Dynamic Systems and Applications 22 (2013) 285-300

BIFURCATION TYPE RESULTS FOR NONLINEAR EIGENVALUE NEUMANN PROBLEMS

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ABSTRACT. We consider a parametric nonlinear elliptic Neumann problem driven by a nonhomogeneous differential operator. Using variational methods combined with truncation and comparison techniques, we prove a bifurcation-type theorem describing the dependence of the set of positive solutions on the parameter $\lambda > 0$.

AMS (MOS) Subject Classification. 35J20, 35J80

1. PRELIMINARIES

Let $\Omega \subseteq \mathbb{R}^{\mathbb{N}}$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following nonlinear Neumann eigenvalue problem

$$(1.1) \qquad \left\{ \begin{array}{l} -\text{div } (a(Du(z)) + \beta(z)|u(z)|^{p-2}u(z) = \lambda f(z, u(z)) \text{ in } \Omega, \\ \frac{\partial u}{\partial n} = 0, \text{ on } \partial\Omega, \ u > 0, \ \lambda > 0. \end{array} \right\}$$

Here $a : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ is a map which is continuous, strictly monotone and satisfies certain other regularity conditions which are presented in hypotheses $H(a)$. These hypotheses incorporate as special cases several differential operators of interest and provide a unifying framework to deal with such equations. So, our setting includes equations driven by the p-Laplacian ($1 < p < \infty$), the (p, q)-Laplacian ($1 < q < p < \infty$) and the generalized p-mean curvature operator. Also, $\beta \in L^{\infty}(\Omega)_+ \backslash \{0\}$, $\lambda > 0$ is a parameter (the "eigenvalue") and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory reaction (i.e., for all $x \in \mathbb{R}$, $z \to f(z, x)$ is measurable and for a.a. $z \in \Omega$, $x \to f(z, x)$ is continuous). Finally $n(\cdot)$ denotes the outward unit normal on $\partial\Omega$. We are looking for positive solutions of problem (1.1) and more precisely, our aim is to investigate the dependence on the parameter $\lambda > 0$ of the set of positive solutions of problem (1.1).

Received December 20, 2012 1056-2176 $15.00

In this direction, we prove a bifurcation-type result for problem (1.1), showing that there exists a critical parameter value $\lambda_* > 0$ such that for all $\lambda > \lambda_*$ problem (1.1) has at least two nontrivial positive smooth solutions, for $\lambda = \lambda_*$ problem (1.1) has at least one nontrivial positive smooth solution and finally for $\lambda \in (0, \lambda_*)$ problem (1.1) has no nontrivial positive solutions.

Nonlinear eigenvalue problems driven by the p-Laplace differential operator with Dirichlet boundary condition, were studied by Brock-Itturiaga-Ubilla [3], Dong [6], Filippakis-O'Regan-Papageorgiou [7], Guo [9], Hu-Papageorgiou [10], Perera [16], and Takeuchi [19]. From the aforementioned works only [3], [7], [10] prove bifurcation-type results describing the precise dependence of the set of positive solutions on the parameter $\lambda > 0$. Parametric equations with Neumann boundary condition, were studied by Motreanu-Motreanu-Papageorgiou [14]. In [14] the eigenvalue problem is different and the authors do not prove a bifurcation type theorem. We should mention that in contrast to the p-Laplacian, our differential operator here is not in general homogeneous and this is the source of difficulties in the analysis of problem (1.1).

Our approach is variational, and uses also suitable truncation and comparison techniques. In the next section, we state the hypotheses on the data of problem (1.1) and we prove some auxiliary results which we will need in the sequel.

2. HYPOTHESES-AUXILIARY RESULTS

Let $h \in C^1(0, +\infty)$ such that

$$\left\{\begin{array}{l} 0 < \frac{th'(t)}{h(t)} \leq c_0 \text{ for all } t > 0 \text{ and some } c_0 > 0, \\ \text{and } c_1 t^{p-1} \leq h(t) \leq c_2(t^{q-1} + t^{p-1}) \\ \text{for all } t > 0 \text{ and some } c_1, c_2 > 0, \ \ 1 < q < p < \infty. \end{array}\right\} \tag{2.1}$$

The hypotheses on the map $a(\cdot)$ are the following:

$\underline{H(a)}$: $a(y) = a_0(\|y\|)y$ for all $y \in \mathbb{R}^{\mathbb{N}}$ with $a_0(t) > 0$ for all $t > 0$ and

(i): $a_0 \in C^1(0, +\infty), ta_0(t) \to 0$ as $t \to 0^+$ and $\lim\limits_{t \to 0^+} \dfrac{ta_0'(t)}{a_0(t)} > -1$;

(ii): $\|\nabla a(y)\| \leq c_3 \frac{h(\|y\|)}{\|y\|}$ for all $y \in \mathbb{R}^{\mathbb{N}} \backslash \{0\}$ and some $c_3 > 0$;

(iii): $(\nabla a(y)\xi, \xi)_{\mathbb{R}^{\mathbb{N}}} \geq \frac{h(\|y\|)}{\|y\|}\|\xi\|^2$ for all $y \in \mathbb{R}^{\mathbb{N}} \backslash \{0\}$.

Remark 2.1. Evidently, hypothesis $H(a)(i)$ implies that $a \in C^1(\mathbb{R}^{\mathbb{N}} \backslash \{0\}, \mathbb{R}^{\mathbb{N}}) \cap C(\mathbb{R}^{\mathbb{N}}, \mathbb{R}^{\mathbb{N}})$ and so hypotheses $H(a)(ii), (iii)$ make sense. We set $G_0(t) = \int_0^t a_0(s)sds$ and consider the potential function $G : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ defined by $G(y) = G_0(\|y\|)$ for all $y \in \mathbb{R}^{\mathbb{N}}$. Then

$$\nabla G(y) = G_0'(\|y\|)\frac{y}{\|y\|} = a_0(\|y\|)y = a(y) \text{ for all } y \in \mathbb{R}^{\mathbb{N}} \backslash \{0\}$$

and $\nabla G(0) = 0$ (see $H(a)(i)$).

