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SQUARE-MEAN ASYMPTOTICALLY ALMOST AUTOMORPHIC PROCESS AND ITS APPLICATION TO STOCHASTIC INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, we introduce a new concept of square-mean asymptotically almost automorphy for stochastic processes. Also, we study the properties on the completeness and the composition of the space that consists of such processes. We then apply the results obtained to investigate the existence of the square-mean asymptotically almost automorphic mild solutions to a class of abstract semi-linear stochastic integro-differential equations. Finally, an example is also given to justify the practical usefulness of the established general theorems. Our main results extend some known ones in the sense of square-mean almost automorphy.

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1. INTRODUCTION

In this paper, we study the existence of square-mean asymptotically almost automorphic solutions for the following abstract stochastic integro-differential equations

$$\begin{cases} dx(t) = \left[Ax(t) + \int_0^t B(t-s)x(s)ds\right] dt + f\left(t, x(t)\right) dW(t), \ t \geq 0, \\ x(0) = x_0, \end{cases} \tag{1.1}$$

where A and $B(t), t \geq 0$ are densely defined and closed linear operators in a Hilbert space $L^2(\mathbb{P}, \mathbb{H})$, $W(t)$ is a two-sided standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathscr{F}, \mathbb{P}, \mathscr{F}_t)$, where $\mathscr{F}_t = \sigma\{W(u) - W(v); u, v \leq t\}$, x_0 is an $\mathscr{F}_0$-adapted, $\mathbb{H}$-valued random variable independent of the Wiener

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process W, and $f : [0, +\infty) \times L^2(\mathbb{P}, \mathbb{H}) \to L^2(\mathbb{P}, \mathbb{H})$ is an appropriate function specified later.

The concept of almost automorphy was introduced by S. Bochner [5, 6] in relation to some aspects of differential geometry. It turns out to be an important generalization of almost periodicity. For more details about this topics and the related works, we refer the reader to [27, 28].

The asymptotically almost automorphic functions were firstly introduced by G. M. N'Guérékata in [29]. Since then these functions have became of great interest to several mathematicians and generated lots of developments and applications, we refer the reader to [7, 18, 19, 20, 31] and the references therein.

Recently, the existence of almost periodic, almost automorphic and pseudo almost automorphic solutions to some stochastic differential equations have been considered in many publications such as [1–4, 8–14, 30] and references therein. In a very recent paper [13], the authors introduced a new concept of S^2-almost automorphy for stochastic processes including a composition theorem. However, to the best of our knowledge, there are no results available in the literature on square-mean asymptotically almost automorphic mild solution to abstract semi-linear stochastic integro-differential equations. Therefore, motivated by the works [10, 13, 19], the main purpose of this paper is to introduce the notion of square-mean asymptotically almost automorphic stochastic process and establish some basic results not only on the completeness of the space that consists of the square-mean asymptotically almost automorphic processes but also on the composition of such processes. Also, we apply this new concept to investigate the existence of square-mean asymptotically almost automorphic mild solutions to the problem (1.1). The obtained result can be seen as a contribution to this emerging field.

The rest of this paper is organized as follows. In section 2, we introduce the notion of square-mean asymptotically almost automorphic processes and study some of their basic properties. In section 3, we prove the existence of square-mean asymptotically almost automorphic mild solutions to the problem (1.1). An example is given in Section 4 to illustrate the results obtained.

2. **PRELIMINARIES**

In this section, we introduce some basic definitions, notations, lemmas and technical results which will be used in the sequel. For more details on this section, we refer the reader to [10, 12, 21].

Throughout the paper, we assume that $(\mathbb{H}, \|\cdot\|, \langle\cdot,\cdot\rangle)$ and $(\mathbb{K}, \|\cdot\|_{\mathbb{K}}, \langle\cdot,\cdot\rangle_{\mathbb{K}})$ are two real separable Hilbert spaces. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space. The notation $L^2(\mathbb{P},\mathbb{H})$ stands for the space of all $\mathbb{H}$-valued random variables x such that

$$E\|x\|^2 = \int_\Omega \|x\|^2 d\mathbb{P} < \infty.$$

For $x \in L^2(\mathbb{P},\mathbb{H})$, let

$$\|x\|_2 = \left(\int_\Omega \|x\|^2 d\mathbb{P}\right)^{\frac{1}{2}}.$$

Then it is routine to check that $L^2(\mathbb{P},\mathbb{H})$ is a Hilbert space equipped with the norm $\|\cdot\|_2$. The notation $C_0\left(\mathbb{R}^+; L^2(\mathbb{P},\mathbb{H})\right)$ stands for the collection of all bounded continuous stochastic processes φ from $\mathbb{R}^+$ into $L^2(\mathbb{P},\mathbb{H})$ such that $\lim_{t\to+\infty} E\|\varphi(t)\|^2 = 0$. It is then easy to check that $C_0\left(\mathbb{R}^+; L^2(\mathbb{P},\mathbb{H})\right)$ is a Banach space when it is endowed with the norm $\|\varphi\|_{C_0} := \sup_{t\in\mathbb{R}^+} \|\varphi(t)\|_2$. Similarly, $C_0\left(\mathbb{R}^+ \times L^2(\mathbb{P},\mathbb{H}); L^2(\mathbb{P},\mathbb{H})\right)$ stands for the space of the continuous stochastic processes $f : \mathbb{R}^+ \times L^2(\mathbb{P},\mathbb{H}) \to L^2(\mathbb{P},\mathbb{H})$ such that

$$\lim_{t\to+\infty} E\|f(t,x)\|^2 = 0$$

uniformly for $x \in K$, where $K \subset L^2(\mathbb{P},\mathbb{H})$ is any bounded subset. In addition, $W(t)$ is a two-sided standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathscr{F}, \mathbb{P}, \mathscr{F}_t)$, where $\mathscr{F}_t = \sigma\{W(u) - W(v); u, v \le t\}$.

Throughout the rest of the paper, $A : D(A) \subset L^2(\mathbb{P},\mathbb{H}) \to L^2(\mathbb{P},\mathbb{H})$ is the infinitesimal generator of a resolvent operator $\{R(t) : t \ge 0\}$ in the Hilbert space $L^2(\mathbb{P},\mathbb{H})$ and $B(t) : D(B(t)) \subset L^2(\mathbb{P},\mathbb{H}) \to L^2(\mathbb{P},\mathbb{H})$, $t \ge 0$ is a bounded linear operator. To obtain our results, we assume that the abstract Cauchy problem

$$\begin{cases} dx(t) = \left[Ax(t) + \int_0^t B(t-s)x(s)ds\right] dt, \; t \ge 0, \\ x(0) = x_0 \in L^2(\mathbb{P},\mathbb{H}), \end{cases} \tag{2.1}$$

has an associated resolvent operator of bounded linear operators $\{R(t) : t \ge 0\}$ on $L^2(\mathbb{P},\mathbb{H})$.

Definition 2.1. A family of bounded linear operators $\{R(t) : t \ge 0\}$ from $L^2(\mathbb{P},\mathbb{H})$ into $L^2(\mathbb{P},\mathbb{H})$ is a resolvent operator family for the problem (2.1) if the following conditions are verified.

