \in (a, b]$ there is $x \in (a, y)$ such that $\gamma(y) = (x, y]$. A γ-fine left partition is a mutually disjoint collection of intervals $\mathcal{P}$ such that $\cup_{I\in\mathcal{P}} I = (a, b]$; if $I \in \mathcal{P}$ then $I = (x, y]$ for some $a \leq x < y \leq b$ and $(x, y] \subset \gamma(y)$.

The definition makes sense for all $-\infty \leq a < b \leq \infty$. If $[a, b]$ is a compact interval then the gauge can be constructed from a positive function δ as is done in Henstock-Kurzweil integration. The intervals in the gauge are then of the form $(x-\delta(x), x]$.

A collection of disjoint intervals in $\mathbb{R}$ is necessarily countable. Note that a γ-fine left partition need not be finite. For example, if for each $x>0$ we have $\gamma(x)\subset(x/2,x]$ then every γ-fine left partition of $[-1,1]$ must be denumerable. Without loss of generality we will assume each γ-fine left partition is denumerable.

Definition 5.4. Let $F\in\mathcal{R}^{lc}([a,b])$ and let $g\in\mathcal{BV}([a,b])$. The integral $\int_a^b F\,dg = A\in\mathbb{R}$ if for each $\epsilon>0$ there is a left gauge γ such that if $\mathcal{P}=\{I_i\}_{i=1}^\infty$ is a γ-fine left partition of $[a,b]$ then for each $\xi_i\in I_i$ we have $|\sum_{i=1}^\infty F(\xi_i)[g(y_i+)-g(x_i+)]-A|<\epsilon$. Here, $I_i=(x_i,y_i]$ for $i\geq 1$.

The main properties of this integral are in the following theorem.

Theorem 5.5. *Let $F, F_i\in\mathcal{R}^{lc}([a,b])$, let $g, g_i\in\mathcal{BV}([a,b])$ and let γ be a left gauge on $[a,b]$. (a) There exists a γ-fine left partition. (b) The integral is unique. (c) Let $a_i, b_i\in\mathbb{R}$. Then $\int_a^b(a_1F_1+a_2F_2)\,d(b_1g_1+b_2g_2) = a_1b_1\int_a^b F_1\,dg_1 + a_1b_2\int_a^b F_1\,dg_2 + a_2b_1\int_a^b F_2\,dg_1 + a_2b_2\int_a^b F_2\,dg_2$. (d) $|\int_a^b F\,dg|\leq\|F\|_\infty Vg$. (e) Suppose each integral $\int_a^b F_n\,dg$ exists and $\|H-F_n\|_\infty\to 0$ for some function H. Then $\int_a^b H\,dg$ exists and equals $\lim_{n\to\infty}\int_a^b F_n\,dg$. (f) Let F be a bounded countably stepped function that is left continuous on $(a,b]$ and right continuous at a. On $(a,b]$ write $F=\sum_{i=1}^\infty a_i\chi_{I_i}$ where $\{I_i\}$ is a left continuous partition of $(a,b]$. Use the notation of Definition 5.4. Then $\int_a^b F\,dg=\sum_{i=1}^\infty a_i[g(y_i+)-g(x_i+)]$. (g) The integral exists for each $F\in\mathcal{R}^{lc}([a,b])$ and $g\in\mathcal{BV}([a,b])$. (h) If $\int_a^c F\,dg$ and $\int_c^b F\,dg$ exist for some $c\in(a,b)$ then $\int_a^b F\,dg$ exists and equals $\int_a^c F\,dg+\int_c^b F\,dg$. (i) If $\int_a^b F\,dg$ exists then $\int_x^y F\,dg$ exists for all $(x,y]\subset(a,b]$. And, $\int_a^b F\,dg=\int_a^c F\,dg+\int_c^b F\,dg$.*

Proof. (a) This follows from the construction in Lemma 2.1. (b) Given two left gauges γ_1 and γ_2, define $\gamma(x)=\gamma_1(x)\cap\gamma_2(x)$. Then γ is a left gauge. The usual uniqueness proof for Henstock-Kurzweil integrals now shows the integral is unique. For example, see [16, p. 39]. (c) This follows from the linearity of the approximating series. (d) $|\sum_{i=1}^\infty F(\xi_i)[g(y_i)-g(x_i)]|\leq\|F\|_\infty Vg$. (e) Using the inequality in (d), the proof is essentially the same as for uniform convergence of sequences of Riemann integrable functions. See [16, p. 84]. (f) Write $I_i=(b_i,c_i]$. Define a left gauge γ so that if $x\in I_i$ then $\gamma(x)\subset I_i$. If $\mathcal{P}=\{(\alpha_i,\beta_i]\}$ is a γ-fine left partition then each I_i is a disjoint union $I_i=\cup\{J_j\mid J_j\subset I_i\}$. There is a signed Borel measure μ_g such that $\mu_g((x,y])=g(y+)-g(x+)$. Let $\xi_i\in I_i$. We then have

$$\begin{cases}\sum_{i=1}^\infty F(\xi_i)[g(\beta_i+)-g(\alpha_i+)]=\sum_{i=1}^\infty a_i\sum_{J_j\subset I_i}[g(\beta_j+)-g(\alpha_j+)]\\ =\sum_{i=1}^\infty a_i\sum_{J_j\subset I_i}\mu_g(J_j)=\sum_{i=1}^\infty a_i\mu_g\left(\cup_{J_j\subset I_i}J_j\right)=\sum_{i=1}^\infty a_i\mu_g\left(I_i\right)\\ =\sum_{i=1}^\infty a_i[g(c_i+)-g(b_i+)].\end{cases}$$

Due to the estimate in (d) all of the series converge absolutely. (g) By Lemma 2.1(b) F is the uniform limit of a sequence of bounded countably stepped functions. The result now follows from (e) and (f). (h) Given $\epsilon > 0$ there exist gauges γ_1 and γ_2 so that sums over respective γ_i-fine partitions approximate $\int_a^c F\,dg$ and $\int_c^b F\,dg$ with error at most $\epsilon/2$. Defining $\gamma(x) = \gamma_1(x)$ if $x \in (a,c]$ and $\gamma_2(x)$ if $x \in (c,b]$ defines a left gauge on $(a,b]$ with the property that sums over γ-fine partitions approximate $\int_a^b F\,dg$ with error at most ϵ. (i) For each $(x,y] \subset (a,b]$, the function $F\chi_{(x,y]}$ is in $R^{lc}([a,b])$. By (h), $\int_a^b F\chi_{(x,y]}\,dg$ exists. Examining the approximating sums shows $\int_a^b F\chi_{(x,y]}\,dg = \int_x^y F\,dg$. Now write $\int_a^b F\,dg = \int_a^b F(\chi_{(a,c]} + \chi_{(c,b]})\,dg$. By linearity this equals $\int_a^c F\,dg + \int_c^b F\,dg$. □

6. EXAMPLES AND REMARKS

In this section we will first construct examples of left regulated functions that are locally integrable in the HK, Lebesgue or Riemann sense, and have at every rational number a discontinuity of the second kind. These functions are also used to define primitives which are locally integrable in $I = [0,\infty)$. Lemmas 2.4 and 2.5 are used to verify the local integrability of the constructed functions.

Example 6.1. Define a mapping $G : \mathbb{R} \to \mathbb{R}$ by

(6.1)
$$G(t) = \sum_{n=1}^{\infty} \frac{1}{n^2}\left(2(nt-[nt])\cos\left(\frac{\pi}{2(nt-[nt])}\right) + \frac{\pi}{2}\sin\left(\frac{\pi}{2(nt-[nt])}\right)\right), \quad t \in \mathbb{R},$$

where $[nt] = m,\ m \le nt < m+1$. For each fixed $m \in \mathbb{N}$, denote by $G_m(t)$ the mth partial sum of the series (6.1) when $t \in \mathbb{R}$. It is easy to verify that the so obtained functions $G_m : \mathbb{R} \to \mathbb{R}$ are left regulated, and that the set of all discontinuity points of G_m is

$$Z_m = \left\{\frac{i}{j} \mid j \in \{1,\dots,m\},\ i \in \mathbb{Z}, \text{ and } i \text{ and } j \text{ are coprime}\right\}.$$

Moreover, the sequence (G_m) converges uniformly to G on each compact subinterval of $\mathbb{R}$. Define a function $F : \mathbb{R} \to \mathbb{R}$ by

$$F(t) = \sum_{n=1}^{\infty} \frac{(nt-[nt])^2}{n^3}\cos\left(\frac{\pi}{2(nt-[nt])}\right), \quad t \in \mathbb{R}. \tag{6.2}$$

The mth partial sums of the series (6.2) define functions $F_m : \mathbb{R} \to \mathbb{R}$. Obviously, each F_m is continuous, and the sequence (F_m) converges uniformly to F on each compact subinterval of $\mathbb{R}$, whence F is continuous. Moreover, $F_m'(t) = G_m(t)$ for each $t \in \mathbb{R} \setminus Z_m$. Consequently, the hypotheses of Lemma 2.5 are valid for F and G, so that $F'(t) = G(t)$ for each $t \in \mathbb{R} \setminus \cup_m Z_m$. Thus G is by Lemma 2.4 locally HK integrable. Because G is locally bounded, it is also locally Riemann integrable. G is discontinuous at every point of the set $\cup_m Z_m$, which is the set $\mathbb{Q}$ of all rational numbers. Moreover, all the discontinuities are of the second kind because of the sine term in the right hand side of (6.1). On the other hand, for each $t \in \mathbb{R} \setminus \mathbb{Q}$, the functions G_m are continuous at t and converge uniformly in $[t-1, t+1]$ to G, whence G is continuous at t. Similarly, since for every $m \in \mathbb{N}$, G_m has a left limit at each point of $\mathbb{R}$, this property holds also for G, i.e., G is left regulated.

