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TOPOLOGICAL PRINCIPLES VIA A CONNECTEDNESS AND COMPACTNESS ARGUMENT

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Dedicated to V. Lakshmikantham with much admiration

ABSTRACT. In this paper we present fixed point principles for equations and inclusions using a simple connectedness and compactness argument.

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1. INTRODUCTION

The aim of this paper is to present a simple and powerful fixed point result based on a connectedness and compactness argument. In particular no knowledge is needed of the theory of fixed points (for example Brouwer's fixed point theorem is not assumed). To motivate our fixed point result we begin by presenting some continuation principles for Fredholm and Volterra integral equations. We remark here that similar continuation principles to those in this paper could be established for integral inclusions and discrete equations and inclusions. We also note (see Section 2) that the existence principles we establish in Section 2 in the Banach space setting are not as general as those established in the literature via fixed point arguments. However the advantage of our approach is that it is elementary and no knowledge of fixed point theory is assumed. Finally we note that the fixed point principle we will establish at the end of Section 2 will be in a Fréchet space setting (indeed it can be trivially adjusted to a more general setting, for example complete gauge spaces).

For the space of continuous functions on the closed interval $[0, T]$, denoted by $C[0, T]$ and norm $|\cdot|_0$ given by

$$|y|_0 = \sup_{t \in [0,T]} |y(t)|,$$

the Arzéla-Ascoli Theorem gives conditions under which a subset M of $C[0, T]$ is compact.

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Theorem 1.1 (Arzéla-Ascoli Theorem). *Let $M \subseteq C([0,T],\mathbf{R})$. If M is uniformly bounded and equicontinuous, then M is relatively compact in $C([0,T],\mathbf{R})$.*

The set of bounded, continuous functions on the half-open interval $[0,T)$, $0 \leq T \leq \infty$, denoted by $BC[0,T)$, is also a normed space with norm given by

$$|y|_0 = \sup_{t\in[0,T)} |y(t)|.$$

We will require compactness criteria for a subset of $BC[0,T)$, namely $C_l[0,T)$. The space $C_l[0,T)$ is the set of all bounded, continuous functions y on $[0,T)$, for which $\lim_{t\to T} y(t)$ exists. We have the following criterion of compactness on $C_l[0,T)$:

Theorem 1.2 (Corduneanu, [2, P. 62]). *Let $M \subset C_l([0,\infty),\mathbf{R})$. Then M is compact in $C_l([0,\infty),\mathbf{R})$ if the following conditions hold:*

(i) *M is bounded in C_l,*
(ii) *the functions belonging to M are equicontinuous on any compact interval of $[0,\infty)$,*
(iii) *the functions from M are equiconvergent, that is, given $\epsilon > 0$, there corresponds $T(\epsilon) > 0$ such that $|f(t) - f(\infty)| < \epsilon$ for any $t \geq T(\epsilon)$ and $f \in M$.*

We now turn our attention from continuous functions to measurable functions. The most important spaces of measurable functions are the Lebesgue spaces $L^p(I)$, $1 \leq p \leq \infty$, where I is an interval of $\mathbf{R}$ (and indeed could be $\mathbf{R}^+$). For $y \in L^p(I)$, the norm is given by

$$\|y\|_p = \left(\int_I |y(t)|^p\,dt\right)^{1/p}, \qquad \text{for } 1 \leq p < \infty,$$
$$\|y\|_p = ess\ sup_{t\in I}\,|y(t)|, \qquad \text{for } p = \infty.$$

If $|I| < \infty$, we have the following compactness criteria for a subset M of $L^p(I)$, $1 \leq p < \infty$:

Theorem 1.3 (Riesz Compactness Criteria). *Let $M \subset L^p([t_0,t_1],\mathbf{R})$, $1 \leq p < \infty$. Necessary and sufficient conditions for the relative compactness of M in L^p are:*

(i) *M is bounded in L^p,*
(ii) *$\int_{t_0}^{t_1} |x(t+h) - x(t)|^p\,dt \to 0$ as $h \to 0$ uniformly for $x \in M$.*

If I is not necessarily finite, compactness of a subset M of $L^p(I)$ is given by

Theorem 1.4 (Yosida, [7, P. 275]). *Let S be the real line, $\mathbf{B}$ the σ-ring of Baire subsets B of S and $m(B) = \int_B dx$ the ordinary Lebesgue measure of B. Then a subset K of $L^p(S,\mathbf{B},m)$, $1 \leq p < \infty$, is strongly relatively compact if and only if it satisfies the following conditions:*

(i) $\sup_{x\in K} \|x\| = \sup_{x\in K} \left(\int_S |x(s)|^p\,ds\right)^{\frac{1}{p}} < \infty$,

(ii) $\lim_{t\to 0} \int_S |x(t+s) - x(s)|^p \, ds = 0$ *uniformly in* $x \in K$,
(iii) $\lim_{\alpha\uparrow\infty} \int_{|s|>\alpha} |x(s)|^p \, ds = 0$ *uniformly in* $x \in K$.

Let I be an interval in $\mathbf{R}$.

Definition 1.5. A function $g : I \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory function if the following conditions hold:

(i) the map $t \mapsto g(t, y)$ is measurable for all $y \in \mathbf{R}$,
(ii) the map $y \mapsto g(t, y)$ is continuous for almost all $t \in I$.

Definition 1.6. A function $g : I \times \mathbf{R} \to \mathbf{R}$ is a L^q-Carathéodory function if the following conditions hold:

(i) the map $t \mapsto g(t, y)$ is measurable for all $y \in \mathbf{R}$,
(ii) the map $y \mapsto g(t, y)$ is continuous for almost all $t \in I$,
(iii) for any $r > 0$, there exists $\mu_r \in L^q(I)$ such that $|y| \le r$ implies that $|g(t, y)| \le \mu_r(t)$ for almost all $t \in I$.