(i) $R(0) = I$ (the identity operator on $L^2(\mathbb{P},\mathbb{H})$) and the map $t \to R(t)x$ is a continuous function on $[0,+\infty) \to L^2(\mathbb{P},\mathbb{H})$ for every $x \in L^2(\mathbb{P},\mathbb{H})$;

(ii) $R(t)D(A) \subset D(A)$ for all $t \ge 0$ and all $x \in D(A)$, $AR(t)x$ is continuous on $[0,+\infty)$ and $R(t)x$ is continuously differentiable on $[0,+\infty)$;

(iii) For every $x \in D(A)$ and $t \ge 0$,

$$\frac{d}{dt}R(t)x = AR(t)x + \int_0^t B(t-s)R(s)xds,$$

$$\frac{d}{dt}R(t)x = R(t)Ax + \int_0^t R(t-s)B(s)xds.$$

For more on resolvent of bounded linear operators and related issues, we refer the reader to [16, 17, 23].

Definition 2.2 ([21]). A stochastic process $x : \mathbb{R} \to L^2(\mathbb{P}, \mathbb{H})$ is said to be stochastically continuous if

$$\lim_{t\to s} E\|x(t) - x(s)\|^2 = 0.$$

Definition 2.3 ([10]). A stochastically continuous stochastic process $x : \mathbb{R} \to L^2(\mathbb{P}, \mathbb{H})$ is said to be square-mean almost automorphic if for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$, there exist a subsequence $\{s_n\}_{n\in\mathbb{N}}$ and a stochastic process $y : \mathbb{R} \to L^2(\mathbb{P}, \mathbb{H})$ such that

$$\lim_{n\to\infty} E\|x(t + s_n) - y(t)\|^2 = 0 \quad \text{and} \quad \lim_{n\to\infty} E\|y(t - s_n) - x(t)\|^2 = 0$$

hold for each $t \in \mathbb{R}$. The collection of all square-mean almost automorphic stochastic processes $x : \mathbb{R} \to L^2(\mathbb{P}, \mathbb{H})$ is denoted by $AA\,(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H}))$.

Definition 2.4 ([10]). A function $f : \mathbb{R}\times L^2(\mathbb{P}, \mathbb{H}) \to L^2(\mathbb{P}, \mathbb{H})$, $(t, x) \to f(t, x)$, which is jointly continuous, is said to be square-mean almost automorphic if $f(t, x)$ is square-mean almost automorphic in $t \in \mathbb{R}$ uniformly for all $x \in K$, where K is any bounded subset of $L^2(\mathbb{P}, \mathbb{H})$. That is to say, for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$, there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ and a function $\widetilde{f} : \mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}) \to L^2(\mathbb{P}, \mathbb{H})$ such that

$$\lim_{n\to\infty} E\|f(t + s_n, x) - \widetilde{f}(t, x)\|^2 = 0 \quad \text{and} \quad \lim_{n\to\infty} E\|\widetilde{f}(t - s_n, x) - f(t, x)\|^2 = 0$$

for each $t \in \mathbb{R}$ and each $x \in K$. Denote by $AA\,(\mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}); L^2(\mathbb{P}, \mathbb{H}))$ the set of all such functions.

Lemma 2.5 ([21]). *$(AA\,(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H}))\,, \|\cdot\|_\infty)$ is a Banach space when it is equipped with the norm*

$$\|x\|_\infty := \sup_{t\in\mathbb{R}} \|x(t)\|_2 = \sup_{t\in\mathbb{R}} (E\|x(t)\|^2)^{\frac{1}{2}},$$

for $x \in AA\,(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H}))$.

Lemma 2.6 ([10]). *Let $f : \mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}) \to L^2(\mathbb{P}, \mathbb{H})$, $(t, x) \to f(t, x)$ be square-mean almost automorphic, and assume that $f(t, \cdot)$ is uniformly continuous on each bounded subset $K \subset L^2(\mathbb{P}, \mathbb{H})$ uniformly for $t \in \mathbb{R}$, that is for all $\varepsilon > 0$, there exists $\delta > 0$ such that $x, y \in K$ and $E\|x - y\|^2 < \delta$ imply that $E\|f(t, x) - f(t, y)\|^2 < \varepsilon$ for all $t \in \mathbb{R}$. Then for any square-mean almost automorphic process $x : \mathbb{R} \to L^2(\mathbb{P}, \mathbb{H})$, the stochastic process $F : \mathbb{R} \to L^2(\mathbb{P}, \mathbb{H})$ given by $F(\cdot) := f\,(\cdot, x(\cdot))$ is square-mean almost automorphic.*

Definition 2.7. A stochastically continuous process $f : \mathbb{R}^+ \to L^2(\mathbb{P}, \mathbb{H})$ is said to be square-mean asymptotically almost automorphic if it can be decomposed as $f = g + h$, where $g \in AA\,(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H}))$ and $h \in C_0\,(\mathbb{R}^+; L^2(\mathbb{P}, \mathbb{H}))$. Denote by

$AAA\,(\mathbb{R}^+; L^2(\mathbb{P},\mathbb{H}))$ the collection of all the square-mean asymptotically almost automorphic processes $f : \mathbb{R}^+ \to L^2(\mathbb{P},\mathbb{H})$.

Definition 2.8. A function $f : \mathbb{R}^+ \times L^2(\mathbb{P},\mathbb{H}) \to L^2(\mathbb{P},\mathbb{H})$, $(t,x) \to f(t,x)$, which is jointly continuous, is said to be square-mean asymptotically almost automorphic if it can be decomposed as $f = g + h$, where $g \in AA\,(\mathbb{R} \times L^2(\mathbb{P},\mathbb{H}); L^2(\mathbb{P},\mathbb{H}))$ and $h \in C_0\,(\mathbb{R}^+ \times L^2(\mathbb{P},\mathbb{H}); L^2(\mathbb{P},\mathbb{H}))$. Denote by $AAA\,(\mathbb{R}^+ \times L^2(\mathbb{P},\mathbb{H}); L^2(\mathbb{P},\mathbb{H}))$ the set of all such functions.

Now, we introduce a few preliminary and important results.

Lemma 2.9. *If f, f_1 and f_2 are all square-mean asymptotically almost automorphic stochastic processes, then the following hold true:*

(I) *$f_1 + f_2$ is square-mean asymptotically almost automorphic;*
(II) *λf is square-mean asymptotically almost automorphic for any scalar λ;*
(III) *there exists a constant $M > 0$ such that $\sup_{t\in\mathbb{R}^+} E\|f(t)\|^2 \le M$.*