The above reasoning shows that (6.1) defines a function $G : \mathbb{R} \to \mathbb{R}$ that has the following properties:

- G is left regulated and locally Riemann integrable;
- G is continuous in $\mathbb{R} \setminus Q$, and each point of $\mathbb{Q}$ is its discontinuity point of the second kind.

The function $t \mapsto tG(t)$ has the above properties, and it is right continuous at the origin. Its restriction to $I = \mathbb{R}_+$ belongs to $\mathcal{R}^{lr}(I)$. The function

$$G_0(t) = \begin{cases} tG(t), & t \in I \setminus \mathbb{Q}_+ \\ tG(t-), & t \in \mathbb{Q}_+, \end{cases} \quad \text{belongs to } \mathcal{R}_0^{lc}(I).$$

Also the function $t \mapsto e^{-|t|}G(t)$ has the properties listed above, and it belongs to $\mathcal{R}^{lr}(\mathbb{R})$. Moreover, it is HK integrable.

In the next example we present locally Lebesgue integrable primitives that are discontinuous at every rational point of their domains, and are not locally Riemann integrable.

Example 6.2. Let G and F be defined by (6.1) and (6.2). Define functions $G^m : \mathbb{R} \to \mathbb{R}$, $m \in \mathbb{N}$, by

$$G^m(t) = G(t) + \sum_{n=1}^{m} \frac{1}{2\sqrt{nt - [nt]}}, \quad t \in \mathbb{R}. \tag{6.3}$$

It is elementary to verify that $\mathbb{Q}$ is the set of discontinuity points of functions G^m, and that these functions are left regulated. Define functions $F^m : \mathbb{R} \to \mathbb{R}$, $m \in \mathbb{N}$, by

$$F^m(t) = F(t) + \sum_{n=1}^{m} \frac{[nt] + \sqrt{nt - [nt]}}{n}, \quad t \in \mathbb{R}_-. \tag{6.4}$$

F^m is continuous, and $(F^m)'(t) = G^m(t)$ for all $t \in \mathbb{R} \setminus \mathbb{Q}$ and $m \in \mathbb{N}$. Lemma 2.4 implies then that functions G^m are locally HK integrable. Because the functions F^m are locally absolutely continuous, then every G^m is locally Lebesgue integrable. But G^m is not locally bounded, and hence not locally Riemann integrable, for any $m \in \mathbb{N}$.

The functions $t \mapsto tG^m(t)$ have the above properties, and they are right continuous at the origin. Their restrictions to $\mathbb{R}_+$ belong to $\mathcal{L}^{lr}(\mathbb{R}_+)$, but not to $\mathcal{B}^{lr}(\mathbb{R}_+)$. The functions $t \mapsto tG^m(t-)$, restricted to $\mathbb{R}_+$, belong to $\mathcal{L}_0^{lc}(\mathbb{R}_+)$, but not to $\mathcal{R}_0^{lc}(\mathbb{R}_+)$.

The functions $t \mapsto e^{-|t|}G^m(t)$ are Lebesgue integrable on $\mathbb{R}$.

Locally HK integrable primitives which are discontinuous at every rational point of their domains, and are not locally Lebesgue integrable, are presented in the next example.

Example 6.3. Let G and F be defined by (6.1) and (6.2). Define functions $G_m : \mathbb{R} \to \mathbb{R}$, $m \in \mathbb{N}$, by

$$G_m(t) = G(t) + \sum_{n=1}^{m} \left(\cos\left(\frac{\pi}{2(nt - [nt])} \right) + \frac{\pi \sin(\frac{\pi}{2(nt-[nt])})}{2(nt - [nt])} \right), \quad t \in \mathbb{R}. \tag{6.5}$$

G_m is left regulated, and $\mathbb{Q}$ is the set of its discontinuity points. Functions $F_m : \mathbb{R} \to \mathbb{R}$, defined by

$$F_m(t) = F(t) + \sum_{n=1}^{m} \frac{1}{n}(nt - [nt]) \cos\left(\frac{\pi}{2(nt - [nt])} \right), \quad t \in \mathbb{R}, \tag{6.6}$$

are continuous, and $F_m'(t) = G_m(t)$ for all $t \in \mathbb{R} \setminus \mathbb{Q}$. It then follows from Lemma 2.4 that the functions G_m are locally HK integrable. On the other hand, G_m is neither locally Lebesgue integrable nor locally Riemann integrable for any $m \in \mathbb{N}$, since F_m is not locally absolutely continuous, and G_m is not locally bounded for any $m \in \mathbb{N}$.

The functions $t \mapsto t^2 G_m(t)$, $t \in \mathbb{R}+$, belong to $\mathcal{D}^{lr}([0,1]) \setminus \mathcal{L}^{lr}([0,1])$, and the functions $t \mapsto t^2 G_m(t-)$, $t \in \mathbb{R}_+$, belong to $\mathcal{D}_0^{lc}([0,1]) \setminus \mathcal{L}_0^{lc}([0,1])$.

The functions $t \mapsto e^t G_m(t)$ are HK integrable on $\mathbb{R}$.

Example 6.4. The function $G : (-\infty, 1] \to \mathbb{R}$, defined by

$$G(t) = \sum_{n=1}^{\infty} \frac{1 + nt - \lfloor nt \rfloor}{n^p}, \quad t \in (-\infty, 1], \tag{6.7}$$

where $\lfloor nt \rfloor = m$, $m - 1 < nt \le m$, $m = 0, 1, \ldots$, and $p \ge 2$, is bounded, left continuous on $(-\infty, 1]$, and has the right limit at every point of $t \in (-\infty, 1]$. The set Z of discontinuity points of G is the set of all nonzero rational numbers of $(-\infty, 1]$. Every number x of Z can be represented as $x = \frac{i}{j}$, where i and j are coprime, i.e., their greatest common divisor $GCD(i,j)$ is 1. It can be shown (cf. [15, (236)]) that $G(x+) - G(x) = -\frac{\zeta(p)}{j^p}$, where ζ is the Riemannian zeta function. The family $(G(x+) - G(x))_{\alpha \in Z}$ is absolutely summable in the sense that (see [4]) for a bijection φ from $\mathbb{N}$ to Z the series $\sum\limits_{n=1}^{\infty} (G(\varphi(n)+) - G(\varphi(n)))$ is absolutely convergent. The sum of the family $(G(x+) - G(x))_{z \in Z \cap (t,1]}$, $t \in (-\infty, 1]$, is

$$H(t) = -\sum_{j=1}^{\infty} \sum_{jt < i < j}^{CGD(i,j)=1} \frac{\zeta(p)}{j^p}, \quad t \in (-\infty, 1]. \tag{6.8}$$

H contains all the discontinuities of G, i.e., the function $W = G - H$ is continuous. The distributional derivative of W is in fact its ordinary derivative:

$$W' = \frac{d}{dt}\sum_{n=1}^{\infty}\frac{1+nt}{n^p} = \zeta(p-1). \tag{6.9}$$

Consequently, the function G, defined by (6.7), is the solution of the impulsive distributional Cauchy problem

$$G' = H' + \zeta(p-1), \quad G(0) = 0, \tag{6.10}$$

where the impulsive part of the equation is contained in H'.

The function G is locally Riemann integrable, whence it has by Theorem 2.7 a CD primitive. When $p > 2$, the CD primitive of G which vanishes at origin is

$$F(t) = t\sum_{n=1}^{\infty}\frac{1-\lfloor nt\rfloor}{n^p} + t^2\sum_{n=1}^{\infty}\frac{1}{2n^{p-1}}, \quad t \in (-\infty, 1]. \tag{6.11}$$

Thus F is a solution of the Cauchy problem

$$F'(t) = G(t) \ \text{ for each irrational } t \in (-\infty, 1], \quad F(0) = 0. \tag{6.12}$$

Because $0 \le 1 + nt - \lfloor nt\rfloor \le 1$ for all $n \in \mathbb{N}$, the series in (6.7) is absolutely and uniformly convergent in $(-\infty, 1]$ for every $p > 1$. Thus, (6.7) defines a bounded and left continuous function G on $(-\infty, 1]$ also when $1 < p \le 2$. For these values of p, G is locally Riemann integrable, has a CD primitive, and can be uniformly approximated by countably stepped functions. However, if $p \in (1, 2]$, then the CD primitive of G cannot be represented as (6.11).

An example of a left regulated function that is not HK integrable at any subinterval of $\mathbb{R}$ that contains origin is

$$G_p(t) = \begin{cases} G(t) + \frac{1}{t}, & t > 0, \\ G(t), & t \le 0, \end{cases}$$

where G is defined by (6.1).

LCP integrable distributions are defined in [9]. The space of their primitives is

$$\mathcal{B}^{lr}(I) = \{H : I \to \mathbb{R} | H \text{ is bounded and left regulated, and } H(\min I+) \text{ exists}\}.$$

LCP integrable distributions have unique primitives in the space

$$\mathcal{B}_0(I) = \begin{cases} \{F : I \to \mathbb{R} | F \text{ is bounded and left continuous, and} \\ F(\min I+) = F(\min I) = 0\}. \end{cases}$$

Because a left regulated function is by Lemma 2.2 locally Riemann integrable if and only if it is locally bounded, then $\mathcal{B}^{lr}(I) = \mathcal{R}^{lr}(I)$ and $\mathcal{B}_0(I) = \mathcal{R}_0^{lc}(I)$.

