Dynamic Systems and Applications 22 (2013) 301-316

ON SOLUTIONS OF SEMILINEAR INFINITE SYSTEMS OF DIFFERENTIAL EQUATIONS

JÓZEF BANAŚ AND MONIKA KRAJEWSKA

Department of Mathematics, Rzeszów University of Technology
al. Powstańców Warszawy 8, 35-959 Rzeszów, Poland
jbanas@prz.edu.pl (J. Banaś)
Department of Mathematics and Natural Sciences, State Higher School of
Technology and Economics in Jaroslaw, Czarnieckiego 16, 37-500 Jaroslaw, Poland
monika.krajewska@pwste.edu.pl (M. Krajewska)

To the memory of Professor V. Lakshmikantham

ABSTRACT. The paper is devoted to the study of the solvability of semilinear infinite systems of ordinary differential equations in the space of real bounded sequences. Using the technique associated with measures of noncompactness and some results from the theory of ordinary differential equations in Banach spaces, we prove a few existence results concerning infinite systems of differential equations.

AMS (MOS) Subject Classification: 34A34, 34G20, 47H08

Keywords. Infinite system of differential equations, Banach space, measure of noncompactness, uniformly convergent function series

1. INTRODUCTION

The theory of ordinary differential equations is one of the most important branches of mathematics since it finds numerous and very essential applications. The classical theory of those equations i.e., the theory of ordinary differential equations in finfite dimensional spaces became almost closed around fiftieth years of the past century. The classical monographs [6, 9] present almost complete state of that theory.

In 1950 theory of ordinary differential equations obtained a new impetus after famous examples of Dieudonné [8] who showed that in the case of infinite dimension the classical results of that theory fail to work. Starting from that moment a great interest has been focused on the theory of ordinary differential equations in Banach (infinite dimensional) spaces (cf. [7, 10, 13, 14, 15]). There have been developed new methods and new tools useful in the study of those equations. Thorough state of the discussed theory up to the year 1977 was presented in the monograph [7].

Received January 15, 2013

The majority of results concerning the theory of differential equations in infinite dimensional Banach spaces were obtained till the end of the eighties of the past century. The most imortant part of those results is devoted to the existence of solutions of considered differential equations. Those results were mainly obtained with the use of the technique of the so-called measuers of noncompactness [1, 2].

It is worthwhile mentioning that although the monograph of Deimling [7] indicated some important applications of the theory of differential equations in Banach spaces, mathematicians have not paid special attention to those applications. To justify such an opinion it is sufficient to notice that in [7] the author indicated among possible applications, the infinite systems of differential equations. Such systems appear both in a natural way in several considerations associated with applications to real world problems and in the study of numerical schemes of solving some problems for partial differential equations.

Up to now there here appeared only a few papers concerning infinite systems of differential equations (cf. [3, 4, 5, 7, 11, 12]). The aim of this paper is to present some further results concerning mentioned infinite systems of differential equations.

We focus here on the so-called semilinear infinite systems of ordinary differential equations i.e., on linear systems with perturbations. We will work in the classical Banach sequence space l^∞, since considerations in this space are conducted very seldom (cf. [5]). It is caused by the fact that l^∞ is not separable, thus the tools of the theory of measures of noncompactness in Banach spaces with bases cannot be utilized in this setting (cf. [2, 11, 12]).

On the other hand the known results concerning infinite systems of differential equations in the space l^∞ obtained in the paper [5] are not correct. Indeed, on page 110 of the mentioned paper the authors made an error in calculations.

In the present paper we correct and improve the results obtained in [5]. Let us note that the corrected formulation of the mentioned results forced us to impose a bit more restrective assumptions which is caused by the method used in our considerations. Obviously, the validity of existence results under more general assumptions is an open problem.

2. NOTATION, DEFINITIONS AND AUXILIARY FACTS

In this section we collect auxiliary facts and results which will be needed in our further considerations.

At the beginning we establish some notation. Let $\mathbb{R}$ denote the set of real numbers while the symbol $\mathbb{N}$ stands for the set of natural numbers. Put $\mathbb{R}_+ = [0, \infty)$.

Further, assume that E is a real Banach space with the norm $\|\cdot\|$ and the zero element θ. If X is a subset of E then $\overline{X}$, $\text{Conv}X$ denote the closure and the convex closure of X, respectivelly. Moreover, by diam$X$ we denote the diameter of X, provided X is a bounded subset of E. The symbol $B(x_0, r)$ will denote the closed ball centered at x_0 and with radius r.

In what follows we will denote by $\mathfrak{M}_E$ the family of all nonempty and bounded subsets of the space E and by $\mathfrak{N}_E$ its subfamily consisting of all relatively compact sets. Finally, let us recall that the standard algebraic operations on sets X, Y will be denoted by $X + Y$, λX for $\lambda \in \mathbb{R}$.

Now, we recall the basic concept which will be the main tool used in our further investigations. The definition of that concept is taken from [2].