The following is a result for Carathéodory functions:

Theorem 1.7 (Krasnoselskii, [3, P. 22, 27])**.** *Let* $g : I \times \mathbf{R} \to \mathbf{R}$ *be a Carathéodory function such that* $y \in L^{p_1}(I)$ *implies that* $g(t, y) \in L^{p_2}(I)$ $(p_1, p_2 \ge 1)$. *Then the operator* $G : L^{p_1}(I) \to L^{p_2}(I)$ *defined by* $Gy(t) = g(t, y(t))$, *is continuous and bounded. In particular, there exists* $a_1 \in L^{p_2}(I)$ *and* $a_2 > 0$ *such that*

$$|g(t, y)| \le a_1(t) + a_2 |y|^{\frac{p_1}{p_2}}.$$

2. EXISTENCE

We begin this section by establishing an existence principle for

$$y(t) = h(t) + \int_0^1 k(t, s)\, g(s, y(s))\, ds, \quad t \in [0, 1], \tag{2.1}$$

using a simple connectedness argument (no knowledge of fixed point theory is needed).

Theorem 2.1. *Let* $1 \le p \le \infty$ *be a constant, and* q *be such that* $1/p + 1/q = 1$. *Assume*

$$h \in C[0, 1], \tag{2.2}$$

$$g : [0, 1] \times \mathbf{R} \to \mathbf{R} \text{ is an } L^q\text{-Carathéodory function}, \tag{2.3}$$

$$k_t(s) = k(t, s) \in L^p[0, 1], \text{ for each } t \in [0, 1] \tag{2.4}$$

and

$$\text{the map } t \mapsto k_t \text{ is continuous from } [0, 1] \to L^p[0, 1] \tag{2.5}$$

hold. In addition suppose

(i) *there exists a constant $M>0$, independent of λ, with $|y|_0 \le M$ for any solution $y \in C[0,1]$ to*

$$(2.6)_\lambda \qquad y(t) = \lambda\left(h(t) + \int_0^1 k(t,s)\,g(s,y(s))\,ds\right), \quad t\in[0,1],$$

for each $\lambda \in [0,1]$,

(ii) *for any $\lambda_0 \in [0,1]$ where $(2.6)_{\lambda_0}$ has a solution in $C[0,1]$ there exists a neighborhood of λ_0 (one-sided neighborhood of λ_0 if $\lambda_0 = 0$ or $\lambda_0 = 1$) so that $(2.6)_\lambda$ has a solution in $C[0,1]$ for all λ in the neighborhood of λ_0.*

Then (2.1) *has at least one solution in $C[0,1]$.*

Proof. Let

$$\Lambda = \{\lambda \in [0,1] : \ (2.6)_\lambda \text{ has a solution in } C[0,1]\}.$$

Note $0 \in \Lambda$. Now we show Λ is closed. To see this let $\{\lambda_n\}_1^\infty$ be a sequence in Λ with $\lambda_n \to \lambda$. Let $u_n \in C[0,1]$ be a solution to $(2.6)_\lambda$ corresponding to $\lambda = \lambda_n$. It is easy to check that (via the Arzéla-Ascoli Theorem, see [5, Theorem 4.2.2]) that $\{u_n\}_1^\infty$ is relatively compact in $C[0,1]$. For completeness we present the proof here (however we note that the compactness arguments in this paper are well known so for our other results in this paper we will just refer the reader to the appropriate theorem in the book [5]). Now there exists $\mu_M \in L^q[0,1]$ such that $|g(s,u_n(s))| \le \mu_M(s)$, for almost every $s\in[0,1]$ and $n \in \{1,2,\dots\}$. Note

$$|u_n|_0 \le |h|_0 + \sup_{t\in[0,1]} \|k_t\|_p\,\|\mu_M\|_q$$

and for any t_1, $t_2 \in [0,1]$ we have

$$|u_n(t_1) - u_n(t_2)| \le |h(t_1) - h(t_2)| + \left(\int_0^1 |k_{t_1}(s) - k_{t_2}(s)|^p\,ds\right)^{\frac{1}{p}} \|\mu_M\|_q,$$

so the Arzéla-Ascoli Theorem implies there is a subsequence S of $\{1,2,\dots\}$ and a $u \in C[0,1]$ with $u_n \to u$ in $C[0,1]$ as $n\to\infty$ in S. Let $N : C[0,1] \to C[0,1]$ be given by

$$Ny(t) = h(t) + \int_0^1 k(t,s)g(s,y(s))\,ds.$$

It is easy to check (see [5, Theorem 4.2.2]) via the Lebesgue dominated convergence theorem that $N : C[0,1] \to C[0,1]$ is continuous. This with

$$u_n(t) = \lambda_n\left(h(t) + \int_0^1 k(t,s)\,g(s,u_n(s))\,ds\right), \quad t\in[0,1],$$

implies

$$u(t) = \lambda\left(h(t) + \int_0^1 k(t,s)\,g(s,u(s))\,ds\right), \quad t\in[0,1].$$

Thus u is a solution of $(2.6)_\lambda$ i.e. $\lambda\in\Lambda$ so Λ is closed.

Finally (ii) guarantees that Λ is open. Since $\Lambda \neq \emptyset$ is both open and closed in $[0,1]$ it follows that $\Lambda = [0,1]$. Since $1 \in \Lambda$ then (2.1) has a solution in $C[0,1]$. □

Remark 2.2. One can put conditions on k and g (see for example [8, pg 156–157]) so that (ii) in Theorem 2.1 holds. In the literature it is usual to write $y(t) - \lambda\left(h(t) + \int_0^1 k(t,s)g(s,y(s))\,ds\right) = 0$, $t \in [0,1]$ as $F(\lambda, y) = 0$ where $F : [0,1] \times C[0,1] \to C[0,1]$ and one approach to guarantee (ii) in Theorem 2.1 is to put conditions so that the implicit function theorem can be applied. Of course if one used an appropriate fixed point result (for example the Leray-Schauder alternative) instead of the connectedness approach then condition (ii) is not needed in Theorem 2.1. This shows how powerful the fixed point approach is. However we remark that the connectedness approach is elementary (for example no knowledge is needed of the Brouwer's fixed point theorem, the starting off point in fixed point theory) and still a powerful and applicable topological existence principle can be established.