Note that $G(\cdot)$ is convex and since $G(0) = 0$, $\nabla G(y) = a(y)$ for all $y \in \mathbb{R}^{\mathbb{N}}$, we have

$$G(y) \leq (a(y), y)_{\mathbb{R}^{\mathbb{N}}} \text{ for all } y \in \mathbb{R}^{\mathbb{N}}. \tag{2.2}$$

From (2.1), (2.2), hypotheses $H(a)$ and the integral form of the mean value theorem, we have the following lemma summarizing the properties of the map $a(\cdot)$.

Lemma 2.2. *If hypotheses $H(a)$ hold, then*

a) *the map $y \to a(y)$ is maximal monotone and strictly monotone;*
b) $\|a(y)\| \leq c_4(\|y\|^{q-1} + \|y\|^{p-1})$ *for all $y \in \mathbb{R}^{\mathbb{N}}$ and some $c_4 > 0$;*
c) $(a(y), y)_{\mathbb{R}^{\mathbb{N}}} \geq \frac{c_1}{p-1}\|y\|^p$ *for all $y \in \mathbb{R}^{\mathbb{N}}$.*

Similarly, this lemma and (2.2) lead to the following growth estimates for the primitive $G(\cdot)$.

Corollary 2.3. *If hypotheses $H(a)$ hold, then $\frac{c_1}{p(p-1)}\|y\|^p \leq G(y) \leq c_5(\|y\|^q + \|y\|^p)$ for all $y \in \mathbb{R}^{\mathbb{N}}$ and some $c_5 > 0$.*

<u>Examples</u> The following maps satisfy hypotheses $H(a)$.

a) $a(y) = \|y\|^{p-2}y$ with $1 < p < \infty$.
This map corresponds to the p-Laplace differential operator

$$\triangle_p u = div(\|Du\|^{p-2}Du) \text{ for all } u \in W^{1,p}(\Omega).$$

The primitive is $G(y) = \frac{1}{p}\|y\|^p$ for all $y \in \mathbb{R}^{\mathbb{N}}$.

b) $a(y) = \|y\|^{p-2}y + \|y\|^{q-2}y$ with $1 < q < p < \infty$.
This map corresponds to the (p, q)-differential operator

$$\triangle_p u + \triangle_q u \text{ for all } u \in W^{1,p}(\Omega).$$

The primitive is $G(y) = \frac{1}{p}\|y\|^p + \frac{1}{q}\|y\|^q$ for all $y \in \mathbb{R}^{\mathbb{N}}$.

This differential operator is important in quantum physics in connection with the problem of existence of solitons, see Benci-D'Avenia-Fortunato-Pisani [2]. Equations driven by such differential operators, were studied recently by Cingolani-Degiovanni [5], Li-Zhang [13], Sun [18].

c) $a(y) = (1 + \|y\|^2)^{\frac{p-2}{p}}y$ with $1 < p < \infty$.
This map corresponds to the generalized p-mean curvature differential operator

$$div[(1 + \|Du\|^2)^{\frac{p-2}{p}}Du] \text{ for all } u \in W^{1,p}(\Omega).$$

The primitive is $G(y) = \frac{1}{p}[(1 + \|y\|^2)^{\frac{p}{2}} - 1]$ for all $y \in \mathbb{R}^{\mathbb{N}}$.

Equations monitored by this differential operator were investigated by Chen-Shen [4].

d) $a(y) = \|y\|^{p-2}y + \frac{\|y\|^{p-2}y}{1+\|y\|^p}$ with $1 < p < \infty$. item[] The primitive is $G(y) = \frac{1}{p}\|y\|^p + \frac{1}{p}ln(1+\|y\|^p)$ for all $y \in \mathbb{R}^{\mathbb{N}}$.

Remark 2.4. We should mention that in [14] the authors also deal with equations driven by a nonhomogeneous differential operator div $a(Du)$. However, the hypotheses on the map $a(\cdot)$ are more restrictive and exclude important cases such as the (p,q)-differential operator and the generalized p-mean curvature differential operator.

Let $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function with subcritical growth, i.e.,

$$|f_0(z,x)| \leq a(z) + c|x|^{r-1} \text{ for all } z \in \Omega, \text{ all } x \in \mathbb{R}$$

with $1 < r < p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } p \geq N. \end{cases}$. Let $F_0(z,x) = \int_0^x f_0(z,s)ds$ and consider the C^1-functional $\varphi_0 : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_0(u) = \int_\Omega G(Du(z))dz - \int_\Omega F_0(z,u(z))dz \text{ for all } u \in W^{1,p}(\Omega).$$

The next result relates Holder and Sobolev local minimizers of φ_0 and can be found in Motreanu-Papageorgiou [15].

Theorem 2.5. *If hypotheses $H(a)$ holds and $u_0 \in W^{1,p}(\Omega)$ is a local $C^1(\overline{\Omega})$-minimizer of φ_0, i.e., there exists $\rho_0 > 0$ such that*

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h) \text{ for all } h \in C^1(\overline{\Omega}) \text{ with } \|h\|_{C^1(\overline{\Omega})} \leq \rho_0$$

then $u_0 \in C^{1,\gamma}(\overline{\Omega})$ for some $\gamma \in (0,1)$ and u_0 is also a local $W^{1,p}(\Omega)$ minimizer of φ_0, i.e., there exists $\rho_1 > 0$ such that

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h) \text{ for all } h \in W^{1,p}(\Omega) \text{ with } \|h\| \leq \rho_1.$$

Let X be a Banach space and $\varphi \in C^1(X)$. We say that φ satisfies the "Palais Smale condition" (the PS-condition), for short), if the following holds:

"Every sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that $\{\varphi(x_n)\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded and

$$\varphi'(x_n) \to 0 \text{ in } X^*,$$

admits a strongly convergent subsequence".