In [8] RP-integrable distributions are defined so that their primitives form the space of all regulated functions on I. These distributions have unique primitives in the space

$$\mathcal{P}_R(I) = \begin{cases} \{F : I \to \mathbb{R} | F \text{ is regulated and left continuous, and} \\ F(\min I) = F(\min I+) = 0\}. \end{cases}$$

$\mathcal{P}_R(I)$ is a proper subset of $B_0(I)$. For instance, the function G_0 defined in Example 6.1, belongs to $\mathcal{B}^{lr}(\mathbb{R}_+)$, but not to $\mathcal{P}_R(\mathbb{R}_+)$. The restriction of the function G, defined by (6.7), to any interval I for which $\min I$ exists and is irrational belongs to $\mathcal{P}_R(I)$.

In [23] a theory is presented for the regulated primitive integral of distributions whose primitives belong to the space

$$\mathcal{B}_R = \begin{cases} \{F : \mathbb{R} \cup \{\pm\infty\} \to \mathbb{R} | F \text{ is regulated and left continuous on } \mathbb{R}, \\ F(-\infty) = 0, F(\infty) \in \mathbb{R}\}. \end{cases}$$

The function F_0, defined by $F_0(t) = e^{-|t|}F(t)$, $t \in \mathbb{R}$, $F_0(\pm\infty) = 0$, belongs to $\mathcal{B}_R$.

Remark 6.5. The proofs of Theorems 3.2, 3.6, 3.7, 4.1, 4.2 and 5.1 are similar to proofs of corresponding results of [23]. The case of regulated primitives considered in this paper yields four different integrals over each of the intervals (a, b), $[a, b]$, $[a, b)$ and $(a, b]$. For example, $\int_{(a,b)} F' = F(b-) - F(a+)$. Since we have left continuity of primitives with the LD, LL and LR primitive integrals, the integral (1.1) can be considered as $\int_a^b F' = \int_{[a,b)} F' = F(b-) - F(a-) = F(b) - F(a)$.

The above examples show that inclusions in $\mathcal{P}_R(I) \subset \mathcal{R}_0^{lc}(I) \subset \mathcal{L}_0^{lc}(I) \subset \mathcal{D}_0^{lc}(I)$ are proper. This result and the bijective correspondence $f \overset{\mathcal{F}}{\leftrightarrow} F$ between integrable distributions and their primitives implies that the RP integrable distributions form a proper subset of LCP integrable distributions, which is equal to the space of LRP integrable distributions. They in turn form a proper subset of LLP integrable distributions, which form a proper subset of LDP integrable distributions.

In defining $\mathcal{D}^{lr}(I)$, $\mathcal{L}^{lr}(I)$ and $\mathcal{R}^{lr}(I)$ we have chosen the primitives to be left regulated. Another obvious choice is to take primitives that are right regulated. One can also use a convex combination of left and right limits. Properties derived in Section 2 for left regulated functions have also analogous counterparts for right regulated functions, also for functions with values in Banach spaces (cf. [10]). Right and left regulated functions are studied also in [11].

The order $\preceq$ defined by (3.7) is not compatible with the usual order on distributions: if $T, U \in \mathcal{D}'$ then $T \geq U$ if and only if $\langle T - U, \phi \rangle \geq 0$ for all $\phi \in \mathcal{D}$ such that $\phi \geq 0$. Nor is it compatible with pointwise ordering in the case of functions in $\mathcal{A}_D(I)$. For example, if $f(t) = H_1(t)\sin(t^2)$, where H_1 is the Heaviside step function, i.e., $H_1(t) = \begin{cases} 1, t > 0, \\ 0, t \leq 0, \end{cases}$ then $0 \leq F$ so $0 \preceq f$ in $\mathcal{A}_D[-1,1])$ but not pointwise. And, f is not positive in the distributional sense. Note, however, that if $f \in \mathcal{A}_D(I)$ is a nonnegative function or distribution then $0 \preceq f$ in $\mathcal{A}_D(I)$.

In Lebesgue and Henstock-Kurzweil integration, we have equivalence classes of functions that agree almost everywhere. In $\mathcal{A}_D(I)$, $\mathcal{A}_L(I)$ and $\mathcal{A}_R(I)$ there are no such equivalence classes, for two distributions are equal if they agree on all test functions.

Except the last remark we assume from now on that I is a compact real interval. If function f is continuous, or in $L^p(I)$ for some $1 \leq p \leq \infty$, or in the Denjoy space $D(I)$ of HK integrable functions from I to $\mathbb{R}$, then $f \in \mathcal{A}_R(I)$ and hence in $\mathcal{A}_L(I)$ and $\mathcal{A}_D(I)$. If primitive function F is continuous then $F' \in \mathcal{A}_R(I)$. Taking F to be continuous but with a pointwise derivative nowhere, we see that $\mathcal{A}_R(I)$ contains distributions that have no pointwise values. Note that the formula $\int_a^b F' = F(b) - F(a)$ holds although this now has no meaning as a Riemann, Lebesgue or Henstock-Kurzweil integral. If F is continuous but $F'(x) = 0$ almost everywhere then the Lebesgue integral $\int_a^b F'(x)\,dx = 0$ but in $\mathcal{A}_R(I)$ we have $\int_a^b F' = F(b) - F(a)$.

The space $\mathcal{A}_R(I)$ contains all signed Borel measures on $(\min I, \max I)$. Suppose μ is a signed Borel measure such that $\mu(\{\min I\}) = \mu(\{\max I\}) = 0$. Define $F(\min I) = 0$ and $F(x) = \int_{(0,x)} d\mu$ for $x \in (\min I, \max I]$. This Lebesgue integral defines primitive F that is of bounded variation and in $\mathcal{R}_0^{lc}(I)$. Hence, $\mu \in \mathcal{A}_R(I)$. For example, the Dirac measure, δ, is in $\mathcal{A}_R([a,b])$ for any $a < 0 < b$.

Lemma 2.1 shows left regulated functions can be approximated uniformly by countably stepped functions. However, countably stepped functions are not dense in $\mathcal{R}^{lc}(I)$. For example, let $F = \sum_{n=1}^{\infty}(-1)^n \chi_{((n+1)^{-1}, n^{-1}]}$. Then $F \in \mathcal{R}^{lc}([-1,1])$ and $\|F\|_\infty = 1$. Suppose σ is a step function. Then $\lim_{x\to 0^+} \sigma(x)$ exists. We have

$$\|F - \sigma\|_\infty \geq \limsup_{x\to 0^+} |F(x) - \sigma(x)| = \limsup_{x\to 0^+} |F(x) - \sigma(0+)| \geq 1. \tag{6.13}$$

To show that $\mathcal{D}_0^{lc}(I)$ is not closed, define $F_n = \sum_{m=1}^{n}(-1)^m \chi_{(1/(m+1), 1/m]}$. Then $F_n \in \mathcal{D}_0^{lc}([0,1])$. Let $F = \sum_{m=1}^{\infty}(-1)^m \chi_{(1/(m+1), 1/m]}$. Then $F \in \mathcal{D}([0,1])$ since $F \in L^1([0,1])$. Note that $\|F - F_n\|_A = 1/(n+1) - 1/(n+2) \to 0$ as $n \to \infty$. So $F_n \to F$ in the Alexiewicz norm. But $\lim_{x\to 0^+} F(x)$ does not exist so F is not regulated on $[0,1]$. Hence, although $\mathcal{D}_0^{lc}(I)$ is a normed space it is not a Banach space. The same functions show $\mathcal{L}_0^{lc}(I)$ is not closed. Now,

$$\|F_n - F\|_1 = \sum_{m=n+1}^{\infty} \left(\frac{1}{m} - \frac{1}{m+1}\right) = \frac{1}{n+1} \to 0 \quad \text{as } n \to \infty.$$

Hence, $\mathcal{L}_0^{lc}(I)$ is also not a Banach space. It now follows that neither $\mathcal{A}_D(I)$ nor $\mathcal{A}_L(I)$ are Banach spaces. Since the continuous functions are dense in $\mathcal{D}^{lc}(I)$ the completion of $\mathcal{D}^{lc}(I)$ in the Alexiewicz norm is $D(I)$ and the completion of $\mathcal{A}_D(I)$ is the space of distributional derivatives of HK integrable functions. This space was studied in [22]. Similarly, the completion of $\mathcal{A}_L(I)$ in the 1-norm is the space of distributional derivatives of Lebesgue integrable functions. This space was studied in [25].

7. APPLICATIONS TO DISTRIBUTIONAL SYSTEMS

In this section we will study the following system of distributional Cauchy problems:

$$y_i' = f_i(y_1, \dots, y_m), \quad y_i(a) = c_i, \quad i = 1, \dots, m. \tag{7.1}$$

Dependence of solutions on f_i and c_i is also studied. Values of the functions f_i are distributions on a half-open real interval $I = [a, b)$, $a < b \leq \infty$.