Next we will look for L^p solutions to

$$y(t) = h(t) + \int_0^1 k(t,s)\,g(s,y(s))\,ds \quad \text{a.e. } t \in [0,1]. \tag{2.7}$$

Theorem 2.3. *Let p, p_1 and p_2 be such that $1 \le p_1 \le p < \infty$ and $\frac{1}{p_1} + \frac{1}{p_2} = 1$. Assume*

$$h \in L^p[0,1], \tag{2.8}$$

$$\begin{cases} g : [0,1] \times \mathbf{R} \to \mathbf{R} \text{ is a Cararthéodory function} \\ \text{and } \ g(t,y(t)) \in L^{p_2}[0,1] \text{ for } y \in L^p[0,1] \end{cases} \tag{2.9}$$

and

$$\begin{cases} k : [0,1] \times [0,1] \to \mathbf{R} \text{ is such that} \\ (t,s) \mapsto k(t,s) \text{ is measurable and} \\ \left(\int_0^1 \left(\int_0^1 |k(t,s)|^p\,dt\right)^{\frac{p_1}{p}} ds\right)^{\frac{1}{p_1}} \equiv M_0 < \infty \end{cases} \tag{2.10}$$

hold. In addition suppose

(i) *there exists a constant $M > 0$, independent of λ, with $\|y\|_p \le M$ for any solution $y \in L^p[0,1]$ to*

$$y(t) = \lambda\left(h(t) + \int_0^1 k(t,s)\,g(s,y(s))\,ds\right) \qquad a.e. \quad t \in [0,1] \tag{2.11$_\lambda$}$$

for each $\lambda \in [0,1]$,

(ii) *for any $\lambda_0 \in [0,1]$ where $(2.11)_{\lambda_0}$ has a solution in $L^p[0,1]$ there exists a neighborhood of λ_0 (one-sided neighborhood of λ_0 if $\lambda_0 = 0$ or $\lambda_0 = 1$) so that $(2.11)_\lambda$ has a solution in $L^p[0,1]$ for all λ in the neighborhood of λ_0.*

Then (2.7) has at least one solution in $L^p[0,1]$.

Proof. Let

$$\Lambda = \{\lambda \in [0,1] : (2.11)_\lambda \text{ has a solution in } L^p[0,1]\}.$$

Note $0 \in \Lambda$. Now we show Λ is closed. To see this let $\{\lambda_n\}_1^\infty$ be a sequence in Λ with $\lambda_n \to \lambda$. Let $u_n \in L^p[0,1]$ be a solution to $(2.11)_\lambda$ corresponding to $\lambda = \lambda_n$. It is easy to check (via the Riesz compactness criterion, see [5, Theorem 4.2.1]) that $\{u_n\}_1^\infty$ is relatively compact in $L^p[0,1]$. Thus there is a subsequence S of $\{1,2,\dots\}$ and a $u \in L^p[0,1]$ with $u_n \to u$ in $L^p[0,1]$ as $n \to \infty$ in S. Let $G : L^p[0,1] \to L^{p_2}[0,1]$ be

$$Gy(t) := g(t, y(t)),$$

$K : L^{p_2}[0,1] \to L^p[0,1]$ be

$$Ky(t) := h(t) + \int_0^1 k(t,s)\, y(s)\, ds$$

and $N : L^p[0,1] \to L^p[0,1]$ be $Ny(t) := KGy(t)$. From Theorem 1.7 we know G is continuous and bounded and also K is continuous since if $y_n \to y$ in $L^{p_2}[0,1]$ then Hölder's inequality guarantees that

$$\|Ky_n - Ky\|_p \le M_0 \, \|y_n - y\|_{p_2} \to 0 \text{ as } n \to \infty.$$

As a result $N : L^p[0,1] \to L^p[0,1]$ is continuous. This with

$$u_n(t) = \lambda_n \left(h(t) + \int_0^1 k(t,s)\, g(s, u_n(s))\, ds \right) \quad \text{a.e.} \quad t \in [0,1],$$

implies

$$u(t) = \lambda \left(h(t) + \int_0^1 k(t,s)\, g(s, u(s))\, ds \right) \quad \text{a.e.} \quad t \in [0,1].$$

Thus u is a solution of $(2.11)_\lambda$ i.e. $\lambda \in \Lambda$ so Λ is closed.

Finally (ii) guarantees that Λ is open. Since $\Lambda \neq \emptyset$ is both open and closed in $[0,1]$ it follows that $\Lambda = [0,1]$. Since $1 \in \Lambda$ then (2.7) has a solution in $L^p[0,1]$. □

More generally we can look for solutions to (2.7) in an Orlicz space. Let P and Q be complementary N-functions [4]. The Orlicz class, denoted by O_P, consists of measurable functions $y : [0,1] \to \mathbf{R}$ for which

$$\rho(y; P) = \int_0^1 P(y(x)) dx < \infty.$$

We shall denote by $L_P([0,1], \mathbf{R})$ the Orlicz space of all measurable functions $y : [0,1] \to \mathbf{R}$ for which

$$|y|_P = \sup_{\substack{\rho(v;Q) \le 1 \\ v \in O_Q}} \left| \int_0^1 y(x) \cdot v(x) dx \right| < \infty.$$

Note also Hölder's inequality [4, p. 74] which says

$$\left|\int_0^1 y(x)\cdot v(x)dx\right| \le |y|_P \cdot |v|_Q.$$

It is known that $(L_P([0,1],\mathbf{R}), |\cdot|_P)$ is a Banach space [4]. Let $E_P([0,1],\mathbf{R})$ be the closure in $L_P([0,1],\mathbf{R})$ of the set of all bounded functions. Note that $E_P \subseteq L_P \subseteq O_P$. We have $E_P = L_P = O_P$ if P satisfies the $(\triangle_2)$ condition, which is

$(\triangle_2)$ there exist ω, $y_0 \ge 0$ such that for $y \ge y_0$, we have $P(2y) \le \omega P(y)$.

For a discussion of the $(\triangle_2)$ condition, we refer the reader to [4, p. 23–29]. For example if P grows faster than a power, then Q satisfies the $(\triangle_2)$ condition.

Using the ideas of [4] we can present many topological existence principles in an Orlicz space; we refer the reader also to [6]. One such result is as follows.