The next theorem is known in the literature as the "Mountain Pass Theorem" and characterizes certain critical values of φ.

Theorem 2.6. *If $\varphi \in C^1(X)$ satisfies the PS-condition, $x_0, x_1 \in X$, $\|x_1 - x_0\| > \rho > 0$*

$$\max\{\varphi(x_0), \varphi(x_1)\} < \inf[\varphi(x) : \|x - x_0\| = \rho] = \eta_\rho$$

and

$$c = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} \varphi(\gamma(t)) \text{ where } \Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = x_0, \gamma(1) = x_1\}$$

then $c \geq \eta_\rho$ and c is a critical value of φ.

Let $\langle \cdot, \cdot \rangle$ denote the duality brackets for the pair $(W^{1,p}(\Omega)^*, W^{1,p}(\Omega))$ and let $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ be the nonlinear map defined by

$$\langle A(u), y\rangle = \int_\Omega (a(Du), Dy)_{\mathbb{R}^N} dz \ \text{ for all } \ u, y \in W^{1,p}(\Omega). \tag{2.3}$$

From Gasinski-Papageorgiou [8, p. 652], we have:

Proposition 2.7. *If hypotheses $H(a)$ hold, then $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ defined by (2.3) is continuous, monotone (hence maximal monotone) and of type $(S)_+$, i.e., if $u_n \xrightarrow{w} u$ in $W^{1,p}(\Omega)$ and $\limsup_{n\to\infty} \langle A(u_n), u_n - u\rangle \leq 0$, then $u_n \to u$ in $W^{1,p}(\Omega)$.*

In the analysis of problem (1.1) in addition to the Sobolev space $W^{1,p}(\Omega)$, we will also use the Banach space $C^1(\overline{\Omega})$. This is an ordered Banach space with positive cone

$$C_+ = \{u \in C^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$int C_+ = \{u \in C_+ : u(z) > 0 \text{ for all } \ z \in \overline{\Omega}\}.$$

In what follows, by $\|\cdot\|$ we denote the norm of $W^{1,p}(\Omega)$. The same notation also denotes the $\mathbb{R}^N$-norm. However, no confusion is possible since it will always be clear from the context which norm is used. For $x \in \mathbb{R}$, we set $x^\pm = \max\{\pm x, 0\}$ and then for $u \in W^{1,p}(\Omega)$ we define $u^\pm(\cdot) = u(\cdot)^\pm$. We know that

$$u^\pm \in W^{1,p}(\Omega), |u| = u^+ + u^-, \ u = u^+ - u^-.$$

By $|\cdot|_{\mathbb{R}^N}$ we denote the Lebesgue measure on $\mathbb{R}^N$ and for any measurable function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ (for example a Caratheodory function), we introduce the map

$$N_g(u)(\cdot) = g(\cdot, u(\cdot)) \ \text{ for all } \ u \in W^{1,p}(\Omega)$$

(the Nemytskii map corresponding to g).

Next we introduce the hypotheses on the functions $z \to \beta(z)$ and $(z, x) \to f(z, x)$.

$\underline{H(\beta)}$: $\beta \in L^\infty(\Omega)$, $\beta \geq 0$, $\beta \neq 0$.

$\underline{H(f)}$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function such that $f(z, 0) = 0$ a.e.in Ω and

(i): for every $\rho > 0$, there exists $a_\rho \in L^\infty(\Omega)_+$ such that

$$f(z, x) \leq a_\rho(z) \ \text{ for a.a. } \ z \in \Omega, \text{ all } \ x \in [0, \rho];$$

(ii): for every $\rho > 0$, we can find $c_\rho > 0$ such that

$$f(z, x) \geq c_\rho \text{ for a.a. } \ z \in \Omega, \text{ all } x \geq \rho.$$

(iii): $\lim\limits_{x\to+\infty} \dfrac{f(z, x)}{x^{p-1}} = \lim\limits_{x\to 0^+} \dfrac{f(z, x)}{x^{p-1}} = 0$ uniformly for a.a. $z \in \Omega$.

(iv): for every $\rho > 0$, there exists $\xi_\rho > 0$ such that for a.a. $z \in \Omega$, $x \to f(z,x) + \xi_\rho x^{p-1}$ is nondecreasing on $[0, \rho]$.

Remark 2.8. Since we are looking for positive solutions and all the above hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0, +\infty]$, we may (and we will) assume that $f(z,x) = 0$ for a.a. $z \in \Omega$, all $x \leq 0$. Evidently hypothesis $H(f)(ii)$ implies that $f(z,x) > 0$ for a.a. $z \in \Omega$ and all $x > 0$.

Example The following function satisfies hypotheses $H(f)$ (for the sake of simplicity we drop the z-dependence):

$$f(x) = \begin{cases} x^{r-1} & \text{if } x \in [0,1], \\ x^{q-1} & \text{if } x > 1, \end{cases}, \text{ with } 1 < q < p < r < \infty.$$

Proposition 2.9. *If hypotheses $H(\beta)$ hold, then there exists $\xi^* > 0$ such that*

$$\frac{c_1}{p-1}\|Du\|_p^p + \int_\Omega \beta |u|^p dz \geq \xi^* \|u\|^p \ \textit{ for all } \ u \in W^{1,p}(\Omega).$$

Proof. Let $\psi : W^{1,p}(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$\psi(u) = \frac{c_1}{p-1}\|Du\|_p^p + \int_\Omega \beta |u|^p dz \ \text{ for all } \ u \in W^{1,p}(\Omega).$$

We need to show that $\psi(u) \geq \xi^* \|u\|^p$ for all $u \in W^{1,p}(\Omega)$.