The regulated primitive integral is studied in detail in [23] when $I = \overline{\mathbb{R}}$, and applied in [8] to problem (7.1) when $I = [a, b]$. The left continuous primitive integral is applied in [9] to problem (7.1) when $I = [a, b]$. Because $\mathcal{B}_0(I) = \mathcal{R}_0^{lc}(I)$ when I is compact, the left continuous primitive integral and the LR primitive integral are equal. Therefore we study only applications of the LD primitive integral and the LL primitive integral to problem (7.1). No continuity hypotheses are imposed on functions f_i.

7.1. **On the smallest and greatest solutions.** We will first study the existence of the smallest and greatest solutions of problem (7.1) and their dependence on f_i and c_i. Component functions $y_i : [a, b) \to \mathbb{R}$ of solutions of (7.1) are assumed to be in the space $\mathcal{D}^{lc}([a, b))$ of those functions from $[a, b)$ to $\mathbb{R}$ which are locally HK integrable, right continuous at a and left continuous on (a, b).

Assume that $\mathcal{D}^{lc}([a, b))$ is ordered pointwise, that the space $\mathcal{A}_D([a, b))$ is equipped with the ordering $\preceq$ defined by (3.7), that the space $HK_{loc}([a, b))$ of locally HK integrable functions on $[a, b)$ is equipped with a.e. pointwise ordering, and that a.e. equal functions of $HK_{loc}([a, b))$ are identified. The product spaces $\mathcal{D}^{lc}([a, b))^m = \times_{i=1}^m \mathcal{D}^{lc}([a, b))$ and $HK_{loc}([a, b))^m = \times_{i=1}^m HK_{loc}([a, b))$ are ordered by componentwise ordering, i.e., if $x = (x_i, \dots, x_m)$ and $y = (y_1, \dots, y_m)$ belong to one of these product spaces, then

$$x \leq y \text{ iff } x_i \leq y_i \text{ for all } i = 1, \dots, m.$$

Definition 7.1. A function $(y_1, \dots, y_m) \in \mathcal{D}^{lc}([a, b))^m$ is called a subsolution of (7.1) if

$$y_i' \preceq f_i(y_1, \dots, y_m) \text{ in } \mathcal{A}_D([a, b)), \text{ and } y_i(a) \leq c_i \text{ for every } i = 1, \dots, m. \tag{7.2}$$

If reversed inequalities hold in (7.2), we say that $(y_1, \dots, y_m)$ is a supersolution of (7.1). If equalities hold in (7.2), then $(y_1, \dots, y_m)$ is called a solution of (7.1).

The following result that transforms the system (7.1) into a system of integral equations is a direct consequence of Theorem 3.3.

Lemma 7.2. *Assume that* $(y_1,\dots,y_m) \in \mathcal{D}^{lc}([a,b))^m$*, and that* $f_i(y_1,\dots,y_m) \in \mathcal{A}_D([a,b))$ *for every* $i = 1,\dots,m$*. Then* $(y_1,\dots,y_m)$ *is a solution of the system* (7.1) *if and only if it is a solution of the following system of integral equations:*

$$y_i(t) = c_i + \int_a^t f_i(y_1,\dots,y_m), \quad t \in [a,b),\ i = 1,\dots,m. \tag{7.3}$$

The application of monotone methods to find solutions of (7.1) is complicated by the fact that the limit function, supremum and/or infimum of a pointwise convergent monotone sequence of $\mathcal{D}^{lc}(I)$ are not necessarily in $\mathcal{D}^{lc}(I)$ even in the case when the interval I is compact. For instance, the sequence of functions $x_n \in \mathcal{D}^{lc}([0,1])$, $n = 0,1,\dots$, defined by

$$x_n(t) = \begin{cases} 0, & 0 \le t \le \frac{1}{2^{n+1}}, \\ 1-(-1)^k, & \frac{1}{2^{k+1}} < t \le \frac{1}{2^k},\ k = 0,\dots,n, \end{cases}$$

is increasing in the pointwise ordering of $\mathcal{D}^{lc}[0,1])$, but neither its pointwise limit nor its supremum is in $\mathcal{D}^{lc}[0,1])$.

Therefore we study in this section the existence of such solutions of the system (7.1) whose components are locally HK integrable on $[a,b)$.

Definition 7.3. Given partially ordered sets $X = (X,\le)$ and $Y = (Y,\preceq)$, we say that a mapping $f : X \to Y$ is *increasing* if $f(x) \preceq f(y)$ whenever $x \le y$ in X, and *order-bounded* if there exist $\underline{y}, \overline{y} \in Y$ such that the range $f[X]$ of f is contained in the order interval $[\underline{y}, \overline{y}] = \{y \in Y : \underline{y} \preceq y \preceq \overline{y}\}$ of Y.

The first existence and comparison theorem for the smallest and greatest solutions of the system (7.1) reads as follows.

Theorem 7.4. *Assume that* $f_i : HK_{loc}([a,b))^m \to \mathcal{A}_D([a,b))$ *is increasing, that the system* (7.1) *has in* $\mathcal{D}^{lc}([a,b))^m$ *a subsolution* $\underline{y} = (\underline{y}_1,\dots,\underline{y}_m)$ *and a supersolution* $\overline{y} = (\overline{y}_1,\dots,\overline{y}_m)$*, and that* $\underline{y}_i \le \overline{y}_i$ *for each* $i = 1,\dots,m$*. Then the system* (7.1) *has in the order interval* $[\underline{y}, \overline{y}]$ *of* $\mathcal{D}^{lc}([a,b))^m$ *the smallest and greatest solutions, and they are increasing with respect to* f_i *and* c_i*,* $i = 1,\dots,m$*.*

Proof. Let $F : HK_{loc}([a,b))^m \to \mathcal{D}^{lc}([a,b))^m$ be defined by

$$\begin{cases} F(x) = (F_1(x_1,\dots,x_m),\dots,F_m(x_1,\dots,x_m)), \quad \text{where} \\ F_i(x_1,\dots,x_m)(t) = c_i + \int_a^t f_i(x_1,\dots,x_m), \quad t \in [a,b),\ i = 1,\dots,m. \end{cases} \tag{7.4}$$

In view of Lemma 7.2, $y = (y_1,\dots,y_m)$ is a solution of the system (7.1) in $\mathcal{D}^{lc}([a,b))^m$ if and only if y is a solution of the fixed point equation $y = F(y)$ in $\mathcal{D}^{lc}([a,b))^m$.

Assume that $x = (x_1, \dots, x_m)$ belongs to the order interval $[\underline{y}, \overline{y}]$ of $HK_{loc}([a, b))^m$. The given hypotheses imply by Definitions 7.1 and 7.3 that for every $i = 1, \dots, m$,

$$\underline{y}_i' \preceq f_i(\underline{y}) \preceq f_i(x_1, \dots, x_m) \preceq f_i(\overline{y}) \preceq \overline{y}_i'$$

for all $i = 1, \dots, m$. Moreover, $\underline{y}_i(a) \le c_i \le \overline{y}_i(a)$ for every $i = 1, \dots, m$. Thus

$$\begin{cases} \underline{y}_i(t) \le \underline{y}_i(a) + \int_a^t f_i(\underline{y}) \le c_i + \int_a^t f_i(x) \le \overline{y}_i(a) + \int_a^t f_i(\overline{y}) \le \overline{y}_i(t), \\ t \in [a, b), \ i = 1, \dots, m. \end{cases}$$

Because $t \mapsto \int_a^t f_i(x)$ belongs to $\mathcal{D}^{lc}([a, b))$, it is HK locally integrable. Thus $F_i(x)$ is Lebesgue measurable and order bounded by functions $\underline{y}_i$ and $\overline{y}_i$ of $HK_{loc}([a, b))$. It then follows from [2, Proposition 9.39 and Remark 9.25] that $F_i(x) \in HK_{loc}([a, b))$. This holds for every $i = 1, \dots, m$, whence $F(x) \in HK_{loc}([a, b))^m$.

The above results imply that F maps order interval $[\underline{y}, \overline{y}]$ of $HK_{loc}([a, b))^m$ into itself. Moreover, F is increasing because the functions f_i are increasing. By [11, Theorem 1.1.1] there is a unique chain C in $HK_{loc}([a, b))^m$ that is well-ordered (every non-empty subset of C has the smallest element), and that satisfies

(I): $\underline{y} = \min C$, and if $\underline{y} < x$, then $x \in C$ iff $x = \sup F[\{y \in C : y < x\}]$.

Since C is well-ordered and F is increasing, then $W = F[C]$ is well-ordered. For every $i = 1, \dots, m$, the set $W_i = \{z_i : (z_1, \dots, z_m) \in W\}$ is a well-ordered chain in the order interval $[\underline{y}_i, \overline{y}_i]$ of $HK_{loc}([a, b))$. Thus $y_{*i} = \sup W_i$ exists in $HK_{loc}([a, b))$ by [2, Proposition 9.39]. Obviously, $(y_{*1}, \dots, y_{*m})$ is the supremum of $W = F[C]$ in $HK_{loc}([a, b))^m$. It then follows from [11, Theorem 1.2.1] that $y_* = (y_{*1}, \dots, y_{*m}) = \max C$, and that y_* is the smallest fixed point of F in the order interval $[\underline{y}, \overline{y}]$ of $HK_{loc}([a, b))^m$. Moreover, every fixed point of F belongs to $\mathcal{D}^{lc}([a, b))^m$. Thus $(y_{*1}, \dots, y_{*m})$ is the smallest solution of the system (7.1) in order interval $[\underline{y}, \overline{y}]$ of $\mathcal{D}^{lc}([a, b))^m$.