Theorem 2.4. *Let P and Q be complementary N-functions. Suppose*

$$\left\{\begin{array}{l} \phi \text{ and } \psi \text{ are complementary } N\text{-functions, and the functions} \\ Q \text{ and } \phi \text{ satisfy the } (\triangle_2) \text{ condition,} \end{array}\right. \tag{2.12}$$

$$\left\{\begin{array}{l} k(t,\cdot) \in E_P \text{ for a.e. } t \in [0,1] \text{ and} \\ \text{the function } t \mapsto |k(t,\cdot)|_P \text{ belongs to } E_\phi, \end{array}\right. \tag{2.13}$$

$$h \in L_\phi[0,1] \text{ and } g \text{ is a Carathéodory function} \tag{2.14}$$

and

$$\left\{\begin{array}{l} \text{for each } r>0 \text{ there exists } \eta_r \in L_Q([0,1],\mathbf{R}) \text{ and } K_r \ge 0 \\ \text{such that } |g(t,u)| \le \eta_r(t) + K_r\, Q^{-1}\left(\phi\left(\frac{u}{r}\right)\right) \\ \text{for a.e. } t\in[0,1] \text{ and every } u \in \mathbf{R}. \end{array}\right. \tag{2.15}$$

In addition assume

(i) *there exists a constant $M > 0$, independent of λ, with $|y|_\phi \le M$ for any solution $y \in L_\phi[0,1]$ to $(2.11)_\lambda$ for each $\lambda \in [0,1]$,*
(ii) *for any $\lambda_0 \in [0,1]$ where $(2.11)_{\lambda_0}$ has a solution in $L_\phi[0,1]$ there exists a neighborhood of λ_0 (one-sided neighborhood of λ_0 if $\lambda_0 = 0$ or $\lambda_0 = 1$) so that $(2.11)_\lambda$ has a solution in $L_\phi[0,1]$ for all λ in the neighborhood of λ_0.*

Then (2.7) has at least one solution in $L_\phi[0,1]$.

Proof. Let

$$\Lambda = \{\lambda \in [0,1] : \ (2.11)_\lambda \ \text{ has a solution in } \ L_\phi[0,1]\}.$$

Note $0 \in \Lambda$. Now we show Λ is closed. To see this let $\{\lambda_n\}_1^\infty$ be a sequence in Λ with $\lambda_n \to \lambda$. Let $u_n \in L_\phi[0,1]$ be a solution to $(2.11)_\lambda$ corresponding to $\lambda = \lambda_n$. Let $G : L_\phi \to L_Q$ be

$$Gy(t) := g(t, y(t)),$$

$K : E_Q = L_Q \to E_\phi = L_\phi$ be

$$Ky(t) := h(t) + \int_0^1 k(t,s)\, y(s)\, ds$$

and $N : L_\phi[0,1] \to L_\phi[0,1]$ be $Ny(t) := KGy(t)$. Now Lemma 16.3 and Theorem 16.3 (take $M_1 = Q$, $M_2 = \phi$ and $N_1 = P$) of [4] guarantees that $K : E_Q = L_Q \to E_\phi = L_\phi$ is continuous and completely continuous and Theorem 17.6 in [4] guarantees that $G : A \to L_Q$ is continuous and G maps bounded sets into bounded sets; here $A = \{u \in L_\phi : |u|_\phi \le M\}$. Thus $N : A \to L_\phi$ is continuous and completely continuous. As a result we see that $\{u_n\}_1^\infty$ is relatively compact in $L_\phi[0,1]$. Thus there is a subsequence S of $\{1,2,\dots\}$ and a $u \in L_\phi[0,1]$ with $u_n \to u$ in $L_\phi[0,1]$ as $n \to \infty$ in S. This also with

$$u_n(t) = \lambda_n \left(h(t) + \int_0^1 k(t,s)\, g(s, u_n(s))\, ds \right) \quad \text{a.e.} \quad t \in [0,1],$$

implies

$$u(t) = \lambda \left(h(t) + \int_0^1 k(t,s)\, g(s, u(s))\, ds \right) \quad \text{a.e.} \quad t \in [0,1].$$

Thus u is a solution of $(2.11)_\lambda$ i.e. $\lambda \in \Lambda$ so Λ is closed.

Finally (ii) guarantees that Λ is open. Since $\Lambda \ne \emptyset$ is both open and closed in $[0,1]$ it follows that $\Lambda = [0,1]$. Since $1 \in \Lambda$ then (2.7) has a solution in $L_\phi[0,1]$. □

Remark 2.5. By placing other conditions on k and g (see [4, Sections 15, 16, 17]) we may deduce other existence principles in an Orlicz space.

Next we establish existence principles for the Volterra equation

$$y(t) = h(t) + \int_0^t k(t,s)\, g(s, y(s))\, ds, \quad t \in [0,T] \tag{2.16}$$

where $T > 0$.

Theorem 2.6. *Let $1 \le p \le \infty$ be a constant, and q be such that $1/p + 1/q = 1$. Assume*

$$h \in C[0,T], \tag{2.17}$$

$$g : [0,T] \times \mathbf{R} \to \mathbf{R} \textit{ is an } L^q\textit{-Carathéodory function }, \tag{2.18}$$

$$\begin{cases} k_t(s) = k(t,s) \in L^p[0,t], \ \textit{for each } t \in [0,T] \\ \textit{and} \ \ \sup_{t\in[0,T]} \int_0^t |k_t(s)|^p\, ds < \infty, \end{cases} \tag{2.19}$$

and

$$\begin{cases} \textit{for any } t, t' \in [0,T], \\ \int_0^{t^\star} |k_t(s) - k_{t'}(s)|^p\, ds \to 0 \ \textit{as } t \to t', \\ \textit{where } t^\star = \min\{t, t'\} \end{cases} \tag{2.20}$$

hold. In addition suppose

(i) *there exists a constant $M > 0$, independent of λ, with $|y|_0 \le M$ for any solution $y \in C[0,T]$ to*

$$(2.21)_\lambda \qquad y(t) = \lambda\left(h(t) + \int_0^t k(t,s)\,g(s,y(s))\,ds\right), \quad t \in [0,T],$$

for each $\lambda \in [0,1]$,

(ii) *for any $\lambda_0 \in [0,1]$ where $(2.21)_{\lambda_0}$ has a solution in $C[0,T]$ there exists a neighborhood of λ_0 (one-sided neighborhood of λ_0 if $\lambda_0 = 0$ or $\lambda_0 = 1$) so that $(2.21)_\lambda$ has a solution in $C[0,T]$ for all λ in the neighborhood of λ_0.*