Arguing by contradiction, suppose that the proposition is not true. Exploiting the p-homogeneity of $\psi(\cdot)$, we can find $\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega)$ such that

$$\|u_n\| = 1 \ \text{ for all } \ n \geq 1 \ \text{ and } \ \psi(u_n) \downarrow 0 \ \text{ as } \ n \to \infty.$$

By passing to a suitable subsequence if necessary, we may assume that

$$u_n \xrightarrow{w} u \ \text{ in } \ W^{1,p}(\Omega) \ \text{ and } \ u_n \to u \ \text{ in } \ L^p(\Omega).$$

It is easy to see that $\psi(\cdot)$ is sequentially weakly lower semicontinuous. So,

$$\begin{aligned} &\psi(u) \leq 0, \\ &\quad \Rightarrow \|Du\|_p = 0, \ \text{hence } u = \theta \in \mathbb{R}. \end{aligned}$$

If $\theta \neq 0$, then $\psi(u) = |\theta|^p \int_\Omega \beta dz \leq 0$, a contradiction.

So, $\theta = 0$ and we have $u \equiv 0$. Hence

$$\begin{aligned} &\|Du_n\|_p \to 0, \\ &\quad \Rightarrow u_n \to 0 \text{ in } W^{1,p}(\Omega), \end{aligned}$$

which contradicts the fact that $\|u_n\| = 1$ for all $n \geq 1$. ∎

Let

$$\widehat{\lambda}_1 = \inf[\frac{\frac{c_1}{p-1}\|Du\|_p^p + \int_\Omega \beta|u|^p dz}{\|u\|_p^p} : u \in W^{1,p}(\Omega), u \neq 0]$$

From Proposition 2.9 it follows that:

Proposition 2.10. *If hypotheses $H(\beta)$ hold, then $\widehat{\lambda}_1 \geq \xi^* > 0$.*

3. BIFURCATION-TYPE THEOREM

We introduce the set

$$\mathcal{P} = \{\lambda > 0 : \text{problem (1.1) has a nontrivial positive solution}\}.$$

We set

$$\lambda_* = \inf \mathcal{P} \ \ (\text{if } \ \mathcal{P} = \emptyset, \ \text{ then } \ \lambda_* = +\infty)$$

Also, if $\lambda \in \mathcal{P}$, then by $S(\lambda)$ we denote the set of nontrivial positive solutions of (1.1).

We start by establishing the properties of $S(\lambda)$ and of λ_*.

Proposition 3.1. *If hypotheses $H(a), H(\beta)$ and $H(f)$ hold, then for all $\lambda > 0$, $S(\lambda) \subseteq intC_+$ and $\lambda_* > 0$.*

Proof. We may assume that $\lambda \in \mathcal{P}$ (if $\lambda \notin \mathcal{P}$, then $S(\lambda) = \emptyset$). Then we can find $u \in W^{1,p}(\Omega)$, $u \geq 0$, $u \neq 0$ such that

$$-\text{div } a(Du(z)) + \beta(z)u(z)^{p-1} = \lambda f(z, u(z)) \ \text{ in } \ \Omega, \ \frac{\partial u}{\partial n} = 0 \ \text{ on } \ \partial\Omega.$$

From Hu-Papageorgiou [11] (see Proposition 2.7) we have $u \in L^\infty(\Omega)$. Then from the nonlinear regularity result of Lieberman [12] (p. 320), we have $u \in C_+\backslash\{0\}$. Let $\rho = \|u\|_\infty$ and let $\xi_\rho > 0$ be as postulated by hypothesis $H(f)(iv)$. Then we have

$$\begin{aligned} &- \text{div } aD(u(z)) + (\beta(z) + \lambda\xi_\rho)u(z)^{p-1} \\ &= \lambda[f(z, u(z)) + \xi_\rho u(z)^{p-1}] \geq 0 \text{ a.e. in } \Omega, \\ &\Rightarrow \text{div } a(Du(z)) \leq (\|\beta\|_\infty + \lambda\xi_\rho)u(z)^{p-1} \ \text{ a.e. in } \Omega. \end{aligned} \tag{3.1}$$

From (3.1) and the strong maximum principle of Pucci-Serrin [17] (p. 34) we can apply the boundary point theorem of Pucci-Serrin [17] (p. 120) and infer that $u \in intC_+$. Therefore we have

$$S(\lambda) \subseteq intC_+.$$

Hypotheses $H(f)(i), (ii), (iii)$ imply that we can find $c_6 > 0$ such that

$$0 \leq f(z, x) \leq c_6 x^{p-1} \ \text{ for a.a. } \ z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{3.2}$$

Let $\tilde{\lambda} < \frac{\widehat{\lambda}_1}{c_6}$ and consider $\mu \in (0, \tilde{\lambda}]$. If $\mu \in \mathcal{P}$, then we can find $u_\mu \in S(\mu) \subseteq intC_+$ such that

$$A(u_\mu) + \beta u_\mu^{p-1} = \mu N_f(u_\mu). \tag{3.3}$$

Acting on (3.3) with u_μ and using Lemma 2.2(c), we obtain

$$\frac{c_1}{p-1}\|Du_\mu\|_p^p + \int_\Omega \beta|u_\mu|^p dz \le \mu \int_\Omega f(z,u_\mu)u_\mu dz$$
$$\Rightarrow \frac{c_1}{p-1}\|Du_\mu\|_p^p + \int_\Omega \beta|u_\mu|^p dz \le \mu c_6\|u_\mu\|_p^p \text{ (see (3.2))}$$
$$< \widehat{\lambda}_1\|u_\mu\|_p^p \text{ (since } \mu \le \tilde{\lambda}),$$
$$\Rightarrow \frac{c_1}{p-1}\|Du_\mu\|_p^p + \int_\Omega \beta|u_\mu|^p dz < \widehat{\lambda}_1\|u_\mu\|_p^p,$$

which contradicts (2.3). Therefore $\mu \notin \mathcal{P}$ and so $\lambda_* \ge \tilde{\lambda} > 0$. □

Next we show the nonemptiness of $\mathcal{P}$. In what follows $F(z,x) = \int_0^x f(z,s)ds$.