According to [11, Proposition 1.2.1] there exists a unique chain D that is inversely well-ordered, and that satisfies

(II): $\overline{y} = \max D$, and if $x < \overline{y}$, then $x \in D$ iff $x = \inf F[\{y \in D : x < y\}]$.

The proof that $y^* = \inf F[D] = \min D$ exists and is the greatest fixed point of F in the order interval $[\underline{y}, \overline{y}]$ of $\mathcal{D}^{lc}([a, b))^m$ is similar to the above proof. Thus $y^* = (y_1^*, \dots, y_m^*)$ is the greatest solution of the system (7.1) in the order interval $[\underline{y}, \overline{y}]$ of $\mathcal{D}^{lc}([a, b))^m$. Moreover, according to [11, Theorem 1.2.1 and Proposition 1.2.1],

$$y_* = \min\{x \in [\underline{y}, \overline{y}] : F(x) \le x\}, \ y^* = \max\{x \in [\underline{y}, \overline{y}] : x \le F(x)\}. \tag{7.5}$$

Applying these relations one can show that the fixed points y_* and y^* of F are increasing with respect to F. Consequently, by (3.7) and (7.4), their components, and hence the smallest and greatest solutions of the system (7.1) in $[\underline{y}, \overline{y}]$, are increasing with respect to f_i and c_i, $i = 1, \dots, m$. □

As a special case of Theorem 7.4 we obtain the following corollary.

Corollary 7.5. *Assume that $f_i : HK_{loc}([a,b))^m \to \mathcal{A}_D([a,b))$ is increasing and order-bounded for every $i = 1,\dots,m$. Then the system (7.1) has in $\mathcal{D}^{lc}([a,b))^m$ the smallest and greatest solutions, and they are increasing with respect to f_i and c_i, $i = 1,\dots,m$.*

Proof. Because functions f_i are order-bounded, there exist distributions $\underline{h}_i$ and $\overline{h}_i$ in $\mathcal{A}_D([a,b))$ such that $\underline{h}_i \preceq f_i(x_1,\dots,x_m) \preceq \overline{h}_i$ for all $(x_1,\dots,x_m) \in HK_{loc}([a,b))^m$ and $i = 1,\dots,m$. Defining for every $i = 1,\dots,m$,

$$\underline{y}_i(t) := c_i + \int_a^t \underline{h}_i, \quad \overline{y}_i(t) := c_i + \int_a^t \overline{h}_i,\ t \in [a,b),$$

it is easy to see that $\underline{y} = (\underline{y}_1,\dots,\underline{y}_m)$ is a lower solution and $\overline{y} = (\overline{y}_1,\dots,\overline{y}_m)$ is an upper solution of (7.1) in $\mathcal{D}^{lc}([a,b))^m$. Thus (7.1) has by Theorem 7.4 the smallest and greatest solutions in the order interval $[\underline{y},\overline{y}]$ of $\mathcal{D}^{lc}([a,b))^m$. If $y = (y_1,\dots,y_m)$ is a solution of (7.1) in $\mathcal{D}^{lc}([a,b))^m$, then

$$\underline{y}_i(t) := c_i + \int_a^t \underline{h}_i \le c_i + \int_a^t f_i(y) \le \overline{c}_i + \int_a^t \overline{h}_i = \overline{y}_i(t),\ t \in [a,b),\ i = 1,\dots,m.$$

Thus y belongs to the order interval $[\underline{y},\overline{y}]$ of $\mathcal{D}^{lc}([a,b))^m$, whence the smallest and greatest solutions of (7.1) in that order interval are the smallest and greatest solutions of (7.1) in the whole $\mathcal{D}^{lc}([a,b))^m$. The last conclusion of Theorem 7.4 implies that these solutions are increasing with respect to f_i and c_i, $i = 1,\dots,m$. □

The following result is a consequence of Corollary 7.5.

Proposition 7.1. *Let $f_i(x_i,\dots,x_m)$ be for all fixed $i = 1,\dots,m$ and $(x_1,\dots,x_m) \in HK_{loc}([a,b))^m$ the distributional derivative of a function*

$$F_i(x_1,\dots,x_m)(t) = \sum_{j=1}^n H_{ij}(t) \int_a^t g_{ij}(x_1,\dots,x_m) + G_i(t), \quad t \in [a,b), \tag{7.6}$$

where G_i belong to $\mathcal{D}^{lc}([a,b))$, $G_i(a) = 0$, functions H_{ij} are bounded and non-negative-valued on $[a,b)$ and left-continuous on (a,b), and the functions $g_{ij}(x) : [a,b) \to \mathbb{R}$ satisfy the following hypotheses:

- **($\mathbf{g}_{ij1}$):** *$g_{ij}(x_1,\dots,x_m)$ is locally HK integrable for $(x_1,\dots,x_m) \in HK_{loc}([a,b))^m$.*
- **($\mathbf{g}_{ij2}$):** *There exist locally HK integrable functions $\underline{g}_{ij}, \overline{g}_{ij} : [a,b) \to \mathbb{R}$ such that $\int_a^t \underline{g}_i \le \int_a^t g_{ij}(x_1,\dots,x_m) \le \int_a^t g_{ij}(y_1,\dots,y_m) \le \int_a^t \overline{g}_{ij}$, whenever $t \in [a,b)$ and $(x_1,\dots,x_m) \le (y_1,\dots,y_m)$ in $HK_{loc}([a,b))^m$.*

Then the system (7.1) has in $\mathcal{D}^{lc}([a,b))^m$ the smallest and greatest solutions, and they are increasing with respect to g_{ij} and c_i.

Proof. The hypotheses ensure that (7.6) defines for every $i = 1, \ldots, m$ and that $(x_1, \ldots, x_m) \in HK_{loc}([a,b))^m$ a function $F_i(x_1, \ldots, x_m) \in \mathcal{D}^{lc}([a,b))$, and that its distributional derivative $f_i(x_1, \ldots, x_m)$ is increasing in $(x_1, \ldots, x_m)$ and is order-bounded by distributions $\underline{h}_i$ and $\overline{h}_i$ whose primitives are

$$\underline{y}_i(t) = c_i + \sum_{j=1}^{n} H_{ij}(t) \int_a^t \underline{g}_{ij} + G_i(t),\ \overline{y}_i(t) = c_i + \sum_{j=1}^{n} H_{ij}(t) \int_a^t \overline{g}_{ij} + G_i(t), \quad t \in [a,b).$$

Thus the conclusions follow from Corollary 7.5. □

Remark 7.6. The smallest elements of the well-ordered chain C determined by (I) are $F^n(\underline{y})$, $n \in \mathbb{N}_0$, as long as $F^n(\underline{y}) = F(F^{n-1}(\underline{y}))$ is defined and $F^{n-1}(\underline{y}) < F^n(\underline{y})$, $n \in \mathbb{N}$. If $F^{n-1}(\underline{y}) = F^n(\underline{y})$ for some $n \in \mathbb{N}$, there is a smallest such n, and $y_* = F^{n-1}(\underline{y})$ is under the hypotheses of Theorem 7.4 the smallest fixed point of F in $[\underline{y}, \overline{y}]$. If $y_\omega = \sup_{n\in\mathbb{N}} F^n(\underline{y})$ is defined in $HK_{loc}([a,b))^m$ and is a strict upper bound of $\{F^n(\underline{y})\}_{n\in\mathbb{N}}$, then y_ω is the next element of C. If $y_\omega = F(y_\omega)$, the $y_* = y_\omega$, otherwise the next elements of C are of the form $F^n(y_\omega)$, $n \in \mathbb{N}$, and so on.

The greatest elements of the inversely well-ordered chain D determined by (II) are n-fold iterates $F^n(\overline{y})$, as long as they are defined and $F^n(\overline{y}) < F^{n-1}(\overline{y})$. If equality holds for some $n \in \mathbb{N}$, then $y^* = F^{n-1}(\overline{y})$ is the greatest fixed point of F in $[\underline{y}, \overline{y}]$.

Example 7.7. Determine the smallest and greatest solution of the system (7.1), where $m = 2$, $c_i = 0$, $f_i(x_1, x_2)$ are for each $(x_1, x_2) \in HK_{loc}([0,\infty))^2$ the distributional derivatives of the functions $F_i(x_1, x_2) : HK_{loc}([0,\infty))^2 \to \mathcal{D}^{lc}([0,\infty))$, defined by

$$F_i(x_1, x_2)(t) = H_1(t) \int_0^t g_{i1}(x_1, x_2) + G_i(t), \quad t \in [0,1],\ i = 1, 2, \tag{7.7}$$

where H_1 is the Heaviside step function, $G_i \in \mathcal{D}^{lc}([0,\infty))$, $G_i(0) = 0$, $g_{i1}(x_1, x_2)(0) = 0$, $i = 1, 2$, and

$$\begin{cases} g_{11}(x_1,x_2)(t) = \arctan\left([10^5 \int_0^1 (x_2(t) - G_2(t))\,dt]10^{-4}\right)\left(\frac{1}{t}\cos(\frac{1}{t}) - \sin(\frac{1}{t}) + 1\right), \\ g_{21}(x_1,x_2)(t) = \tanh\left([3\cdot 10^4 \int_0^1 (x_1(t) - G_1(t))\,dt]10^{-4}\right)\left(\frac{1}{t}\sin(\frac{1}{t}) + \cos(\frac{1}{t}) + 1\right), \end{cases}$$

$[z]$ denoting the greatest integer $\le z \in \mathbb{R}$.