Then (2.16) *has at least one solution in $C[0,T]$.*

Remark 2.7. In Theorem 2.6 the condition (2.20) can be replaced by

$$\begin{cases} \text{for any } t, t' \in [0,T], \\ \int_0^{t^\star} |k_t(s) - k_{t'}(s)|^p\,ds + \int_{t^\star}^{t^{\star\star}} |k_{t^{\star\star}}(s)|^p\,ds \to 0 \text{ as } t \to t', \\ \text{where } t^\star = \min\{t,t'\} \text{ and } t^{\star\star} = \max\{t,t'\}. \end{cases}$$

Note this condition implies $\sup_{t\in[0,T]} \int_0^t |k_t(s)|^p\,ds < \infty$ in (2.19).

Proof. Let

$$\Lambda = \{\lambda \in [0,1] : \ (2.21)_\lambda \ \text{ has a solution in } \ C[0,T]\}\,.$$

Note $0 \in \Lambda$. Now we show Λ is closed. To see this let $\{\lambda_n\}_1^\infty$ be a sequence in Λ with $\lambda_n \to \lambda$. Let $u_n \in C[0,T]$ be a solution to $(2.11)_\lambda$ corresponding to $\lambda = \lambda_n$. Let $N : C[0,T] \to C[0,T]$ be given by

$$Ny(t) = h(t) + \int_0^t k(t,s)\,g(s,y(s))\,ds$$

and it is easy to check [5] that $N : C[0,T] \to C[0,T]$ is continuous and completely continuous. Thus there is a subsequence S of $\{1,2,\dots\}$ and a $u \in C[0,T]$ with $u_n \to u$ in $C[0,T]$ as $n \to \infty$ in S and we can conclude immediately that

$$u(t) = \lambda\left(h(t) + \int_0^t k(t,s)\,g(s,u(s))\,ds\right), \quad t \in [0,T].$$

Thus u is a solution of $(2.21)_\lambda$ i.e. $\lambda \in \Lambda$ so Λ is closed.

Finally (ii) guarantees that Λ is open. Since $\Lambda \neq \emptyset$ is both open and closed in $[0,1]$ it follows that $\Lambda = [0,1]$. Since $1 \in \Lambda$ then (2.16) has a solution in $C[0,T]$. □

We can also obtain immediately the following two existence principles (using the results in [4, 5]) for

$$(2.22) \qquad y(t) = h(t) + \int_0^t k(t,s)\,g(s,y(s))\,ds \ \text{ for a.e. } \ t \in [0,T]$$

where $T > 0$.

Theorem 2.8. *Let p, p_1 and p_2 be such that $1 \leq p_1 \leq p < \infty$ and $\frac{1}{p_1} + \frac{1}{p_2} = 1$. Assume*

$$h \in L^p[0,T], \tag{2.23}$$

$$\begin{cases} g : [0,T] \times \mathbf{R} \to \mathbf{R} \text{ is a Cararthéodory function} \\ \text{and } g(t,y(t)) \in L^{p_2}[0,T] \text{ for } y \in L^p[0,T] \end{cases} \tag{2.24}$$

and

$$\begin{cases} k : [0,T] \times [0,t] \to \mathbf{R} \text{ is such that} \\ (t,s) \mapsto k(t,s) \text{ is measurable and} \\ \left(\int_0^T \left(\int_s^T |k(t,s)|^p \, dt \right)^{\frac{p_1}{p}} ds \right)^{\frac{1}{p_1}} < \infty \end{cases} \tag{2.25}$$

hold. In addition suppose

(i) *there exists a constant $M > 0$, independent of λ, with $\|y\|_p \leq M$ for any solution $y \in L^p[0,T]$ to*

$$y(t) = \lambda \left(h(t) + \int_0^t k(t,s)\, g(s,y(s))\, ds \right) \quad a.e. \quad t \in [0,T] \tag{2.26$_\lambda$}$$

for each $\lambda \in [0,1]$,

(ii) *for any $\lambda_0 \in [0,1]$ where $(2.26)_{\lambda_0}$ has a solution in $L^p[0,T]$ there exists a neighborhood of λ_0 (one-sided neighborhood of λ_0 if $\lambda_0 = 0$ or $\lambda_0 = 1$) so that $(2.26)_\lambda$ has a solution in $L^p[0,T]$ for all λ in the neighborhood of λ_0.*

Then (2.22) has at least one solution in $L^p[0,T]$.

As above let P and Q be complementary N-functions. The Orlicz class, denoted by O_P, consists of measurable functions $y : [0,T] \to \mathbf{R}$ for which

$$\rho(y;P) = \int_0^1 P(y(x))dx < \infty.$$

We shall denote by $L_P([0,T],\mathbf{R})$ the Orlicz space of all measurable functions $y : [0,T] \to \mathbf{R}$ for which

$$|y|_P = \sup_{\substack{\rho(v;Q) \leq 1 \\ v \in O_Q}} \left| \int_0^T y(x) \cdot v(x) dx \right| < \infty.$$

Theorem 2.9. *Let P and Q be complementary N-functions. Suppose*

$$\begin{cases} \phi \text{ and } \psi \text{ are complementary } N\text{-functions, and the functions} \\ Q \text{ and } \phi \text{ satisfy the } (\triangle_2) \text{ condition,} \end{cases} \tag{2.27}$$

$$\begin{cases} k(t,\cdot) \in E_P \text{ for a.e. } t \in [0,T] \text{ and} \\ \text{the function } t \mapsto |k(t,\cdot)|_P \text{ belongs to } E_\phi, \end{cases} \tag{2.28}$$

$$h \in L_\phi[0,T] \quad \textit{and} \quad g \quad \textit{is a Carathéodory function} \tag{2.29}$$

and

$$\begin{cases} \textit{for each } r > 0 \textit{ there exists } \eta_r \in L_Q([0,T],\mathbf{R}) \textit{ and } K_r \geq 0 \\ \textit{such that } \ |g(t,u)| \leq \eta_r(t) + K_r\, Q^{-1}\left(\phi\left(\frac{u}{r}\right)\right) \\ \textit{for a.e. } \ t \in [0,T] \textit{ and every } u \in \mathbf{R}. \end{cases} \tag{2.30}$$