Proof. The proof of statements (I) and (II) can be performed along the direction of the proof of Theorem 2.5.3 in [27]. So, we only prove (III). Since $f \in AAA\,(\mathbb{R}^+; L^2(\mathbb{P},\mathbb{H}))$, we have by definition that $f = g + h$, where $g \in AA\,(\mathbb{R}; L^2(\mathbb{P},\mathbb{H}))$ and $h \in C_0(\mathbb{R}^+; L^2(\mathbb{P},\mathbb{H}))$. Then, by [21, Lemma 2.3.(3)], there exists a constant $M_1 > 0$ such that

$$E\|g(t)\|^2 \le M_1$$

for each $t \in \mathbb{R}$. On the other hand, since $\lim_{t\to+\infty} E\|h(t)\|^2 = 0$, then for any given $\varepsilon > 0$, there exists a constant $T > 0$ such that

$$E\|h(t)\|^2 < \varepsilon$$

for each $t \in (T, +\infty)$. Note that $E\|h(t)\|^2$ is uniformly continuous on $[0, T]$, therefore there exists a constant $M_2 > \varepsilon$ such that

$$E\|h(t)\|^2 \le M_2$$

for each $t \in [0, T]$. Now let $M = 2(M_1 + M_2)$. Then, for each $t \in \mathbb{R}^+$, we have

$$E\|f(t)\|^2 \le 2E\|g(t)\|^2 + 2E\|h(t)\|^2 \le 2(M_1 + M_2) = M.$$

□

Lemma 2.10. *Suppose that $f \in AAA\,(\mathbb{R}^+; L^2(\mathbb{P},\mathbb{H}))$ admits a decomposition $f = g + h$, where $g \in AA\,(\mathbb{R}; L^2(\mathbb{P},\mathbb{H}))$ and $h \in C_0\,(\mathbb{R}^+; L^2(\mathbb{P},\mathbb{H}))$. Then $\{g(t) : t \in \mathbb{R}\} \subset \overline{\{f(t) : t \in \mathbb{R}^+\}}$.*

Proof. By the definition of $AA\,(\mathbb{R}; L^2(\mathbb{P},\mathbb{H}))$, there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ of $\{s'_n\}_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} s_n = +\infty$ such that for a certain stochastic process $\widetilde{g}$

$$\lim_{n\to\infty} E\|g(t + s_n) - \widetilde{g}(t)\|^2 = 0 \quad \text{and} \quad \lim_{n\to\infty} E\|\widetilde{g}(t - s_n) - g(t)\|^2 = 0 \tag{2.2}$$

holds for each $t \in \mathbb{R}$. For any fixed $t_0 \in \mathbb{R}$, we note that the sequence $t_0 + s_n \to +\infty$ as $n \to +\infty$. Hence, we obtain

$$\begin{aligned} \lim_{n\to+\infty} E\|f(t_0+s_n) - \widetilde{g}(t_0)\|^2 &= \lim_{n\to+\infty} E\|g(t_0+s_n) + h(t_0+s_n) - \widetilde{g}(t_0)\|^2 \\ &\leq 2\lim_{n\to+\infty} E\|g(t_0+s_n) - \widetilde{g}(t_0)\|^2 \\ &\quad +2\lim_{n\to+\infty} E\|h(t_0+s_n)\|^2 \\ &= 0. \end{aligned}$$

Therefore, $\widetilde{g}(t_0) \in \overline{\{f(t) : t \in \mathbb{R}^+\}}$, which shows that $\{\widetilde{g}(t) : t \in \mathbb{R}\} \subset \overline{\{f(t) : t \in \mathbb{R}^+\}}$. On the other hand, it follows immediately from (2.2) that $\overline{\{\widetilde{g}(t) : t \in \mathbb{R}\}} = \overline{\{g(t) : t \in \mathbb{R}\}}$. Thus, $\{g(t) : t \in \mathbb{R}\} \subset \overline{\{f(t) : t \in \mathbb{R}^+\}}$. □

Using Lemma 2.10, the following key property can be proved. One can refer to Theorem 2.5.4 in [27] for a detailed proof.

Corollary 2.11. *The decomposition of a square-mean asymptotically almost automorphic process is unique.*

Lemma 2.12. $AAA(\mathbb{R}^+; L^2(\mathbb{P}, \mathbb{H}))$ *is a Banach space when it is equipped with the norm:*

$$\|f\|_{AAA(\mathbb{R}^+;L^2(\mathbb{P},\mathbb{H}))} := \sup_{t\in\mathbb{R}} \|g(t)\|_2 + \sup_{t\in\mathbb{R}^+} \|h(t)\|_2,$$

where $f = g+h \in AAA(\mathbb{R}^+; L^2(\mathbb{P}, \mathbb{H}))$ *with* $g \in AA(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H}))$, $h \in C_0(\mathbb{R}^+; L^2(\mathbb{P}, \mathbb{H}))$.

Proof. Let $\{f_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $AAA(\mathbb{R}^+; L^2(\mathbb{P}, \mathbb{H}))$, then by definition that there exist two sequences of functions $\{g_n\}_{n=1}^{\infty}$ and $\{h_n\}_{n=1}^{\infty}$ such that $f_n = g_n + h_n$, $n = 1, 2, \ldots$, where $g_n \in AA(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H}))$ and $h_n \in C_0(\mathbb{R}^+; L^2(\mathbb{P}, \mathbb{H}))$ for $n = 1, 2, \ldots$. From Lemma 2.10, we easily deduce that $\{g_n\}_{n=1}^{\infty}$ is also a Cauchy sequence of square-mean almost automorphic functions with respect to the norm of the space $AA(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H}))$. Thus there exists a function $g \in AA(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H}))$ such that $\lim_{n\to\infty} \|g_n - g\|_2 = 0$ uniformly. Furthermore, the second terms of the functions f_n : $\{h_n\}_{n=1}^{\infty}$ form a Cauchy sequence of continuous functions with respect to the norm sup. Hence, there exists a continuous function h such that $\|h_n - h\|_2 \to 0$ uniformly on $\mathbb{R}^+$, as $n \to \infty$.

Now, using the fact that for each $n \in \mathbb{N}$, $\lim_{t\to+\infty} \|h_n(t)\|_2 = 0$ and the equality $h(t) = h(t) - h_n(t) + h_n(t)$ for $t \in \mathbb{R}^+$, we obtain

$$\lim_{t\to+\infty} \|h(t)\|_2 = 0.$$

Thus $f := g + h \in AAA(\mathbb{R}^+; L^2(\mathbb{P}, \mathbb{H}))$ and $\lim_{n\to\infty} \|f_n - f\|_2 = 0$, so the space $AAA(\mathbb{R}^+; L^2(\mathbb{P}, \mathbb{H}))$ is a Banach space. □

Lemma 2.13. *$AAA\left(\mathbb{R}^{+};L^{2}(\mathbb{P},\mathbb{H})\right)$ is a Banach space with the norm:*

$$\|f\|_{\infty}:=\sup_{t\in\mathbb{R}^{+}}\|f(t)\|_{2}=\sup_{t\in\mathbb{R}^{+}}\left(E\|f(t)\|^{2}\right)^{\frac{1}{2}}.$$

Proof. The proof can be performed along the same line of the proof of [19, Lemma 1.8.], and we omit the details here. □

Remark 2.14. In view of the previous Lemmas it is clear that the two norms are equivalent in $AAA\left(\mathbb{R}^{+};L^{2}(\mathbb{P},\mathbb{H})\right)$.