Proposition 3.2. *If hypotheses $H(a), H(\beta)$ and $H(f)$ hold, then $\mathcal{P} \neq \emptyset$.*

Proof. Hypotheses $H(f)(i),(ii),(iii)$ imply that given $\varepsilon > 0$ we can find $c_7 = c_7(\varepsilon) > 0$ such that

$$0 \le F(z,x) \le \frac{\varepsilon}{p}|x|^p + c_7 \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{3.4}$$

Then for every $u \in W^{1,p}(\Omega)$, we have

$$\varphi_\lambda(u) = \int_\Omega G(Du)dz + \frac{1}{p}\int_\Omega \beta|u|^p dz - \lambda\int_\Omega F(z,u)dz$$
$$\ge \frac{\xi^*}{p}\|u\|^p - \frac{\lambda\varepsilon}{p}\|u\|^p - \lambda c_7|\Omega|_N \tag{3.5}$$

(see Corollary 2.3, Proposition 2.9 and (3.4))

$$= \frac{1}{p}[\xi^* - \lambda\varepsilon]\|u\|^p - \lambda c_7|\Omega|_N. \tag{3.6}$$

Choosing $\varepsilon \in (0, \frac{\xi^*}{\lambda})$, from (3.5) we infer that φ_λ is coercive. Also, using the Sobolev embedding theorem, we check easily that φ_λ is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find $u_0 \in W^{1,p}(\Omega)$ such that

$$\varphi_\lambda(u_0) = \inf[\varphi_\lambda(u) : u \in W^{1,p}(\Omega)]. \tag{3.7}$$

If $\tilde{u} \in C_+\backslash\{0\}$, then $\int_\Omega F(z,\tilde{u})dz > 0$ and so for $\lambda > 0$ big, we have

$$\lambda\int_\Omega F(z,\tilde{u})dz > \int_\Omega G(D\tilde{u})dz + \frac{1}{p}\int_\Omega \beta|\tilde{u}|^p dz,$$
$$\Rightarrow \varphi_\lambda(\tilde{u}) < 0 = \varphi_\lambda(0),$$
$$\Rightarrow \varphi_\lambda(u_0) < 0 = \varphi_\lambda(0) \text{ (see (3.7)), hence } u_0 \neq 0.$$

From (3.7) and for $\lambda > 0$ big we have

$$\varphi_\lambda'(u_0) = 0 \text{ with } u_0 \neq 0,$$
$$\Rightarrow A(u_0) + \beta|u_0|^{p-2}u_0 = \lambda N_f(u_0). \tag{3.8}$$

On (3.8) we act with $-u_0^- \in W^{1,p}(\Omega)$ and obtain

$$\frac{c_1}{p-1}\|Du_0^-\|_p^p + \int_\Omega \beta(u_0^-)^p dz \leq 0 \ \text{ (see Lemma 2.2(c))},$$
$$\Rightarrow \xi^*\|u_0^-\|^p \leq 0$$

(see Proposition 2.9), hence $u_0 \geq 0$, $u_0 \neq 0$. Therefore $u_0 \in S(\lambda) \subseteq intC_+$ (see Proposition 3.1) for $\lambda > 0$ big and so $\mathcal{P} \neq \emptyset$. □

Proposition 3.3. *If hypotheses $H(a), H(\beta)$ and $H(f)$ hold and $\lambda \in \mathcal{P}$, then $[\lambda, +\infty) \subseteq \mathcal{P}$.*

Proof. Let $\mu > \lambda$. Since $\lambda \in \mathcal{P}$, we can find $u_\lambda \in S(\lambda) \subseteq intC_+$ (see Proposition 3.1). We consider the following truncation of the reaction in problem (1.1).

$$h_\mu(z,x) = \begin{cases} \mu f(z, u_\lambda(z)) & \text{if } x \leq u_\lambda(z), \\ \mu f(z,x) & \text{if } u_\lambda(z) < x, \end{cases}. \tag{3.9}$$

Clearly this is a Caratheodory function. We set $H_\mu(z,x) = \int_0^x h_\mu(z,s)ds$ and consider the C^1-functional $\psi_\mu : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi_\mu(u) = \int_\Omega G(Du)dz + \frac{1}{p}\int_\Omega \beta|u|^p dz - \int_\Omega H_\mu(z,u)dz \text{ for all } u \in W^{1,p}(\Omega).$$

As in the proof of Proposition 3.2, using hypotheses $H(f)(i),(iii)$, we show that ψ_μ is coercive. Also it is sequentially weakly lower semicontinuous.