Solution: The validity of the hypotheses (g_{i11}) and (g_{i12}) is easy to verify. Thus, the system (7.1) has by Proposition 7.1 the smallest and greatest solutions in $\mathcal{D}^{lc}([0,\infty))^2$. To determine these solutions, denote

$$\begin{cases} \underline{y}_1(t) := G_1(t) - 4t(1 + \cos(\frac{1}{t})),\ t \in (0,1], \quad \underline{x}_1(0) = 0, \\ \overline{y}_1(t) := G_1(t) + 4t(1 + \cos(\frac{1}{t})),\ t \in (0,1], \quad \overline{x}_1(0) = 0, \\ \underline{y}_2(t) := G_2(t) - 4t(1 - \sin(\frac{1}{t})),\ t \in (0,1], \quad \underline{x}_2(0) = 0, \\ \overline{y}_2(t) := G_2(t) + 4t(1 - \sin(\frac{1}{t})),\ t \in (0,1], \quad \overline{x}_0(0) = 0. \end{cases}$$

Calculating the successive approximations

$$\begin{cases} (x_{n+1},y_{n+1}) = (F_1((x_n,y_n),F_2(x_n,y_n)), \quad (x_0,y_0) = (\underline{y}_1,\underline{y}_2) \text{ and} \\ (z_{n+1},w_{n+1}) = (F_1(z_n,w_n),F_2(z_n,w_n), \quad (z_0,w_0) = (\overline{y}_1,\overline{y}_2), \end{cases}$$

we see that (x_n,y_n) form an increasing and (z_n,w_n) a decreasing sequence. Moreover, $(x_{16},y_{16}) = (F_1(x_{16},y_{16}),F_2(x_{16},y_{16}))$, and $(z_{16},w_{16}) = (F_1(z_{16},w_{16}),F_2(z_{16},w_{16}))$. According to (I), (II) and Remark 7.6 we then have

$$\begin{cases} C = \{(x_n,y_n)\}_{n=0}^{16}, \ \sup F[C] = (x_{16},y_{16}), \ D = \{(z_n,w_n)\}_{n=0}^{16}, \\ \text{and} \quad \inf F[D] = (z_{16},w_{16}). \end{cases}$$

Thus $y_* = (x_{16},y_{16})$ and $y^* = (z_{16},w_{16})$ are the smallest and greatest solutions of (7.1) when $f_i(x_1,x_2)$ are for each $(x_1,x_2) \in HK_{loc}([0,\infty))^2$ the distributional derivatives of the functions $F_i(x_1,x_2) : HK_{loc}([0,\infty))^2 \to \mathcal{D}^{lc}([0,\infty))$, defined by (7.7). The exact formulas of y_* and y^*, calculated by using simple Maple programs, are

$$\begin{cases} y_*(t) = \left(G_1(t) - \arctan\left(\frac{5139}{5000}\right)\left(t + t\cos(\frac{1}{t})\right), G_2(t) - \tanh\left(\frac{12421}{10000}\right)\left(t - t\sin(\frac{1}{t})\right)\right), \\ y^*(t) = \left(G_1(t) + \arctan\left(\frac{2569}{2500}\right)\left(t + t\cos(\frac{1}{t})\right), G_2(t) + \tanh\left(\frac{12419}{10000}\right)\left(t - t\sin(\frac{1}{t})\right)\right). \end{cases}$$

7.2. **Uniqueness results.** Denoting $\mathcal{H}_+([a,b)) = \{\lceil x\rceil \mid x \in \mathcal{D}^{lc}([a,b))^m\}$, where $a < b \le \infty$, $\theta(t) \equiv 0$, and

$$\lceil x\rceil = t \mapsto \max\{|x_i(t)| : i = 1,\dots,m\}, \ x = (x_1,\dots,x_m) \in \mathcal{D}^{lc}([a,b))^m, \tag{7.8}$$

we shall prove the following uniqueness Lemma.

Lemma 7.8. *Given $F : \mathcal{D}^{lc}([a,b))^m \to \mathcal{D}^{lc}([a,b))^m$, assume that*

$$\lceil F(y) - F(z)\rceil \le G(\lceil y - z\rceil) \quad \textit{whenever} \quad y,\, z \in \mathcal{D}^{lc}([a,b))^m, \tag{7.9}$$

where $G : \mathcal{H}_+([a,b)) \to \mathcal{H}_+([a,b))$ has the following properties.

- **(G):** *G is increasing, i.e., $G(u) \le G(v)$ whenever $u \le v$, and for each $u \in \mathcal{H}_+([a,b))$ there exists a $w_0 \in \mathcal{H}_+([a,b))$, $u \le w_0$, such that $\inf G[W] = \theta$, where W is the chain in $\mathcal{H}_+([a,b))$ that is inversely well-ordered, i.e., every nonempty subset of W has the greatest element, and that satisfies the following condition.*
- **(W):** *$\max W = w_0$, and if $u < w_0$, then $u \in W$ if and only if $u = \inf G[\{w \in W : u < w\}]$.*

Then F has at most one fixed point y, i.e., $y \in \mathcal{D}^{lc}([a,b))^m$ and $y = F(y)$.

Proof. Assume that $y,\, z \in \mathcal{D}^{lc}([a,b))^m$, $y = F(y)$, and $z = F(z)$. Choose $w_0 \in \mathcal{D}^{lc}([a,b))$ such that $\lceil y - z\rceil \le w_0$, and that $\inf G[W] = \theta$, where W is the chain in $\mathcal{H}_+([a,b))$ that is inversely well-ordered and satisfies condition (W). By [11, Proposition 1.2.1] W exists and is uniquely determined. If the inequality $\lceil y - z\rceil \le w$ does not hold for all $w \in W$, there is the greatest element in W, say u, for which

$\lceil y-z\rceil \not\leq u$. If $w\in W$ and $u<w$, then $\lceil y-z\rceil \leq w$. This inequality, property (7.9), monotonicity of G, and equations $y=F(y)$ and $z=F(z)$ imply that

$$\lceil y-z\rceil = \lceil F(y)-F(z)\rceil \leq G(\lceil y-z\rceil) \leq G(w). \tag{7.10}$$

This result holds for all $w\in W$, $u<w$, whence $\lceil y-z\rceil$ is a lower bound of the set $G[\{w\in W : u<w\}]$. But u is by (W) the greatest lower bound of $G[\{w\in W : u<w\}]$, so that $\lceil y-z\rceil \leq u$; a contradiction.

By the above proof $\lceil y-z\rceil \leq w$, and hence (7.10) holds for every $w\in W$, whence $\lceil y-z\rceil$ is a lower bound of $G[W]$ in $\mathcal{H}_+([a,b))$. Thus $\theta \leq \lceil y-z\rceil \leq \inf G[W]=\theta$, i.e., $y=z$. □

Remark 7.9. Note that $\mathcal{H}_+([a,b))$ is not a subset of $\mathcal{D}^{lc}([a,b))^m$ because the functions of $\mathcal{D}^{lc}([a,b))$ are not absolutely integrable.

The first elements of the chain W satisfying (W) are iterates $w_n=G^n(\max W)$, $n\in\mathbb{N}_0$. If $w_\omega=\inf\{w_n\}_{n=1}^\infty$ exists and is a strict lower bound of $\{w_n\}_{n=1}^\infty$, then w_ω is the next element of W. The next possible elements of W are of the form $G^n(w_\omega)$, $n\in\mathbb{N}$, and so on.

Example 7.10. Given $H\in\mathcal{D}^{lc}([0,1]$, define $F:\mathcal{D}^{lc}([0,\infty))\to\mathcal{D}^{lc}([0,\infty))$ by

$$F(x)(t)=\begin{cases} H(t)+t(x(t)-H(1)), & 0\leq t\leq 1,\\ x(i)+i+(t-i)(x(t)-x(i)-i), & i<t\leq i+1,\ i\in\mathbb{N}. \end{cases} \tag{7.11}$$

(a) Show that F has at most one fixed point $y\in\mathcal{D}^{lc}([0,\infty))$.
(b) Determine the fixed point of F when $H'_-(1)=\lim\limits_{t\to 1-}\frac{H(1)-H(t)}{1-t}$ exists.