In addition assume

(i) *there exists a constant $M > 0$, independent of λ, with $|y|_\phi \leq M$ for any solution $y \in L_\phi[0,T]$ to $(2.26)_\lambda$ for each $\lambda \in [0,1]$,*

(ii) *for any $\lambda_0 \in [0,1]$ where $(2.26)_{\lambda_0}$ has a solution in $L_\phi[0,T]$ there exists a neighborhood of λ_0 (one-sided neighborhood of λ_0 if $\lambda_0 = 0$ or $\lambda_0 = 1$) so that $(2.26)_\lambda$ has a solution in $L_\phi[0,T]$ for all λ in the neighborhood of λ_0.*

Then (2.22) has at least one solution in $L_\phi[0,T]$.

Next we turn our attention to finding solutions to

$$y(t) = h(t) + \int_0^\infty k(t,s)\, g(s,y(s))\, ds, \quad t \in [0,\infty). \tag{2.31}$$

Theorem 2.10. *Assume that $1 \leq p \leq \infty$ and let q be such that $\frac{1}{p} + \frac{1}{q} = 1$. Suppose that*

$$h \in C_l[0,\infty), \tag{2.32}$$

$$g \textit{ is } L^q\textit{-Carathéodory}, \tag{2.33}$$

$$k_t \in L^p[0,\infty) \textit{ for each } t \in [0,\infty), \tag{2.34}$$

$$\textit{the map } t \mapsto k_t \textit{ is continuous from } [0,\infty) \textit{ to } L^p[0,\infty) \tag{2.35}$$

and

$$\begin{cases} \textit{there exists } \ \tilde{k} \in L^p[0,\infty) \textit{ such that} \\ k_t \to \tilde{k} \textit{ in } L^p[0,\infty) \textit{ as } t \to \infty \end{cases} \tag{2.36}$$

hold. In addition assume

(i) *there exists a constant $M > 0$, independent of λ, with $|y|_0 = \sup_{t\in[0,\infty)} |y(t)| \leq M$ for any solution $y \in C_l[0,\infty)$ to*

$$y(t) = \lambda\left(h(t) + \int_0^\infty k(t,s)\, g(s,y(s))\, ds\right), \quad t \in [0,\infty) \tag{2.37_λ}$$

for each $\lambda \in [0,1]$,

(ii) *for any $\lambda_0 \in [0,1]$ where $(2.37)_{\lambda_0}$ has a solution in $C_l[0,\infty)$ there exists a neighborhood of λ_0 (one-sided neighborhood of λ_0 if $\lambda_0 = 0$ or $\lambda_0 = 1$) so that $(2.37)_\lambda$ has a solution in $C_l[0,\infty)$ for all λ in the neighborhood of λ_0.*

Then (2.31) *has at least one solution in* $C_l[0,\infty)$.

Proof. Let

$$\Lambda = \{\lambda \in [0,1] : \ (2.37)_\lambda \ \text{ has a solution in } \ C_l[0,\infty)\}.$$

Note $0 \in \Lambda$. Now we show Λ is closed. To see this let $\{\lambda_n\}_1^\infty$ be a sequence in Λ with $\lambda_n \to \lambda$. Let $u_n \in C_l[0,\infty)$ be a solution to $(2.37)_\lambda$ corresponding to $\lambda = \lambda_n$. Let $N : C_l[0,\infty) \to C_l[0,\infty)$ be given by

$$Ny(t) = h(t) + \int_0^\infty k(t,s)\, g(s,y(s))\, ds$$

and it is easy to check [5, Theorem 5.2.3] (we use Theorem 1.2 and the Lebesgue dominated convergence theorem) that $N : C_l[0,\infty) \to C_l[0,\infty)$ is continuous and completely continuous. Thus there is a subsequence S of $\{1,2,\dots\}$ and a $u \in C_l[0,\infty)$ with $u_n \to u$ in $C_l[0,\infty)$ as $n \to \infty$ in S and we can conclude immediately that

$$u(t) = \lambda \left(h(t) + \int_0^\infty k(t,s)\, g(s,u(s))\, ds \right), \quad t \in [0,\infty).$$

Thus u is a solution of $(2.37)_\lambda$ i.e. $\lambda \in \Lambda$ so Λ is closed.

Finally (ii) guarantees that Λ is open. Since $\Lambda \neq \emptyset$ is both open and closed in $[0,1]$ it follows that $\Lambda = [0,1]$. Since $1 \in \Lambda$ then (2.31) has a solution in $C_l[0,\infty)$. ☐

Next we look for L^p solutions to

$$y(t) = h(t) + \int_0^\infty k(t,s) g(s,y(s))\, ds \ \text{ a.e. } t \in [0,\infty). \tag{2.38}$$

Theorem 2.11. *Assume that* p, p_1 *and* p_2 *are such that* $1 \le p_1 \le p < \infty$ *and* $\frac{1}{p_1} + \frac{1}{p_2} = 1$ *are satisfied. Suppose that*

$$h \in L^p[0,\infty), \tag{2.39}$$

$$\begin{cases} g : [0,\infty) \times \mathbf{R} \to \mathbf{R} \textit{ is a Cararthéodory function, and} \\ g(t,y(t)) \in L^{p_2}[0,\infty) \textit{ for } y \in L^p[0,\infty) \end{cases} \tag{2.40}$$

and

$$\begin{cases} k : [0,\infty) \times [0,\infty) \to \mathbf{R} \textit{ is such that} \\ (t,s) \mapsto k(t,s) \textit{ is measurable and} \\ \left(\int_0^\infty \left(\int_0^\infty |k(t,s)|^p\, dt \right)^{\frac{p_1}{p}} ds \right)^{\frac{1}{p_1}} < \infty \end{cases} \tag{2.41}$$