Lemma 2.15. *Let $f\in AA\left(\mathbb{R}\times L^{2}(\mathbb{P},\mathbb{H});L^{2}(\mathbb{P},\mathbb{H})\right)$ and let $f(t,x)$ be uniformly continuous in any bounded subset $K\subset L^{2}(\mathbb{P},\mathbb{H})$ uniformly for $t\in\mathbb{R}^{+}$. Then $f(t,x)$ is uniformly continuous in any bounded subset $K\subset L^{2}(\mathbb{P},\mathbb{H})$ uniformly for $t\in\mathbb{R}$.*

Proof. By the definition of $AA\left(\mathbb{R}\times L^{2}(\mathbb{P},\mathbb{H});L^{2}(\mathbb{P},\mathbb{H})\right)$, there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ of $\{s'_n\}_{n\in\mathbb{N}}$ with $\lim_{n\to\infty}s_n=+\infty$ such that for a certain stochastic process $\widetilde{f}$

$$\lim_{n\to\infty}E\|f(t+s_n,x)-\widetilde{f}(t,x)\|^{2}=0 \tag{2.3}$$

and

$$\lim_{n\to\infty}E\|\widetilde{f}(t-s_n,x)-f(t,x)\|^{2}=0 \tag{2.4}$$

holds for each $t\in\mathbb{R}$ and each $x\in K$. Since $f(t,x)$ is uniformly continuous in any bounded subset $K\subset L^{2}(\mathbb{P},\mathbb{H})$ uniformly for $t\in\mathbb{R}^{+}$, then for any $\varepsilon>0$, there exists $\delta>0$ such that for all $x,y\in K$ with $E\|x-y\|^{2}<\delta$ and all $t\in\mathbb{R}^{+}$

$$E\|f(t,x)-f(t,y)\|^{2}<\varepsilon.$$

Take any $t\in\mathbb{R}$. For sufficiently large n, we have $t+s_n>0$. Thus

$$E\|f(t+s_n,x)-f(t+s_n,y)\|^{2}<\varepsilon.$$

Now, by (2.3), we get that

$$E\|\widetilde{f}(t,x)-\widetilde{f}(t,y)\|^{2}<\varepsilon,$$

which yields that $\widetilde{f}(t,x)$ is uniformly continuous in any bounded subset $K\subset L^{2}(\mathbb{P},\mathbb{H})$ uniformly for $t\in\mathbb{R}$. Next by (2.4), analogously to the above proof, we can prove that $f(t,x)$ is uniformly continuous in any bounded subset $K\subset L^{2}(\mathbb{P},\mathbb{H})$ uniformly for $t\in\mathbb{R}$. □

Theorem 2.16. *Let $f\in AAA\left(\mathbb{R}^{+}\times L^{2}(\mathbb{P},\mathbb{H});L^{2}(\mathbb{P},\mathbb{H})\right)$ and suppose that $f(t,x)$ is uniformly continuous in any bounded subset $K\subset L^{2}(\mathbb{P},\mathbb{H})$ uniformly for $t\in\mathbb{R}^{+}$. If $u(t)\in AAA\left(\mathbb{R}^{+};L^{2}(\mathbb{P},\mathbb{H})\right)$, then $f\left(\cdot,u(\cdot)\right)\in AAA\left(\mathbb{R}^{+};L^{2}(\mathbb{P},\mathbb{H})\right)$.*

Proof. Suppose that f and u have the following decompositions: $f = g + h$ and $u = \alpha + \beta$, where $g \in AA\left(\mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}); L^2(\mathbb{P}, \mathbb{H})\right)$, $h \in C_0\left(\mathbb{R}^+ \times L^2(\mathbb{P}, \mathbb{H}); L^2(\mathbb{P}, \mathbb{H})\right)$, $\alpha \in AA\left(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H})\right)$ and $\beta \in C_0\left(\mathbb{R}^+; L^2(\mathbb{P}, \mathbb{H})\right)$.

Now we write

$$\begin{aligned} f(t, u(t)) &= f(t, u(t)) - f(t, \alpha(t)) + g(t, \alpha(t)) + h(t, \alpha(t)) \\ &= I(t) + J(t) + N(t), \end{aligned}$$

where $I(t) = f(t, u(t)) - f(t, \alpha(t))$, $J(t) = g(t, \alpha(t))$ and $N(t) = h(t, \alpha(t))$. Combining $\lim_{t \to +\infty} E\|u(t) - \alpha(t)\|^2 = 0$ with $f(t, \cdot)$ is uniformly continuous in any bounded subset $K \subset L^2(\mathbb{P}, \mathbb{H})$ uniformly for $t \in \mathbb{R}^+$, we get

$$\lim_{t \to +\infty} E\|I(t)\|^2 = \lim_{t \to +\infty} E\|f(t, u(t)) - f(t, \alpha(t))\|^2 = 0.$$

On the other hand, from the definition of $C_0\left(\mathbb{R}^+ \times L^2(\mathbb{P}, \mathbb{H}); L^2(\mathbb{P}, \mathbb{H})\right)$, it is easy to see that h is uniformly continuous on K uniformly for $t \in \mathbb{R}^+$, where $K \subset L^2(\mathbb{P}, \mathbb{H})$ is any bounded subset. Thus, the function g is uniformly continuous on any bounded subset $K \subset L^2(\mathbb{P}, \mathbb{H})$ uniformly for $t \in \mathbb{R}^+$. Now, Lemma 2.15 yields that $g(t, x)$ is uniformly continuous in any bounded subset $K \subset L^2(\mathbb{P}, \mathbb{H})$ uniformly for $t \in \mathbb{R}$. By Lemma 2.6, $J(\cdot) = g(\cdot, \alpha(\cdot)) \in AA\left(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H})\right)$. Furthermore, from $h \in C_0\left(\mathbb{R}^+ \times L^2(\mathbb{P}, \mathbb{H}); L^2(\mathbb{P}, \mathbb{H})\right)$, we see that $\lim\limits_{t \to +\infty} E\|N(t)\|^2 = \lim\limits_{t \to +\infty} E\|h(t, \alpha(t))\|^2 = 0$. Hence, we have $f(\cdot, u(\cdot)) \in AAA\left(\mathbb{R}^+; L^2(\mathbb{P}, \mathbb{H})\right)$. □

We now give the following concept of mild solution of system (1.1).

Definition 2.17. An $\mathscr{F}_t$-adapted stochastic process $x : [0, +\infty) \to L^2(\mathbb{P}, \mathbb{H})$ is called a mild solution of problem (1.1) if $x(0) = x_0$ is $\mathscr{F}_0$-measurable and $x(t)$ satisfies the corresponding stochastic integral equation

$$x(t) = R(t)x_0 + \int_0^t R(t-s) f(s, x(s))\, dW(s)$$

for all $t \geq 0$.

3. MAIN RESULTS

In this section, we establish the existence of square-mean asymptotically almost automorphic mild solutions to (1.1). For that, we need the following technical result.

Lemma 3.1. *Let $\{R(t) : t \geq 0\}$ be a family of bounded linear operators on $L^2(\mathbb{P}, \mathbb{H})$ satisfying $\|R(t)\| \leq Me^{-\delta t}$ for all $t \geq 0$, where $M, \delta > 0$ are fixed constants, and let $f \in AAA\left(\mathbb{R}^+; L^2(\mathbb{P}, \mathbb{H})\right)$. If Υ is the function defined by $\Upsilon(t) := \int_0^t R(t-s)f(s)dW(s), t \geq 0$, then $\Upsilon(\cdot) \in AAA\left(\mathbb{R}^+; L^2(\mathbb{P}, \mathbb{H})\right)$.*

Proof. Suppose that $f = g+h$, where $g \in AA\,(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H}))$ and $h \in C_0\,(\mathbb{R}^+; L^2(\mathbb{P}, \mathbb{H}))$. Then

$$\begin{aligned}\Upsilon(t) &= \int_0^t R(t-s)g(s)dW(s) + \int_0^t R(t-s)h(s)dW(s)\\ &= \int_{-\infty}^t R(t-s)g(s)dW(s) - \int_{-\infty}^0 R(t-s)g(s)dW(s) + \int_0^t R(t-s)h(s)dW(s)\\ &= \Upsilon_1(t) + \Upsilon_2(t),\end{aligned}$$

where $\Upsilon_1(t) = \int_{-\infty}^t R(t-s)g(s)dW(s)$ and $\Upsilon_2(t) = \int_0^t R(t-s)h(s)dW(s) - \int_{-\infty}^0 R(t-s)g(s)dW(s)$.