Solution: We will show that the hypotheses of Lemma 7.8 hold when $G:\mathcal{H}_+(([0,\infty))\to\mathcal{H}_+(([0,\infty))$ is defined by

$$G(u)(t)=\begin{cases} tu(t), & 0\leq t\leq 1,\\ (i+1-t)u(i)+(t-i)u(t), & i<t\leq i+1,\ i\in\mathbb{N}. \end{cases} \tag{7.12}$$

It is easy to verify that G is increasing, and that (7.9) holds. Given $u\in\mathcal{H}_+([0,\infty))$, define

$$w_0(t)=\begin{cases} u(t), & 0\leq t\leq 1,\\ \max\{u(t),u(i)\}, & i<t\leq i+1,\ i\in\mathbb{N}. \end{cases}$$

Routine calculations and induction imply that

$$G^n(w_0)(t)=\begin{cases} t^n u(t), 0\leq t\leq 1,\\ u(i)+(t-i)^n(w_0(t)-u(i)), i<t\leq i+1,\ i\in\mathbb{N}, \end{cases} \qquad n\in\mathbb{N}.$$

Thus

$$\lim_{n\to\infty} G^n(w_0)(t)=\begin{cases} 0, & 0\leq t<1,\\ u(i), & i<t\leq i+1,\ i\in\mathbb{N}. \end{cases}$$

Redefining the limit so that the obtained function is left-continuous at $t = 1$ we get the infimum of the set $\{G^n(w_0)\}$ in $\mathcal{H}_+([0,\infty))$:

$$z_1(t) = \inf_{n\in\mathbb{N}}\{G^n(w_0)\}(t) = \begin{cases} 0, & 0 \le t \le 1, \\ u(i), & i < t \le i+1,\ i \in \mathbb{N}. \end{cases}$$

Similar calculations and induction show that for every $i = 2, 3, \dots,$

$$G^n(z_{i-1})(t) = \begin{cases} 0, & 0 \le t \le i-1, \\ (t-i-1)^n u(i-1), & i-1 < t \le i, \\ u(j), & j < t \le j+1,\ j = i, i+1, \dots, \end{cases} \qquad n \in \mathbb{N},$$

and

$$z_i(t) = \inf_{n\in\mathbb{N}}\{G^n(z_{i-1})\}(t) = \begin{cases} 0, & 0 \le t \le i, \\ u(j), & j < t \le j+1,\ j = i, i+1, \dots. \end{cases}$$

Finally, we obtain, as $i \to \infty$,

$$z_\infty(t) = 0, \quad 0 \le t < \infty.$$

Consequently, if $u(i) > 0$, $i = 1, \dots, b$, the members of the inversely well-ordered chain W satisfying (W) are

$$G^n(w_0), n = 0, 1, \dots,\ G^n(z_i), n = 1, 2, \dots,\ i = 1, 2, \dots, z_\infty.$$

In particular, $\inf W = z_\infty = \theta$.

The above calculations and monotonicity of G imply that the hypotheses of Lemma 7.8 hold. Thus F has at most one fixed point in $\mathcal{D}^{lc}([0,\infty))$.

If y is a fixed point of F in $\mathcal{D}^{lc}([0,\infty))$, it follows from (7.11) that

$$y(t) = \begin{cases} H(t) + t(y(t) - f(1)), & 0 \le t \le 1, \\ y(i) + i + (t-i)(y(t) - y(i) - i), & i < t \le i+1,\ i \in \mathbb{N}. \end{cases}$$

Thus

$$y(t) = H(t) - t\frac{H(1) - H(t)}{1-t}, \quad 0 \le t < 1.$$

Assuming that $H'_-(1) = \lim_{t\to 1-} \frac{H(1)-H(t)}{1-t}$ exists, then x is left-continuous at $t = 1$, and $y(1) = H(1) - H'_-(1)$. Moreover,

$$y(t) = \frac{(1-t+i)(y(i)+i)}{1-t+i} = y(i) + i,\ i < t < i+1,\ i \in \mathbb{N}.$$

This result and left-continuity of y at $t = i$, $i = 1, \dots, b$, imply that $y(i+1) = y(i) + i$, $i \in \mathbb{N}$. Since $y(1) = H(1) - H'_-(1)$, then

$$y(t) = H(1) - H'_-(1) + \frac{i(i+1)}{2}, \quad i < t \le i+1,\ i \in \mathbb{N}.$$

Thus the fixed point of F in $\mathcal{D}^{lc}([0,\infty))$ is

$$y(t) = \begin{cases} H(t) - t\frac{H(1)-H(t)}{1-t}, & 0 \le t < 1, \\ H(1) - H'_-(1), & t = 1, \\ H(1) - H'_-(1) + \frac{i(i+1)}{2}, & i < t \le i+1,\ i \in \mathbb{N}. \end{cases} \tag{7.13}$$

Consequence: Let F be defined by (7.11) with $H(0) = 0$, and y by (7.13), then $F(y) \in \mathcal{D}_0^{lc}([0,\infty))$. Denoting by $f(x)$ the distributional derivative of $F(x)$, $x \in \mathcal{D}^{lc}([0,\infty))$, then

$$y(t) = F(y)(t) = \int_0^t f(y), \quad t \in [0,\infty).$$

Because $y(0) = 0$ then y is the solution of the Cauchy problem

$$y' = f(y), \quad y(0) = 0. \tag{7.14}$$

7.3. **Existence of minimal and maximal solutions.** In this section sufficient conditions are introduced for the existence of local or global minimal and maximal solutions to the distributional Cauchy system

$$y_i' = f_i(y_1,\dots,y_m), \quad y_i(a) = 0,\ i = 1,\dots,m. \tag{7.15}$$

We assume that $I = [a,b]$, $a < b < \infty$, or $I = [a,b)$, $a < b \le \infty$. The space $L^1(I)$ of Lebesgue integrable functions on I, ordered a.e. pointwise and normed by L^1-norm: $\|x\|_1 = \int_a^b |x(s)|\,ds$, is an ordered Banach space. It is easy to verify that the product space $L^1(I)^m = \times_{i=1}^m L^1(I)$, ordered by componentwise ordering and a norm $\|x\| = \{\max \|x_i\|_1 : i = 1,\dots,m\}$, $x = (x_1,\dots,x_m) \in L^1(I)^m$, has the following properties.

(L0): Bounded and monotone sequences of $L^1(I)^m$ converge.

(L1): $x^+ = \sup\{(\theta,\dots,\theta), x\}$ exists, and $\|x^+\| \le \|x\|$ for every $x \in L^1(I)^m$.

Denote

$$B(R) = \{x \in L^1(I)^m : \|x\| \le R\},\ R > 0. \tag{7.16}$$

Because of the properties (L0) and (L1) the following result is a consequence of [2, Theorem 2.44].

Lemma 7.11. *Given a subset P of $L^1(I)^m$, assume that $F : P \to P$ is increasing, and that $F[P] \subseteq B(R) \subseteq P$ for some $R > 0$. Then F has minimal and maximal fixed points.*

The next result is a special case of Lemma 7.11.

Proposition 7.2. *Assume that mappings $f_i : L^1(I)^m \to \mathcal{A}_L(I)$ are increasing, and that for some $R > 0$ the LL primitive integrals $F_i(x_1,\dots,x_m)(t) = \int_a^t f_i(x_1,\dots,x_m)$, $t \in I$, of $f_i(x_1,\dots,x_m)$, $i = 1,\dots,m$, satisfy the following hypothesis.*

(f0): $\|F_i(x_1,\dots,x_m)\|_1 \le R$ *for all* $x_i \in L^1(I)$, $\|x_i\|_1 \le R$, $i = 1,\dots,m$.

Then the system (7.15) *has minimal and maximal solutions in* $B(R) \cap \mathcal{L}^{lc}(I)^m$.

Proof. By definition, the functions $F_i(x_1,\dots,x_m)$ belong to $\mathcal{L}^{lc}(I)$ which is a subset of $L^1(I)$. Thus $F = (F_1,\dots,F_m)$ maps $L^1(I)^m$ into $L^1(I)^m$. The given hypotheses imply that F satisfies the hypotheses of Lemma 7.11 when $P = B(R)$. Thus, by Lemma 7.11, F has in $B(R)$ minimal and maximal fixed points. Their components form minimal and maximal solutions of (7.15) in $B(R) \cap \mathcal{L}^{lc}(I)^m$. □

The following result deals with the existence of minimal and maximal solutions of the system (7.15) in the whole $\mathcal{L}^{lc}(I)^m$.

Theorem 7.12. *Assume that mappings* $f_i : L^1(I)^m \to \mathcal{A}_L(I)$ *are increasing, and that the integrals* $F_i(x)(t) = \int_a^t f_i(x)$, $t \in I$, *satisfy the following hypothesis.*

(f1): $\|F_i(x)\|_1 \le Q(\|x\|)$ *for all* $x \in L^1(I)^m$, *where* $Q : \mathbb{R}_+ \to \mathbb{R}_+$ *is increasing,* $R = Q(R)$ *for some* $R > 0$, *and* $r \le Q(r)$ *implies* $r \le R$.

Then the Cauchy system (7.15) *has minimal and maximal solutions in* $\mathcal{L}^{lc}(I)^m$.

Proof. The hypothesis (f1) implies that $F = (F_1,\dots,F_m)$ has the following property.

$$\|F(x)\|_1 \le Q(\|x\|) \le Q(R) = R \ \text{ for every } \ x \in B(R).$$

Thus the hypothesis (f0) holds, whence (7.15) has the by Proposition 7.2 minimal and maximal solutions in $B(R) \cap \mathcal{L}^{lc}(I)^m$, and they are increasing with respect to f_i.

If $y = (y_1,\dots,y_m) \in \mathcal{L}^{lc}(I)^m$ is a solution of (7.15), then y is a fixed point of F. The hypothesis (f1) with $r = \|y\|$ implies that

$$\|y\| = \|F(y)\| \le Q(\|y\|) \le Q(R) = R.$$

Thus $y \in B(R)$, whence all the solutions of (7.15) are in $B(R) \cap \mathcal{L}^{lc}(I)^m$.

The assertion follows from the above results. □

Definition 7.13. Given a nonempty set X and a normed space $Y = (Y, \|\cdot\|)$, we say that a mapping $f : X \to Y$ is norm bounded if $\sup\{\|f(x)\| : x \in X\} < \infty$.

It follows from Theorem 4.2 that $\mathcal{A}_L(I)$, equipped with the 1-norm:

$$\|g\|_1 = \int_a^b |G(t)|\,dt \ \text{ where } \ G(t) = \int_a^t g,\ g \in \mathcal{A}_L(I),\ t \in I, \tag{7.17}$$

and the partial order $\preceq$ defined by (3.7), is a normed Riesz space.