So, we can find $u_\mu \in W^{1,p}(\Omega)$ such that

$$\psi_\mu(u_\mu) = \inf[\psi_\mu(u) : u \in W^{1,p}(\Omega)],$$
$$\Rightarrow \psi'_\mu(u_\mu) = 0,$$
$$\Rightarrow A(u_\mu) + \beta|u_\mu|^{p-2}u_\mu = N_{h_\mu}(u_\mu). \tag{3.10}$$

On (3.10) we act with $(u_\lambda - u_\mu)^+ \in W^{1,p}(\Omega)$. Then

$$\langle A(u_\mu), (u_\lambda - u_\mu)^+\rangle + \int_\Omega \beta|u_\mu|^{p-2}u_\mu(u_\lambda - u_\mu)^+ dz$$
$$= \int_\Omega \mu f(z,u_\lambda)(u_\lambda - u_\mu)^+ dz \ \text{ (see (3.9))}$$
$$\geq \int_\Omega \lambda f(z,u_\lambda)(u_\lambda - u_\mu)^+ dz \ \text{ (since } \lambda < \mu \text{ and } f \geq 0)$$
$$= \langle A(u_\lambda), (u_\lambda - u_\mu)^+\rangle + \int_\Omega \beta u_\lambda^{p-1}(u_\lambda - u_\mu)^+ dz$$
$$\Rightarrow \int_{\{u_\lambda > u_\mu\}} (a(Du_\lambda) - a(Du_\mu), Du_\lambda - Du_\mu)_{\mathbb{R}^N} dz$$
$$+ \int_{\{u_\lambda > u_\mu\}} \beta(u_\lambda^{p-1} - |u_\mu|^{p-2}u_\mu)(u_\lambda - u_\mu)dz \leq 0,$$
$$\Rightarrow |\{u_\lambda > u_\mu\}|_N = 0 \ \text{ (see Lemma 2.2(a)), hence } u_\lambda \leq u_\mu.$$

Therefore (3.10) becomes

$$\begin{aligned} a(u_\mu) + \beta u_\mu^{p-1} &= \mu N_f(u_\mu) \ \text{ (see (3.9))}, \\ &\Rightarrow u_\mu \in S(\mu) \subseteq intC_+ \text{ (see Proposition 3.1)}, \\ &\Rightarrow [\lambda, +\infty) \subseteq \mathcal{P}. \end{aligned}$$

□

As a consequence of Proposition 3.3, we have that $(\lambda_*, +\infty) \subseteq \mathcal{P}$.

Proposition 3.4. *If hypotheses* $H(a), H(\beta)$ *and* $H(f)$ *hold and* $\lambda > \lambda_*$ *then problem* (1.1) *has at least two nontrivial positive solutions*

$$u_0, \widehat{u} \in intC_+.$$

Proof. Let $\lambda_* < \theta < \lambda < \mu$ and let $u_\theta \in S(\theta) \subseteq intC_+$. We have

$$A(u_\theta) + \beta u_\theta^{p-1} = \theta N_f(u_\theta) \leq \mu N_f(u_\theta) \ \text{ in } \ W^{1,p}(\Omega)^* \tag{3.11}$$

(recall $\theta < \mu$ and $f \geq 0$).

Reasoning as in the proof of Proposition 3.3, we truncate $f(z, \cdot)$ at $u_\theta(z)$ and employ the direct method. Using (3.11), we obtain $u_\mu \in S(\mu) \subseteq intC_+$ such that $u_\theta \leq u_\mu$. Since $\theta < \lambda < \mu$ and $f \geq 0$, we have

$$A(u_\theta) + \beta u_\theta^{p-1} \geq \lambda N_f(u_\theta) \ \text{ in } \ W^{1,p}(\Omega)^*, \tag{3.12}$$

$$A(u_\mu) + \beta u_\mu^{p-1} \geq \lambda N_f(u_\mu) \ \text{ in } \ W^{1,p}(\Omega)^*, \tag{3.13}$$

Then we introduce the following Caratheodory fuction

$$e_\lambda(z, x) = \begin{cases} \lambda f(z, u_\theta(z)) & \text{if } x < u_\theta(z), \\ \lambda f(z, x) & \text{if } u_\theta(z) \leq x \leq u_\mu(z), \\ \lambda f(z, u_\mu(z) & \text{if } u_\mu(z) < x. \end{cases} \tag{3.14}$$

We set $E_\lambda(z, x) = \int_0^x e_\lambda(z, s) ds$ and consider the C^1-functional $\gamma_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\gamma_\lambda(u) = \int_\Omega G(Du) dz + \frac{1}{p} \int_\Omega \beta |u|^p dz - \int_\Omega E_\lambda(z, u) dz \ \text{ for all } \ u \in W^{1,p}(\Omega).$$

Evidently γ_λ is coercive (see Proposition 2.9 and (3.14)). Also, it is sequentially weakly lower semicontinuous. So, we can find $u_0 \in W^{1,p}(\Omega)$ such that

$$\begin{aligned} \gamma_\lambda(u_0) &= \inf[\gamma_\lambda(u) : u \in W^{1,p}(\Omega)], \\ &\Rightarrow \gamma_\lambda'(u_0) = 0, \\ &\Rightarrow A(u_0) + \beta |u_0|^{p-2} u_0 = N_{e_\lambda}(u_0). \end{aligned} \tag{3.15}$$

As in the proof of Proposition 3.3, acting on (3.15) with $(u_\theta - u_0)^+ \in W^{1,p}(\Omega)$ and using (3.12) and then acting on (3.15) with $(u_0 - u_\mu)^+ \in W^{1,p}(\Omega)$ and using (3.13), we show that

$$u_0 \in [u_\theta, u_\mu] = \{u \in W^{1,p}(\Omega) : u_\theta(z) \leq u(z) \leq u_\mu(z) \text{ a.e. in } \Omega\},$$
$$\Rightarrow u_0 \in S(\lambda) \subseteq intC_+ \text{ (see (3.14), (3.15))}.$$

Let $\rho = \|u_\mu\|_\infty$ and let $\xi_\rho > 0$ be as postulated by hypothesis $H(f)(iv)$. For $\delta > 0$, we set $u_0^\delta(z) = u_0(z) + \delta$ for all $z \in \overline{\Omega}$. Then $u_0^\delta \in intC_+$ and we have