First we prove that $\Upsilon_1(t) \in AA\,(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H}))$. Let $\{s'_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence of real numbers. Since $g \in AA\,(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H}))$, there exists a subsequence $\{s_n\}_{n\in\mathbb{N}}$ of $\{s'_n\}_{n\in\mathbb{N}}$ such that for a certain stochastic process $\widetilde{g}$

$$\lim_{n\to\infty} E\|g(t+s_n) - \widetilde{g}(t)\|^2 = 0 \quad\text{and}\quad \lim_{n\to\infty} E\|\widetilde{g}(t-s_n) - g(t)\|^2 = 0. \tag{3.1}$$

hold for each $t \in \mathbb{R}$. Now, let $\widetilde{W}(\sigma) := W(\sigma+s_n) - W(s_n)$ for each $\sigma \in \mathbb{R}$. Note that $\widetilde{W}$ is also a Brownian motion and has the same distribution as W. Moreover, if we let $\widetilde{\Upsilon_1}(t) = \int_{-\infty}^t R(t-s)\widetilde{g}(s)dW(s)$, then by making a change of variables $\sigma = s - s_n$ to get (please see equation (10.6.6) in [25])

$$\begin{aligned}&E\|\Upsilon_1(t+s_n) - \widetilde{\Upsilon_1}(t)\|^2\\ &= E\left\|\int_{-\infty}^{t+s_n} R(t+s_n-s)g(s)dW(s) - \int_{-\infty}^t R(t-s)\widetilde{g}(s)dW(s)\right\|^2\\ &= E\left\|\int_{-\infty}^t R(t-\sigma)[g(\sigma+s_n) - \widetilde{g}(\sigma)]d\widetilde{W}(\sigma)\right\|^2\end{aligned}$$

and hence, using the Ito's isometry property of stochastic integral, we have the following estimations

$$\begin{aligned}E\|\Upsilon_1(t+s_n) - \widetilde{\Upsilon_1}(t)\|^2 &\leq E\left(\int_{-\infty}^t \|R(t-\sigma)[g(\sigma+s_n) - \widetilde{g}(\sigma)]\|^2 d\sigma\right)\\ &\leq M^2 \int_{-\infty}^t e^{-2\delta(t-\sigma)} E\|g(\sigma+s_n) - \widetilde{g}(\sigma)\|^2 d\sigma\\ &\leq \frac{M^2}{2\delta} \sup_{t\in\mathbb{R}} E\|g(t+s_n) - \widetilde{g}(t)\|^2.\end{aligned}$$

Thus, by (3.1), we immediately obtain that

$$\lim_{n\to\infty} E\|\Upsilon_1(t+s_n) - \widetilde{\Upsilon_1}(t)\|^2 = 0$$

for each $t \in \mathbb{R}$. A similar reasoning establishes that

$$\lim_{n\to\infty} E\|\widetilde{\Upsilon_1}(t-s_n) - \Upsilon_1(t)\|^2 = 0$$

for each $t \in \mathbb{R}$. Thus we conclude that $\Upsilon_1(\cdot) \in AA\,(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H}))$.

Next, let us show that $\Upsilon_2 \in C_0(\mathbb{R}^+; L^2(\mathbb{P},\mathbb{H}))$. Since $h \in C_0(\mathbb{R}^+; L^2(\mathbb{P},\mathbb{H}))$, for any sufficiently small $\varepsilon > 0$, there exists a constant $T > 0$ such that $E\|h(s)\|^2 \leq \varepsilon$ for all $s \geq T$. Then, for all $t \geq 2T$, we obtain

$$\begin{aligned}
E\|\Upsilon_2(t)\|^2 &= E\left\|\int_0^{\frac{t}{2}} R(t-s)h(s)dW(s) + \int_{\frac{t}{2}}^t R(t-s)h(s)dW(s)\right. \\
&\quad \left. - \int_{-\infty}^0 R(t-s)g(s)dW(s)\right\|^2 \\
&\leq 3E\left(\int_0^{\frac{t}{2}} \|R(t-s)h(s)\|^2 ds\right) + 3E\left(\int_{\frac{t}{2}}^t \|R(t-s)h(s)\|^2 ds\right) \\
&\quad + 3E\left(\int_{-\infty}^0 \|R(t-s)g(s)\|^2 ds\right) \\
&\leq 3M^2 \int_0^{\frac{t}{2}} e^{-2\delta(t-s)} E\|h(s)\|^2 ds + 3M^2 \int_{\frac{t}{2}}^t e^{-2\delta(t-s)} E\|h(s)\|^2 ds \\
&\quad + 3M^2 \int_{-\infty}^0 e^{-2\delta(t-s)} E\|g(s)\|^2 ds \\
&\leq \frac{3M^2}{2\delta}[e^{-\delta t} - e^{-2\delta t}] \sup_{t\in\mathbb{R}^+} E\|h(t)\|^2 \\
&\quad + \frac{3M^2}{2\delta}[1 - e^{-\delta t}]\varepsilon + \frac{3M^2}{2\delta} e^{-2\delta t} \sup_{t\in\mathbb{R}} E\|g(t)\|^2 \\
&\leq 3M^2 2\delta M_h e^{-\delta t} + \frac{3M^2}{2\delta}\varepsilon + \frac{3M^2}{2\delta} M_g e^{-2\delta t},
\end{aligned}$$

where $M_h = \sup_{t\in\mathbb{R}^+} E\|h(t)\|^2$ and $M_g = \sup_{t\in\mathbb{R}} E\|g(t)\|^2$. Therefore, the last estimation converges to zero as $t \to +\infty$ since ε is arbitrary. Thus, it leads to $\lim_{t\to+\infty} E\|\Upsilon_2(t)\|^2 = 0$. Recalling that $\Upsilon(t) = \Upsilon_1(t) + \Upsilon_2(t)$ for all $t \geq 0$, we get $\Upsilon(t) \in AAA(\mathbb{R}^+; L^2(\mathbb{P},\mathbb{H}))$. The proof is completed. □

Let us list the following assumptions:

(H1) There exists a resolvent operator $\{R(t) : t \geq 0\}$ of Eq. (1.1) and $\{R(t) : t \geq 0\}$ is uniformly exponentially stable, that is, there are constants $M, \delta > 0$ such that $\|R(t)\| \leq Me^{-\delta t}$ for all $t \geq 0$.

(H2) The function $f \in AAA(\mathbb{R}^+ \times L^2(\mathbb{P},\mathbb{H}); L^2(\mathbb{P},\mathbb{H}))$ and there exists a continuous and nondecreasing function $L_f : [0, +\infty) \to [0, +\infty)$ such that for each $r \geq 0$ and for all $E\|x\|^2 \leq r$, $E\|y\|^2 \leq r$,

$$E\|f(t,x) - f(t,y)\|^2 \leq L_f(r)E\|x-y\|^2$$

for all $t \in \mathbb{R}^+$.