The following result is a consequence of Theorem 7.12.

Corollary 7.14. *Assume that mappings* $f_i : L^1(I)^m \to \mathcal{A}_L(I)$, $i = 1,\dots,m$, *are increasing and norm bounded. Then the distributional Cauchy system* (7.15) *has minimal and maximal solutions in* $\mathcal{L}^{lc}(I)^m$.

Proof. Equivalent to the norm boundedness of mappings f_i is that equations

$$F_i(x_1,\dots,x_m)(t)=\int_a^t f_i(x_1,\dots,x_m),\quad t\in I,\quad i=1,\dots,m,$$

define norm-bounded mappings $F_i : L^1(I)^m \to \mathcal{L}^{lc}(I)$, $i=1,\dots,m$. Thus there exist $R_i>0$ such that

$$\|F_i(x)\|_1\le R_i \text{ for all } x\in L^1(I)^m,\ i=1,\dots,m.$$

These inequalities imply that the hypothesis (f1) is valid when

$$Q(r)\equiv R:=\max\{R_i|i=1,\dots,m\}.$$

Because the mappings f_i are also increasing, the conclusion follows from Theorem 7.12. □

Remark 7.15. It follows from [2, Theorem 2.44] that the system (7.15) has under the hypotheses of Proposition 7.2 or Theorem 7.12 the smallest solution $y_*=(y_{*1},\dots,y_{*m})$ and the greatest solution $y^*=(y_1^*,\dots,y_m^*)$ in the order interval $[\underline{y},\overline{y}]$ of $B(R)$, where $\underline{y}=(\underline{y}_1,\dots,\underline{y}_m)$ is the greatest solution of the system

$$y_i(t)=\min\{0,\int_a^t f_i(y_1,\dots,y_m)\},\quad t\in I,\ i=1,\dots,m,$$

and $\overline{y}=(\overline{y}_1,\dots,\overline{y}_m)$ is the smallest solution of

$$y_i(t)=\max\{0,\int_a^t f_i(y_1,\dots,y_m)\},\quad t\in I,\ i=1,\dots,m.$$

Moreover, y^*, y_*, $\underline{y}$ and $\overline{y}$ are all increasing with respect to f_i, $i=1,\dots,m$.

Existence of continuous solutions of distributional Cauchy problems is studied in [6].

8. HIGHER ORDER DIFFERENTIAL EQUATIONS

In this section we will study the following mth order order distributional Cauchy problem

$$(8.1)\qquad y^{(m)}=g(y,y',\dots,y^{(m-1)}),\ y(0)=c_1,\ y'(0)=c_2,\ \dots,\ y^{(m-1)}(0)=c_m.$$

Existence and comparison results for the smallest and greatest solutions of problem (8.1) are then proved by using results of subsection 7.1.

Definition 8.1. We say that $y:[0,b)\to\mathbb{R}$, $b\in(0,\infty]$, is a *solution* of (8.1) if $y^{(m-1)}\in\mathcal{D}^{lc}([0,b))$, if $g(y,y',\dots,y^{(m-1)})\in\mathcal{A}_D([0,b))$, and if (8.1) holds.

The Cauchy problem (8.1) can be transformed into a system of first order Cauchy problems as follows:

Lemma 8.2. *If $g : \mathcal{D}^{lc}([0,b))^m \to \mathcal{A}_D([0,b))$, then y is a solution of the Cauchy problem (8.1) if and only if $(y_1,\dots,y_m) = (y,y',\dots,y^{(m-1)})$ is a solution of the following Cauchy system:*

$$(8.2)\qquad y_i' = y_{i+1},\ i=1,\dots,m-1,\ y_m' = g(y_1,\dots,y_m),\ y_i(0)=c_i,\ i=1,\dots,m.$$

We will present conditions under which problem (8.1) has the smallest and greatest solutions in the set

$$S_D = \{y : [0,b) \to \mathbb{R} : y^{(m-1)} \in \mathcal{D}^{lc}([0,b))\}$$

or in its order interval when S_D is ordered by

$$y \le z \text{ iff } y(t) \le z(t) \text{ and } y^{(i)}(t) \le z^{(i)}(t) \text{ for all } t \in [0,b) \text{ and } i=1,\dots,m-1.$$

We study also dependence of solutions of (8.1) on the functions g and on the initial values c_i, $i=1,\dots,m$.

Definition 8.3. We say that a function $y \in S_D$ is a subsolution of problem (8.1) if

$$(8.3)\qquad y^{(m)} \preceq g(y,y',y^{(m-1)}),\ y(0) \le c_1,\ y'(0) \le c_2,\dots,y^{(m-1)}(0) \le c_m.$$

If reversed inequalities hold in (8.3), we say that y is a supersolution of (8.1). If equalities hold in (8.3), then y is called a solution of (8.1).

As a consequence of Theorem 7.4 we obtain an existence comparison theorem for solutions of problem (8.1).

Theorem 8.4. *Assume that $g : D([0,b))^m \to \mathcal{A}_D([0,b))$ is increasing, and that (8.1) has a subsolution $\underline{y} \in S_D$ and a supersolution $\overline{y} \in S_D$, and $\underline{y} \le \overline{y}$, $\underline{y}' \le \overline{y}',\dots,\underline{y}^{(m-1)} \le \overline{y}^{(m-1)}$. Then the Cauchy problem (8.1) has the smallest and greatest solutions in the order interval $[\underline{y},\overline{y}]$ of S_D, and they are increasing with respect to g and c_i, $i = 1,\dots,m$.*

Proof. The function $\underline{z} = (\underline{y}_1,\dots,\underline{y}_m) = (\underline{y},\underline{y}',\dots,\underline{y}^{(m-1)})$ is a subsolution, and the function $\overline{z} = (\overline{y}_1,\dots,\overline{y}_m) = (\overline{y},\overline{y}',\dots,\overline{y}^{(m-1)})$ is a supersolution of the system (7.1) when the functions $f_i : \mathcal{D}([0,b))^m \to \mathcal{A}_D([0,b))$ are defined by

$$(8.4)\quad f_i(x) = x_i,\ i=1,\dots,m-1,\ f_m(x) = g(x),\quad x=(x_1,\dots,x_m) \in \mathcal{D}^{lc}([0,b))^m,$$

These functions f_i are also increasing because g is. It then follows from Theorem 7.4 that the so obtained system (7.1) has the smallest solution $(y_{*1},\dots,y_{*m})$ and the greatest solution $(y_1^*,\dots,y_m^*)$ in the order interval $[\underline{z},\overline{z}]$ of $\mathcal{D}^{lc}([0,b))^m$. Moreover, they are increasing with respect to f_i and c_i, $i=1,\dots,m$. This result implies by Lemma 8.2 that y_{1*} and y_1^* are the smallest and the greatest solutions of the Cauchy problem (8.1) in the order interval $[\underline{y},\overline{y}]$ of S_D, and they are increasing with respect to g and c_i, $i=1,\dots,m$. □

Next we consider the existence of the smallest and greatest solutions of the Cauchy problem (8.1) in the whole S_D. As a special case of Theorem 8.4 we obtain the following result.

Corollary 8.5. *The Cauchy problem* (8.1) *has the smallest and greatest solutions in* S_D*, and they are increasing with respect to* g *and* c_i*,* $i = 1, \ldots, m$*, if* $g : D([0,b))^m \to \mathcal{A}_D([0,b))$ *is increasing and order-bounded.*

Proof. Because function g is order-bounded, there exist distributions $\underline{h}$ and $\overline{h}$ in $\mathcal{A}_D[0,b))$ such that $\underline{h} \preceq g(x_1, \ldots, x_m) \preceq \overline{h}$ for all $(x_1, \ldots, x_m) \in D([0,b))^m$. Defining,

$$\begin{cases} \underline{y}_m(t) := c_m + \int_0^t \underline{h}, \quad \overline{y}_m(t) := c_m + \int_0^t \overline{h}_i, \ t \in [0,b), \\ \underline{y}_i(t) = c_i + \int_0^t \underline{y}_{i+1}(s)\,ds, \quad \overline{y}_i(t) = c_i + \int_0^t \overline{y}_{i+1}(s)\,ds, \ i = 1, \ldots, m-1, \ t \in [0,b), \end{cases}$$

it is easy to see that $\underline{y}_1$ is a lower solution and $\overline{y}_1$ is an upper solution of (8.1) in $\mathcal{D}^{lc}([0,b))$. Thus (8.1) has by Theorem 8.4 the smallest and greatest solutions in the order interval $[\underline{y}_1, \overline{y}_1]$ of S_D. If y is a solution of (8.1) in S_D, then

$$\underline{y}_1^{(m-1)}(t) = c_m + \int_0^t \underline{h} \le c_m + \int_0^t g(y, y', \ldots, y^{(m-1)}) \le c_m + \int_0^t \overline{h} = \overline{y}_1^{(m-1)}(t), \ t \in [0,b).$$

This result and the definitions of $\underline{y}_1$ and $\overline{y}_1$ can be used to show that y belongs to the order interval $[\underline{y}_1, \overline{y}_1]$ of S_D. Thus the smallest and greatest solutions of (8.1) in that order interval are the smallest and greatest solutions of (8.1) in the whole S_D. The last conclusion of Theorem 8.4 implies that these solutions are increasing with respect to g and c_i, $i = 1, \ldots, m$. □

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