hold. In addition assume

(i) *there exists a constant* $M > 0$, *independent of* λ, *with* $\|y\|_p \neq M$, *for any solution* $y \in L^p[0,\infty)$ *to*

$$y(t) = \lambda \left(h(t) + \int_0^\infty k(t,s) g\left(s, y(s)\right) ds \right) \quad \textit{a.e. } t \in [0,\infty) \tag{2.42_λ}$$

for each $\lambda \in [0,1]$,

(ii) *for any* $\lambda_0 \in [0,1]$ *where* $(2.42)_{\lambda_0}$ *has a solution in* $L^p[0,\infty)$ *there exists a neighborhood of* λ_0 *(one-sided neighborhood of* λ_0 *if* $\lambda_0 = 0$ *or* $\lambda_0 = 1$*) so that* $(2.42)_\lambda$ *has a solution in* $L^p[0,\infty)$ *for all* λ *in the neighborhood of* λ_0.

Then (2.38) *has at least one solution in* $L^p[0,\infty)$.

Proof. Let

$$\Lambda = \{\lambda \in [0,1] : \ (2.42)_\lambda \ \text{ has a solution in } \ L^p[0,\infty)\}.$$

Note $0 \in \Lambda$. Now we show Λ is closed. To see this let $\{\lambda_n\}_1^\infty$ be a sequence in Λ with $\lambda_n \to \lambda$. Let $u_n \in L^p[0,\infty)$ be a solution to $(2.42)_\lambda$ corresponding to $\lambda = \lambda_n$. Let $N : L^p[0,\infty) \to L^p[0,\infty)$ be given by

$$Ny(t) = h(t) + \int_0^\infty k(t,s)\, g(s,y(s))\, ds$$

and it is easy to check [5, Theorem 5.2.1] (we use Theorem 1.4) that $N : L^p[0,\infty) \to L^p[0,\infty)$ is continuous and completely continuous. Thus there is a subsequence S of $\{1,2,\dots\}$ and a $u \in L^p[0,\infty)$ with $u_n \to u$ in $L^p[0,\infty)$ as $n \to \infty$ in S and we can conclude immediately that

$$u(t) = \lambda\left(h(t) + \int_0^\infty k(t,s)\, g(s,u(s))\, ds\right) \quad \text{a.e. } t \in [0,\infty).$$

Thus u is a solution of $(2.42)_\lambda$ i.e. $\lambda \in \Lambda$ so Λ is closed.

Finally (ii) guarantees that Λ is open. Since $\Lambda \neq \emptyset$ is both open and closed in $[0,1]$ it follows that $\Lambda = [0,1]$. Since $1 \in \Lambda$ then (2.38) has a solution in $L^p[0,\infty)$. □

We can also obtain the following two existence principles (using the results in [5]) for the Volterra equation

$$y(t) = h(t) + \int_0^t k(t,s)\, g(s,y(s))\, ds, \quad t \in [0,\infty). \tag{2.43}$$

Theorem 2.12. *Assume that* $1 \le p < \infty$, *and let* q *be such that* $\frac{1}{p} + \frac{1}{q} = 1$. *Suppose that*

$$h \in C_l[0,\infty), \tag{2.44}$$

$$g \text{ is } L^q\text{-Carathéodory}, \tag{2.45}$$

$$\begin{cases} k_t(s) = k(t,s) \in L^p[0,t] \text{ for each } t \in [0,\infty) \\ \text{and } \ \sup_{t\in[0,\infty)} \int_0^t |k_t(s)|^p\, ds < \infty, \end{cases} \tag{2.46}$$

$$\begin{cases} \text{for any } t, t' \in [0,\infty), \\ \int_0^{t^\star} |k_t(s) - k_{t'}(s)|^p\, ds \to 0 \text{ as } t \to t', \\ \text{where } t^\star = \min\{t,t'\} \end{cases} \tag{2.47}$$

and

$$(2.48)\qquad \begin{cases} \text{there exists } \tilde{k} \in L^p[0,\infty) \text{ such that} \\ \lim_{t\to\infty} \left(\int_0^t |k_t(s) - \tilde{k}(s)|^p \, ds \right)^{\frac{1}{p}} = 0 \end{cases}$$

hold. In addition assume

(i) *there exists a constant $M > 0$, independent of λ, with $|y|_0 \le M$ for any solution $y \in C_l[0,\infty)$ to*

$$(2.49)_\lambda \qquad y(t) = \lambda \left(h(t) + \int_0^t k(t,s)\, g(s, y(s))\, ds \right), \quad t \in [0,\infty)$$

for each $\lambda \in [0,1]$,

(ii) *for any $\lambda_0 \in [0,1]$ where $(2.49)_{\lambda_0}$ has a solution in $C_l[0,\infty)$ there exists a neighborhood of λ_0 (one-sided neighborhood of λ_0 if $\lambda_0 = 0$ or $\lambda_0 = 1$) so that $(2.49)_\lambda$ has a solution in $C_l[0,\infty)$ for all λ in the neighborhood of λ_0.*

Then (2.43) has at least one solution in $C_l[0,\infty)$.

Now we consider

$$(2.50)\qquad y(t) = h(t) + \int_0^t k(t,s)\, g(s, y(s))\, ds \quad \text{a.e. } t \in [0,\infty).$$