(H3) $\Theta := \sup_{r>0}\left[\frac{\delta r}{M^2} - 2rL_f(r)\right] > 2\delta E\|x_0\|^2 + 2\sup_{s\in\mathbb{R}^+} E\|f(s,0)\|^2$.

Now, we are ready to establish our main results.

Theorem 3.2. *Assume that* (H1)–(H3) *hold. Then the problem* (1.1) *has a unique square-mean asymptotically almost automorphic mild solution.*

Proof. We define the operator $\Lambda : AAA\left(\mathbb{R}^{+}; L^{2}(\mathbb{P}, \mathbb{H})\right) \to AAA\left(\mathbb{R}^{+}; L^{2}(\mathbb{P}, \mathbb{H})\right)$ by

$$\begin{aligned}(\Lambda x)(t) &= R(t)x_0 + \int_0^t R(t-s) f\left(s, x(s)\right) dW(s) \\ &= \Lambda_1(t) + \Lambda_2(t), \quad t \geq 0,\end{aligned}$$

where $\Lambda_1(t) = R(t)x_0$ and $\Lambda_2(t) = \int_0^t R(t-s) f\left(s, x(s)\right) dW(s)$.

First we prove that $\Lambda\left(AAA\left(\mathbb{R}^{+}; L^{2}(\mathbb{P}, \mathbb{H})\right)\right) \subset AAA\left(\mathbb{R}^{+}; L^{2}(\mathbb{P}, \mathbb{H})\right)$. Given $x \in AAA\left(\mathbb{R}^{+}; L^{2}(\mathbb{P}, \mathbb{H})\right)$, it is not difficult to prove that Λx is continuous. Since $x(t)$ is bounded, we can choose a bounded subset K of $L^2(\mathbb{P}, \mathbb{H})$ such that $x(t) \in K$ for all $t \in \mathbb{R}^{+}$. It follows from (H2) that $f(t,x)$ is uniformly continuous on the bounded subset K uniformly for $t \in \mathbb{R}^{+}$. Then, Theorem 2.16 yields that $f\left(\cdot, x(\cdot)\right) \in AAA\left(\mathbb{R}^{+}; L^{2}(\mathbb{P}, \mathbb{H})\right)$. Now, by Lemma 3.1, we have

$$\Lambda_2(t) := \int_0^t R(t-s) f\left(s, x(s)\right) dW(s) \in AAA\left(\mathbb{R}^{+}; L^{2}(\mathbb{P}, \mathbb{H})\right).$$

On the other hand, since $R(\cdot)$ is uniformly exponentially stable, it follows that

$$\lim_{t\to+\infty} E\|\Lambda_1(t)\|^2 = 0.$$

Thus, $\Lambda x \in AAA\left(\mathbb{R}^{+}; L^{2}(\mathbb{P}, \mathbb{H})\right)$.

Now, by (H4), there exists a constant $r > 0$ such that

$$\frac{\delta r}{M^2} - 2rL_f(r) > 2\delta E\|x_0\|^2 + 2\sup_{s\in\mathbb{R}^{+}} E\|f(s,0)\|^2. \tag{3.2}$$

Let $\mathbb{D} = \{x \in AAA\left(\mathbb{R}^{+}; L^{2}(\mathbb{P}, \mathbb{H})\right) : \|x\|_\infty \leq r\}$. Then $\mathbb{D}$ is a closed subspace of $AAA\left(\mathbb{R}^{+}; L^{2}(\mathbb{P}, \mathbb{H})\right)$. We claim that $\Lambda\mathbb{D} \subseteq \mathbb{D}$. In fact, for any given $x \in \mathbb{D}$ and $t \in \mathbb{R}^{+}$, we get

$$\begin{aligned}E\|(\Lambda x)(t)\|^2 &\leq 2M^2 E\|x_0\|^2 + 2E\left\|\int_0^t R(t-s) f\left(s, x(s)\right) dW(s)\right\|^2 \\ &\leq 2M^2 E\|x_0\|^2 + 2E\left(\int_0^t \|R(t-s) f\left(s, x(s)\right)\|^2 ds\right) \\ &\leq 2M^2 E\|x_0\|^2 + 2M^2 \int_0^t e^{-2\delta(t-s)} E\|f\left(s, x(s)\right)\|^2 ds \\ &\leq 2M^2 E\|x_0\|^2 + 4M^2 \left[\int_0^t e^{-2\delta(t-s)} L_f(r) r ds\right. \\ &\quad \left. + \int_0^t e^{-2\delta(t-s)} E\|f\left(s, 0\right)\|^2 ds\right] \\ &\leq 2M^2 E\|x_0\|^2 + \frac{2M^2}{\delta}\left[L_f(r) r + \sup_{s\in\mathbb{R}^{+}} E\|f(s,0)\|^2\right],\end{aligned}$$

which from (3.2) implies that $\|\Lambda x\|_\infty \leq r$ and so that $\Lambda\mathbb{D} \subseteq \mathbb{D}$.

Next, we prove that $\Lambda(\cdot)$ is a contraction mapping on $\mathbb{D}$. From (3.2) we know that $\frac{\delta r}{M^2} - 2rL_f(r) > 0$, i.e., $\frac{\delta r}{M^2} > 2rL_f(r)$. Therefore, we have

$$\frac{2M^2}{\delta}L_f(r) < 1. \tag{3.3}$$

For any $x, y \in \mathbb{D}$ and $t \geq 0$, we see that

$$\begin{aligned} E\|(\Lambda x)(t) - (\Lambda y)(t)\|^2 &\leq E\left\|\int_0^t R(t-s)[f(s,x(s)) - f(s,y(s))]dW(s)\right\|^2 \\ &\leq M^2\int_0^t e^{-2\delta(t-s)}E\|f(s,x(s)) - f(s,y(s))\|^2 ds \\ &\leq \frac{M^2}{2\delta}L_f(r)\sup_{t\in\mathbb{R}^+}E\|x(t)-y(t)\|^2 \end{aligned}$$

Hence

$$\|\Lambda x - \Lambda y\|_\infty = \sup_{t\in\mathbb{R}^+}\left(E\|(\Lambda x)(t) - (\Lambda y)(t)\|^2\right)^{\frac{1}{2}} \leq \sqrt{\frac{M^2}{2\delta}L_f(r)}\,\|x-y\|_\infty .$$

By (3.3), Λ is a contraction from $\mathbb{D}$ into $\mathbb{D}$. So by the Banach contraction principle, we draw a conclusion that there exists a unique fixed point $x(\cdot)$ for Λ in $\mathbb{D}$. It is clear that x is a square-mean asymptotically almost automorphic mild solution of Eq. (1.1). The proof is now complete. $\square$

Corollary 3.3. *Assume that* (H1)–(H2) *hold. If $L_f(\cdot) \equiv L_f$ and $2L_f < \frac{\delta}{M^2}$, then there exists a unique square-mean asymptotically almost automorphic mild solution to Eq.* (1.1).