(3.16)
$$\begin{aligned} & -\text{div } a(Du_0^\delta(z)) + (\beta(z) + \mu\xi_\rho)u_0^\delta(z)^{p-1} \\ & \leq -\text{div } a(Du_0(z)) + (\beta(z) + \mu\xi_\rho)u_0(z)^{p-1} + \zeta(\delta) \text{ with } \zeta(\delta) \to 0^+ \text{ as } \delta \to 0^+ \\ & = \lambda f(z, u_0(z)) + \mu\xi_\rho u_0(z)^{p-1} \\ & = \mu[f(z, u_0(z)) + \xi_\rho u_0(z)^{p-1}] - (\mu - \lambda)f(z, u_0(z)) + \zeta(\delta) \\ & \leq \mu f(z, u_0(z)) + \mu\xi_\rho u_0(z)^{p-1} - (\mu - \lambda)c_{\rho_0} + \zeta(\delta) \end{aligned}$$

where $\rho_0 = \min_{\overline{\Omega}} u_0$ (recall $u_0 \in intC_+$ and see $H(f)(iii)$).

Since $\zeta(\delta) \to 0+$ as $\delta \to 0^+$, we can find $\delta^* > 0$ such that

$$\zeta(\delta) - (\mu - \lambda)c_\rho \leq 0 \text{ for all } \delta \in (0, \delta^*]$$

. So, from (3.16) we have for all $\delta \in (0, \delta^*]$

$$\begin{aligned} & -\text{div } a(Du_0^\delta(z)) + (\beta(z) + \mu\xi_\rho)u_0^\delta(z) \\ & \leq \mu f(z, u_\mu(z)) + \mu\xi_\rho u_\mu(z)^{p-1} \\ & = -\text{div } a(Du_\mu(z)) + (\beta(z) + \mu\xi_\rho)u_\mu(z) \text{ a.e. in } \Omega, \\ & \Rightarrow u_0^\delta \leq u_\mu \text{ (by acting with } (u_0^\delta - u_\mu)^+(\Omega)), \\ & \Rightarrow u_\mu - u_0 \in intC_+. \end{aligned}$$

In a similar fashion, we show that

$$u_0 - u_\theta \in intC_+.$$

So, we have proved that

$$u_0 \in int_{C^1(\overline{\Omega})}[u_\theta, u_\mu]. \tag{3.17}$$

From (3.14) we see that

$$\gamma_\lambda|_{[u_\theta, u_\mu]} = \varphi_\lambda|_{[u_\theta, u_\mu]} + \widehat{\xi}_\lambda \text{ with } \widehat{\xi}_\lambda \in \mathbb{R}$$

(see the proof of Proposition 3.2 for φ_λ).

Using (3.17) if follows that

$$u_0 \text{ is a local } C^1(\overline{\Omega})\text{-minimizer of } \varphi_\lambda,$$

$$\Rightarrow u_0 \text{ is a local } W^{1,p}(\Omega)\text{-minimizer of } \varphi_\lambda \text{ (see Theorem 2.5).}$$

By virtue of hypothesis $H(f)(iii)$, given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) > 0$ such that

$$F(z,x) \le \frac{\varepsilon}{p}x^p \text{ for a.a. } z \in \Omega, \text{ all } x \in [0,\delta]. \tag{3.18}$$

For $u \in C^1(\overline{\Omega})$ with $\|u\|_{C^1(\overline{\Omega})} \le \delta$, we have

$$\begin{aligned} \varphi_\lambda(u) &= \int_\Omega G(Du)dz + \frac{1}{p}\int_\Omega \beta|u|^p dz - \lambda\int_\Omega F(z,u)dz \\ &\ge \frac{1}{p}[\xi^* - \lambda\varepsilon]\|u\|^p \end{aligned} \tag{3.19}$$

(see Proposition 2.9 and (3.18)).

Choosing $\varepsilon \in (0, \frac{\xi^*}{\lambda})$, from (3.19) we infer that

$$\begin{aligned} &\varphi_\lambda(u) > 0 = \varphi_\lambda(0) \text{ for all } u \in C^1(\overline{\Omega}) \text{ with } 0 < \|u\|_{C^1(\overline{\Omega})} \le \delta, \\ &\Rightarrow u = 0 \text{ is a local } C^1(\overline{\Omega})\text{-minimizer of } \varphi_\lambda, \\ &\Rightarrow u = 0 \text{ is a local } W^{1,p}(\Omega)\text{-minimizer of } \varphi_\lambda \text{ (see Theorem 2.5).} \end{aligned}$$

Without any loss of generality we may assume that $\varphi_\lambda(0) = 0 \le \varphi_\lambda(u_0)$ (the reasoning is similar, if the opposite inequality holds) and that u_0 is an isolated critical point of φ_λ (otherwise we have a whole sequence of distinct positive solutions and so we are done). Then as in Aizicovicci-Papageorgiou-Staicu [1] (see the proof of Proposition 29), we can find $\widehat{\rho} \in (0, \|u_0\|)$ such that

$$\varphi_\lambda(0) = 0 \le \varphi_\lambda(u_0) < \inf[\varphi_\lambda(u) : \|u\| = \widehat{\rho}] = \eta_{\widehat{\rho}}. \tag{3.20}$$

Recall that φ_λ is coercive (see the proof of Proposition 3.2). Using this fact, we can easily check that φ_λ satisfies the PS-condition. Combining this with (3.20), we can apply Theorem 2.6 and obtain $\widehat{u} \in W^{1,p}(\Omega)$ such that

$$\eta_{\widehat{\rho}} \le \varphi_\lambda(\widehat{u}) \tag{3.21}$$

$$\varphi_\lambda'(\widehat{u}) = 0. \tag{3.22}$$

From (3.20) and (3.21) we see that $\widehat{u} \notin \{0, u_0\}$, while from (3.22) it follows that $\widehat{u} \in S(\lambda) \subseteq intC_+$. ☐

The next proposition examines what happens at the critical parameter value $\lambda_* > 0$.