Theorem 2.13. *Assume that p, p_1 and p_2 satisfy $1 \le p_1 \le p < \infty$ and $\frac{1}{p_1} + \frac{1}{p_2} = 1$. Suppose that*

$$(2.51)\qquad h \in L^p[0,\infty),$$

$$(2.52)\qquad \begin{cases} g : [0,\infty) \times \mathbf{R} \to \mathbf{R} \text{ is a Cararthéodory function, and} \\ g(t, y(t)) \in L^{p_2}[0,\infty) \text{ for } y \in L^p[0,\infty) \end{cases}$$

and

$$(2.53)\qquad \begin{cases} k : [0,\infty) \times [0,t] \to \mathbf{R} \text{ is such that} \\ (t,s) \mapsto k(t,s) \text{ is measurable and} \\ \left(\int_0^\infty \left(\int_s^\infty |k(t,s)|^p \, dt \right)^{\frac{p_1}{p}} ds \right)^{\frac{1}{p_1}} < \infty \end{cases}$$

hold. In addition assume

(i) *there exists a constant $M > 0$, independent of λ, with $\|y\|_p \ne M$, for any solution $y \in L^p[0,\infty)$ to*

$$(2.54)_\lambda \qquad y(t) = \lambda \left(h(t) + \int_0^t k(t,s) g\left(s, y(s)\right) ds \right) \quad a.e.\ t \in [0,\infty)$$

for each $\lambda \in [0,1]$,

(ii) *for any $\lambda_0 \in [0,1]$ where $(2.54)_{\lambda_0}$ has a solution in $L^p[0,\infty)$ there exists a neighborhood of λ_0 (one-sided neighborhood of λ_0 if $\lambda_0 = 0$ or $\lambda_0 = 1$) so that $(2.54)_\lambda$ has a solution in $L^p[0,\infty)$ for all λ in the neighborhood of λ_0.*

Then (2.50) *has at least one solution in* $L^p[0,\infty)$.

Remark 2.14. All the results in this section extend in an straightforward way to systems.

Remark 2.15. One can obtain similar existence principles to those in this section for Fredholm and Volterra integral inclusions.

Our next result (which contains all our previous principles in this section) was motivated partly from our previous work; see [1] and the references therein. In our next theorem $E = (E, \{|\cdot|_m\}_{m\in\mathbf{N}})$ (here $\mathbf{N} = \{1, 2, \dots\}$) will be a Fréchet space generated by the family of semi-norms $\{|\cdot|_m : m \in \mathbf{N}\}$. Recall a subset X of E is bounded if for every $m \in \mathbf{N}$ there exists $r_m > 0$ with $|x|_m \le r_m$ for all $x \in X$. We consider the operator equation

$$x = N\,x. \tag{2.55}$$

Theorem 2.16. *Let E be a Fréchet space and assume*

$$N : E \to E \quad \text{is continuous and completely continuous.} \tag{2.56}$$

In addition suppose

(i) *for each $m \in \mathbf{N}$ there exists a constant $M_m > 0$, independent of λ, with $|y|_m \le M_m$ for any solution $y \in E$ to*

$$y = \lambda\, N\, y \tag*{(2.57)$_\lambda$}$$

for each $\lambda \in [0,1]$,

(ii) *for any $\lambda_0 \in [0,1]$ where $(2.57)_{\lambda_0}$ has a solution in E there exists a neighborhood of λ_0 (one-sided neighborhood of λ_0 if $\lambda_0 = 0$ or $\lambda_0 = 1$) so that $(2.57)_\lambda$ has a solution in E for all λ in the neighborhood of λ_0.*

Then (2.55) *has at least one solution in* E.

Proof. Let

$$\Lambda = \{\lambda \in [0,1] : \ (2.57)_\lambda \ \text{ has a solution in } \ E\}\,.$$

Note $0 \in \Lambda$. Now we show Λ is closed. To see this let $\{\lambda_n\}_1^\infty$ be a sequence in Λ with $\lambda_n \to \lambda$. Let $u_n \in E$ be a solution to $(2.57)_\lambda$ corresponding to $\lambda = \lambda_n$. Note for each $m \in \mathbf{N}$ that $|u_n|_m \le M_m$ for each $n \in \mathbf{N}$. Now (2.56) guarantees that there is a subsequence S of $\mathbf{N}$ and a $u \in E$ with $u_n \to u$ in E as $n \to \infty$ in S. This with (2.56) and $u_n = \lambda_n\, N\, u_n$ implies $u = \lambda\, N\, u$. Thus u is a solution of $(2.57)_\lambda$ i.e. $\lambda \in \Lambda$ so Λ is closed.

Finally (ii) guarantees that Λ is open. Since $\Lambda \neq \emptyset$ is both open and closed in $[0,1]$ it follows that $\Lambda = [0,1]$. Since $1 \in \Lambda$ then (2.55) has a solution in E. □

Remark 2.17. It is clear from the above proof that one could replace (2.56) with the condition

$$N : A \to E \quad \text{is continuous and compact;}$$

here $A = \{x \in E : |x|_m \le M_m \ \text{ for all } \ m \in \mathbf{N}\}$. Indeed one could replace $N : A \to E$ compact with the condition:

$$\begin{cases} \text{for any sequence } \{\lambda_n\}_{n=1}^{\infty} \subseteq [0,1] \text{ with } x_n \text{ a} \\ \text{solution to } (2.57)_\lambda \text{ corresponding to } \lambda_n \text{ the} \\ \text{sequence } \{x_n\}_{n=1}^{\infty} \text{ has a convergent subsequence.} \end{cases}$$

Remark 2.18. We stated the previous result when E is a Fréchet space but it is clear that one could consider more general spaces.

Our previous result extends in a straightforward way to the inclusion

$$x \in N\,x. \tag{2.58}$$

Theorem 2.19. *Let E be a Fréchet space and assume*

$$N : E \to 2^E \quad \textit{is a closed and completely continuous map.} \tag{2.59}$$

In addition suppose

(i) *for each $m \in \mathbf{N}$ there exists a constant $M_m > 0$, independent of λ, with $|y|_m \le M_m$ for any solution $y \in E$ to*

$$y \in \lambda\,N\,y \tag*{$(2.60)_\lambda$}$$

for each $\lambda \in [0,1]$,

(ii) *for any $\lambda_0 \in [0,1]$ where $(2.60)_{\lambda_0}$ has a solution in E there exists a neighborhood of λ_0 (one-sided neighborhood of λ_0 if $\lambda_0 = 0$ or $\lambda_0 = 1$) so that $(2.60)_\lambda$ has a solution in E for all λ in the neighborhood of λ_0.*

Then (2.58) *has at least one solution in E.*

Remark 2.20. Note one could replace (2.59) with the condition

$$N : A \to 2^E \quad \text{is a closed and compact map;}$$

here $A = \{x \in E : |x|_m \le M_m \ \text{ for all } \ m \in \mathbf{N}\}$.

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