Proof. Consider the nonlinear operator Λ given by

$$(\Lambda x)(t) \;:=\; R(t)x_0 + \int_0^t R(t-s)f(s,x(s))\,dW(s), \quad t \geq 0.$$

Then, from the proof of Theorem 3.2, we can see that Λ maps $AAA(\mathbb{R}^+; L^2(\mathbb{P},\mathbb{H}))$ into itself. Note that

$$2\delta E\|x_0\|^2 + 2\sup_{s\in\mathbb{R}^+}E\|f(s,0)\|^2 < +\infty,$$

since $2L_f < \frac{\delta}{M^2}$, there exists a constant $r_0 > 0$ such that for all $r \geq r_0$, we have

$$\frac{\delta r}{M^2} - 2rL_f(r) > 2\delta E\|x_0\|^2 + 2\sup_{s\in\mathbb{R}^+}E\|f(s,0)\|^2.$$

In the following, using the same proof as in Theorem 3.2, we know that there exists a unique mild solution $x \in AAA(\mathbb{R}^+; L^2(\mathbb{P},\mathbb{H}))$ to Eq. (1.1). $\square$

4. APPLICATIONS

In this section, we apply our previous existence results to study the existence of square-mean asymptotically almost automorphic mild solutions for the stochastic partial integro-differential equation

$$\begin{aligned} \theta''(t) + \beta(0)\theta'(t) &= \alpha(0)\Delta\theta(t) - \int_0^t \beta'(t-s)\theta'(s)ds \\ &\quad + \int_0^t \alpha'(t-s)\Delta\theta(s)ds + a_1(t)a_2(\theta(t))W(t), \end{aligned} \tag{4.1}$$

where $\alpha(\cdot)$, $\beta(\cdot)$ are real valued functions of class C^2 on $[0, +\infty)$ with $\alpha(0) > 0$, $\beta(0) > 0$, and $W(t)$ is a two-sided standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathscr{F}, \mathbb{P}, \mathscr{F}_t)$.

As was observed in [23, 26], the deterministic system of the type (4.1) arise in the study of heat conduction in materials with fading memory. In [24], the authors established the existence of almost periodic and asymptotically almost periodic solutions for the partial integro-differential equation

$$\begin{aligned} C\theta''(t) + \beta(0)\theta'(t) &= \alpha(0)\Delta\theta(t) - \int_0^t \beta'(t-s)\theta'(s)ds \\ &\quad + \int_0^t \alpha'(t-s)\Delta\theta(s)ds + a_1(t)a_2(\theta(t)), \end{aligned}$$

where $\alpha(\cdot), \beta(\cdot)$ are $\mathbb{R}$-valued functions of class C^2 on $[0, +\infty)$ with $\alpha(0) > 0, \beta(0) > 0$. Recently, in [19], the authors studied the existence of asymptotically almost automorphic mild solutions to the partial integro-differential equation of the form

$$\begin{aligned} \theta''(t) + \beta(0)\theta'(t) &= \alpha(0)\Delta\theta(t) - \int_0^t \beta'(t-s)\theta'(s)ds \\ &\quad + \int_0^t \alpha'(t-s)\Delta\theta(s)ds + a(t)b(\theta(t)), \end{aligned}$$

where Ω is a bounded open connected subset of $\mathbb{R}^3$ with C^∞ boundary and $\alpha, \beta \in C^2([0, +\infty), \mathbb{R})$ with $\alpha(0)$ and $\beta(0)$ positive. Now, we go back to investigate the existence of square-mean asymptotically almost automorphic mild solutions to (4.1).

Throughout the rest of this section, we take $\mathbb{X} = H_0^1(\Omega) \times L^2(\Omega)$ where $\Omega \subset \mathbb{R}^3$ is a bounded open connected subset with smooth boundary of class C^∞. We consider the linear operator $A : D(A) = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega) \to \mathbb{X}$ is defined by

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ \alpha(0)\Delta - \beta(0)y \end{pmatrix}$$

where Δ is the Laplacian on Ω with boundary conditions $\theta|_{\partial\Omega} = 0$. It follows from [15] that A is the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ on $\mathbb{X}$ and that there are constants $\overline{M}, \varpi > 0$ such that $\|T(t)\| \leq \overline{M}e^{-\varpi t}$ for all $t \geq 0$.

We suppose that $B(t) = F(t)A$ where $F : \mathbb{X} \to \mathbb{X}$ is defined by

$$F(t) = [F_{ij}(t)] = \begin{pmatrix} 0 & 0 \\ -\beta'(t) + \beta(0)\frac{\alpha'(t)}{\alpha(0)} & \frac{\alpha'(t)}{\alpha(0)} \end{pmatrix}$$

and assume that $\alpha'(\cdot), \alpha''(\cdot)$, $\beta'(\cdot), \beta''(\cdot)$ are bounded and uniformly continuous, and for all $t \geq 0$

$$\max\{\|F_{21}(t)\|, \|F_{22}(t)\|\} \leq \frac{\varpi e^{-\varpi t}}{2\overline{M}},$$
$$\max\{\|F'_{21}(t)\|, \|F'_{22}(t)\|\} \leq \frac{\varpi^2 e^{-\varpi t}}{4\overline{M}^2}.$$

Then, Eq. (4.1) takes the following abstract form

$$dx(t) = \left[Ax(t) + \int_0^t B(t-s)x(s)ds\right]dt + f\left(t, x(t)\right)dW(t),\ t \geq 0, \tag{4.2}$$

where

$$x(t) = \begin{pmatrix} \theta(t) \\ \eta(t) \end{pmatrix} \quad \text{and} \quad f(t,x) = \begin{pmatrix} 0 \\ a_1(t)a_2(\theta) \end{pmatrix} \quad \text{for} \quad x = \begin{pmatrix} \theta \\ \eta \end{pmatrix} \in \mathbb{X}.$$

Moreover, it follows from [23] that the abstract integro-differential system has an associated uniformly exponentially stable resolvent of operator $\{R(t)\}_{t\geq 0}$ on $\mathbb{X}$ with $\|R(t)\| \leq \overline{M}e^{-\frac{\varpi t}{2}}$ for $t \geq 0$.

Let $a_1(t) \in AAA(\mathbb{R}^+;\mathbb{R})$ and $a_2 : L^2\left(\mathbb{P}, H_0^1(\Omega)\right) \to L^2\left(\mathbb{P}, L^2(\Omega)\right)$ satisfies

$$E\|a_2(\theta_1) - a_2(\theta_2)\|^2 \leq \frac{\varpi}{6\overline{M}^2 \sup_{t\in\mathbb{R}^+}\|a_1(t)\|^2} E\|\theta_1 - \theta_2\|^2 \tag{4.3}$$

for all $\theta_1, \theta_2 \in L^2\left(\mathbb{P}, H_0^1(\Omega)\right)$. We claim that for each $\theta_0 \in L^2\left(\mathbb{P}, H_0^1(\Omega)\right)$ and $\eta_0 \in L^2\left(\mathbb{P}, L^2(\Omega)\right)$, Eq. (4.2) with the initial value condition

$$x(0) = \begin{pmatrix} \theta_0 \\ \eta_0 \end{pmatrix} \tag{4.4}$$

satisfy all the conditions of Corollary 3.1. Clearly, (H1) holds. Since $a_1(t) \in AAA(\mathbb{R}^+;\mathbb{R})$, $f \in AAA\left(\mathbb{R}^+ \times L^2(\mathbb{P},\mathbb{X}); L^2(\mathbb{P},\mathbb{X})\right)$. Take

$$x = \begin{pmatrix} \theta_1 \\ \eta_1 \end{pmatrix}, \quad y = \begin{pmatrix} \theta_2 \\ \eta_2 \end{pmatrix} \in L^2(\mathbb{P},\mathbb{X}).$$

By (4.3), we have

$$\begin{aligned} E\|f(t,x) - f(t,y)\|^2 &= E\|a_1(t)[a_2(\theta_1) - a_2(\theta_2)]\|^2 \\ &\leq \sup_{t\in\mathbb{R}^+}\|a_1(t)\|^2 E\|a_2(\theta_1) - a_2(\theta_2)\|^2 \\ &\leq \frac{\varpi}{6\overline{M}^2} E\|\theta_1 - \theta_2\|^2 \\ &\leq \frac{\varpi}{6\overline{M}^2} E\|x - y\|^2 \end{aligned}$$

for all $t \geq 0$. Hence, (H2) holds with $L_f \equiv \frac{\varpi}{6\overline{M}^2}$.