Proposition 3.5. *If hypotheses $H(a), H(\beta)$ and $H(f)$ hold, then $\lambda_* \in \mathcal{P}$.*

Proof. Let $\{\lambda_n\}_{n\ge1} \subseteq \mathcal{P}$ such that $\lambda_n \downarrow \lambda_*$ and $\lambda_n > \lambda_*$ for all $n \ge 1$. Then we can find $u_n \in S(\lambda_n) \subseteq intC_+$ such that

$$\{u_n\}_{n\ge1} \tag{3.23}$$

is decreasing (see the proof of Proposition 3.4) and

$$A(u_n) + \beta u_n^{p-1} = \lambda_n N_f(u_n) \text{ for all } n \geq 1. \tag{3.24}$$

Recall that

$$0 \leq f(z,x) \leq c_6 x^{p-1} \text{ for a.a. } z \in \Omega, \text{ all } x \geq 0 \text{ (see (3.2))}.$$

From this estimate, Proposition 2.9 and (3.23), (3.24), we infer that

$$\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega) \text{ is bounded.}$$

Hence we may assume that

$$u_n \xrightarrow{w} u_* \text{ in } W^{1,p}(\Omega) \text{ and } u_n \to u \text{ in } L^p(\Omega). \tag{3.25}$$

On (3.24) we act with $u_n - u_* \in W^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (3.25). We obtain

$$\lim_{n\to\infty} \langle A(u_n), u_n - u_* \rangle = 0,$$
$$\Rightarrow u_n \to u_* \text{ in } W^{1,p}(\Omega) \text{ (see Proposition 2.7)}. \tag{3.26}$$

So, by passing to the limit as $n \to \infty$ in (3.24) and using (3.26), we have

$$A(u_*) + \beta u_*^{p-1} = \lambda_* N_f(u_*),$$
$$\Rightarrow u_* \in C_+ \text{ (nonlinear regularity, see [11], [12])}.$$

If we show that $u_* \neq 0$, then $u_* \in S(\lambda_*) \subseteq intC_+$ and so $\lambda_* \in \mathcal{P}$. To this end, we have

$$-\text{div } a(Du_n(z)) + \beta(z) u_n(z)^{p-1} = \lambda_n f(z, u_n(z)) \text{ a.e. in } \Omega, \ \frac{\partial u_n}{\partial n} = 0 \text{ on } \partial\Omega.$$

Since $0 \leq u_n \leq u_1$ for all $n \geq 1$, from Hu-Papageorgiou [11] (see Proposition 2.7), we can find $M_1 > 0$ such that

$$\|u_n\|_\infty \leq M_1 \text{ for all } n \geq 1.$$

Then the regularity result of Lieberman [12] (p. 320) implies that there exist $\eta \in (0,1)$ and $M_2 > 0$ such that

$$u_n \in C^{1,\eta}(\overline{\Omega}) \text{ and } \|u_n\|_{C^{1,\eta}(\overline{\Omega})} \leq M_2 \text{ for all } n \geq 1. \tag{3.27}$$

Since $C^{1,\eta}(\overline{\Omega})$ is embedded compactly in $C^1(\overline{\Omega})$, from (3.27) and (3.26), we have

$$u_n \to u_* \text{ in } C^1(\overline{\Omega}).$$

Suppose that $u_* = 0$. Then

$$u_n \to 0 \text{ in } C^1(\overline{\Omega}). \tag{3.28}$$

By virtue of hypothesis $H(f)(iii)$, given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) > 0$ such that

$$f(z,x) \leq \varepsilon x^{p-1} \text{ for a.a. } z \in \Omega, \text{ all } x \in [0, \delta]. \tag{3.29}$$

From (3.28) it follows that there exists an integer $n_0 \geq 1$ such that

$$\begin{aligned} & u_n(z) \in [0,\delta] \text{ for all } z \in \overline{\Omega}, \text{ all } n \geq n_0, \\ & \quad \Rightarrow -\text{div } a(Du_n(z)) + \beta(z)u_n(z)^{p-1} \leq \lambda_n \varepsilon u_n(z)^{p-1} \\ & \quad \text{a.e. in } \Omega\,, n \geq n_0 \text{ (see (3.29))} \\ & \quad \Rightarrow \xi^* \|u_n\|^p \leq \lambda_n \varepsilon \|u_n\|_p^p, \ n \geq n_0 \text{ (see Proposition 2.9)}, \\ & \quad \Rightarrow \frac{\xi^*}{\varepsilon} \leq \lambda_n \text{ for all } n \geq n_0, \\ & \quad \Rightarrow \frac{\xi^*}{\varepsilon} \leq \lambda_*. \end{aligned}$$

But $\varepsilon > 0$ is arbitrary. So if we let $\varepsilon \to 0^+$, we reach a contradiction. This proves that $u_* \neq 0$ and so $\lambda_* \in \mathcal{P}$. □

Summarizing the situation for problem (1.1), we can state the following bifurcation-type theorem.

Theorem 3.6. *If hypotheses $H(a)$, $H(\beta)$ and $H(f)$ hold, then there exists $\lambda_* > 0$ such that*

(a) *for all $\lambda \in (\lambda_*, +\infty)$ problem* (1.1) *has at least two nontrivial positive solutions*

$$u_0, \widehat{u} \in intC_+;$$

(b) *for $\lambda = \lambda_*$ problem* (1.1) *has at least one nontrivial positive solution $u_* \in intC_+$;*
(c) *for $\lambda \in (0, \lambda_*)$ problem* (1.1) *has no nontrivial positive solution.*

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