Notice that $\delta = \frac{\varpi}{2}$, $M = \overline{M}$. Therefore

$$2L_f = \frac{\varpi}{3\overline{M}^2} < \frac{\varpi}{2\overline{M}^2} = \frac{\delta}{M^2}.$$

Thus, by Corollary 3.1, Eq. (4.2) with the initial value condition (4.4) has a unique square-mean asymptotically almost automorphic mild solution.

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REFERENCES

[1] P. Bezandry and T. Diagana, Existence of almost periodic solutions to some stochastic differential equations, *Appl. Anal.* 86:819–827, 2007.

[2] P. Bezandry, Existence of almost periodic solutions to some functional integro-differential stochastic evolution equations, *Statistics and Probability Letters*, 78: 2844–2849, 2008.

[3] P. Bezandry and T. Diagana, Square-mean almost periodic solutions nonautonomous stochastic differential equations, *Electron. J. Differential Equations*, 2007:1–10, 2007.

[4] P. Bezandry, Existence of almost periodic solutions to some functional integro-differential stochastic evolution equations, *Statistics and Probability Letters*, 78:2844–2849, 2008.

[5] S. Bochner, A new approach to almost automorphicity, *Proc. Natl. Acad. Sci. USA*, 48:2039–2043, 1962.

[6] S. Bochner, Continuous mappings of almost automorphic and almost periodic functions, *Proc. Natl. Acad. Sci. USA*, 52:907–910, 1964.

[7] D. Bugajewski and G. M. N'Guérékata, On the topological structure of almost automorphic and asymptotically almost automorphic solutions of differential and integral equations in abstract spaces, *Nonlinear Anal.*, 59:1333–1345, 2004.

[8] J. F. Cao, Q. G. Yang and Z. T. Huang, Existence and exponential stability of almost automorphic mild solutions for stochastic functional differential equations, *Stochastics*, 83:259–275, 2011.

[9] J. F. Cao, Q. G. Yang and Z. T. Huang, On almost periodic mild solutions for stochastic functional differential equations, *Nonlinear Anal.*, 13:275–286, 2012.

[10] Y. K. Chang, Z.H. Zhao and G. M. N'Guérékata, A new composition theorem for square-mean almost automorphic functions and applications to stochastic differential equations, *Nonlinear Anal.*, 74:2210–2219, 2011.

[11] Y. K. Chang, Z.H. Zhao and G. M. N'Guérékata, Square-mean almost automorphic mild solutions to non-autonomous stochastic differential equations in Hilbert spaces, *Comput. Math. Appl.*, 61:384–391, 2011.

[12] Y. K. Chang, Z.H. Zhao and G. M. N'Guérékata, Square-mean almost automorphic mild solutions to some stochastic differential equations in a Hilbert space, *Advances in Difference Equations*, 2011, 2011:9.

[13] Y. K. Chang, Z.H. Zhao, G. M. N'Guérékata and R. Ma, Stepanov-like almost automorphy for stochastic processes and applications to stochastic differential equations, *Nonlinear Anal. RWA*, 12:1130–1139, 2011.

[14] Z. Chen and W. Lin, Square-mean pseudo almost automorphic process and its applications to stochastic evolution equations, *J. Funct. Anal.*, 261:69–89, 2011.

[15] G. Chen, Control and stabilization for the wave equation in a bounded domain, *SIAM J. Control*, 17:66–81, 1979.

[16] W. Desch, R. Grimmer and W. Schappacher, Some considerations for linear integro-differential equations, *J. Math. Anal. Appl.*, 104:219–234, 1984.

[17] W. Desch, R. Grimmer and W. Schappacher, Well-posedness and wave propagation for a class of integro-differential equations in Banach space, *J. Differential Equations*, 74:391–411, 1988.

[18] T. Diagana, E. Hernández and J. P. C. dos Santos, Existence of asymptotically almost automorphic solutions to some abstract partial neutral integro-differential equations, *Nonlinear Anal.*, 71:248–257, 2009.

[19] H. S. Ding, T. J. Xiao and J. Liang, Asymptotically almost automorphic solutions for some integro-differental equations with nonlocal initial conditions, *J. Math. Anal. Appl.*, 338:141–151, 2008.

[20] J. P. C. dos Santos and C. Cuevas, Asymptotically almost automorphic solutions of abstract fractional integro-differential neutral equations, *Appl. Math. Lett.*, 23:960–965, 2010.

[21] M. M. Fu and Z. X. Liu, Square-mean almost automorphic solutions for some stochastic differential equations, *Proc. Amer. Math. Soc.*, 138:3689–3701, 2010.

[22] M. M. Fu, Almost automorphic solutions for nonautonomous stochastic differential equations, *J. Math. Anal. Appl.*, 393:231–238, 2012.

[23] R. Grimmer, Resolvent operators for integral equations in a Banach space, *Trans. Amer. Math. Soc.* 273:333–349, 1982.

[24] E. Hernández and J. P. C. dos Santos, Asymptotically almost periodic and almost periodic solutions for a class of partial integro-differential equations, *Electron. J. Differential Equations*, 2006:1–8, 2006.

[25] H. H. Kuo, *Introduction to Stochastic Integration*, Springer, New York, Boston, Dordrecht, London, Moscow, 2006.

[26] R. K. Miller, An integro-differential equation for rigid heat conductors with memory, *J. Math. Anal. Appl.* 66:313–332, 1978.

[27] G. M. N'Guérékata, *Almost automorphic and almost periodic functions in abstract space*, Kluwer Academic Plenum Publishers, New York, London, Moscow, 2001.

[28] G. M. N'Guérékata, *Topics in Almost Automorphy*, Springer, New York, Boston, Dordrecht, London Moscow, 2005.

[29] G. M. N'Guérékata, Sur les solutions presqu'automorphes d'équations différentielles abstraites, *Ann. Sci. Math. Québec*, 5:69–79, 1981.

[30] R. Sakthivel, P. Revathi and S. Marshal Anthoni, Existence of pseudo almost automorphic mild solutions to stochastic fractional differential equations, *Nonlinear Anal.*, 75:3339–3347, 2012.

[31] J. Q. Zhao, Y. K. Chang and G. M. N'Guérékata, Existence of asymptotically almost automorphic solutions to nonlinear delay integral equations, *Dynam. Syst. Appl.*, 21:339–350, 2012.