Definition 2.1. A function $\mu : \mathfrak{M}_E \to \mathbb{R}_+$ is called a measure of noncompactness in the space E if it satisfies the following conditions:

1^o The family $\ker \mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker \mu \subset \mathfrak{N}_E$.
2^o $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.
3^o $\mu(\overline{X}) = \mu(\text{Conv}X) = \mu(X)$.
4^o $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0, 1]$.
5^o If $X_n \in \mathfrak{M}_E$, $X_n = \overline{X}_n$, $X_{n+1} \subset X_n$ for $n = 1, 2, \dots$ and if $\lim_{n\to\infty} \mu(X_n) = 0$ then the set $X_\infty = \bigcap_{n=1}^{\infty} X_n \neq \emptyset$.

The family $\ker \mu$ described in axiom 1^o is said to be *the kernel* of the measure of noncompactness μ.

Sometimes we distinguish measures of noncompactness satisfying some extra conditions [2]. For example, a measure of noncompactness μ is called *sublinear* if it satisfies the following two conditions:

6^o $\mu(X + Y) \leq \mu(X) + \mu(Y)$.
7^o $\mu(cX) = c\mu(X)$ for $c \in \mathbb{R}$.

For further facts connected with measures of noncompactness we refer to [2].

Now, let us consider the space l^∞ consisting of all real and bounded sequences $x = (x_i)$, equipped with the standard supremum norm

$$\|x\| = \|(x_i)\| = \sup\{|x_i| : i = 1, 2, \dots\}.$$

Obviously, l^∞ is nonseparable Banach space.

We recall certain measure of noncompactness in the space l^∞ [2]. To this end take a set $X \in \mathfrak{M}_{l^\infty}$. For a fixed natural number n let us donote

$$X_n = \{x_n : x = (x_i) \in X\}.$$

Observe that X_n represents the intersection of the set X at n-th coordinate.

Further, let us put

$$\mu(X) = \limsup_{n\to\infty} \operatorname{diam} X_n, \tag{2.1}$$

where the symbol $\operatorname{diam} X_n$ stands for the diameter of the set X_n i.e., $\operatorname{diam} X_n = \sup\{|x_n - y_n| : x, y \in X\}$.

It can be shown [2] that the function μ defined by (2.1) is a sublinear measure of noncompactness in the space l^∞ (in the sense of Definition 2.1). The kernel $\ker\mu$ of this measure contains all bounded and nonempty subsets X of l^∞ such that the thickness of the bundle formed by sequences of the set X tends to zero at infinity.

This observation plays an important role in characterization of solutions of some operator equations.

In what follows let $I = [0,T]$ be a given interval. For the transparency we will write I_T to denote the interval $[0,T]$. Next, consider the function $f(t,x) = f : I_T \times B(x_0, r) \to E$, where E is a fixed real Banach space and $B(x_0, r)$ is a ball in E. Let us take into account the ordinary differential equation

$$x' = f(t,x) \tag{2.2}$$

with the initial condition

$$x(0) = x_0. \tag{2.3}$$

Our considerations in the next section will be based on the following existence result concerning the initial value problem (2.2)–(2.3) [3].

Theorem 2.2. *Assume that $f = f(t,x)$ is a function defined on the set $I_T \times E$ with values in E such that $\|f(t,x)\| \le A\|x\| + B$ for $t \in I_T$ and $x \in E$, where A and B are nonnegative constants. Further, let f be uniformly continuous on the set $I_T \times B(x_0, r)$, where $AT < 1$ and $r = (BT + AT\|x_0\|)/(1 - AT)$. Assume that μ is a sublinear measure of noncompactness in a Banach space E with $\{x_0\} \in \ker\mu$ and such that for each nonempty set $X \subset B(x_0, r)$ and for almost all $t \in I$ the following inequality holds*

$$\mu(f(t,X)) \le p(t)\mu(X),$$

where $p = p(t)$ is an integrable function on I_T. Then problem (2.2)–(2.3) has a solution $x = x(t)$ defined on the interval I_T and such that $x(t) \in \ker\mu$ for any $t \in I_T$.

Remark 2.3. Under assumptions of Theorem 2.2 it can be show (cf. [2]) that if $y = y(t)$ is an arbitrary solution of problem (2.2)–(2.3) such that $y(t) \in B(x_0, r)$ for $t \in I_T$, then $y(t) \in \ker\mu$ for $t \in I_T$.

3. EXISTENCE THEOREMS FOR A SEMILINEAR INFINITE SYSTEM OF DIFFERENTIAL EQUATIONS

In this section we present main results of the paper. Namely, we will consider the following semilinear infite system of differential equations

$$x_i' = \sum_{j=1}^{\infty} a_{ij}(t)x_j + g_j(t, x_1, x_2, \dots) \tag{3.1}$$

with the initial conditions

$$x_i = (0) = x_i^0 \tag{3.2}$$

for $i = 1, 2, \dots$ and $t \in I_T = [0, T]$.

The above written problem (3.1)–(3.2) will be considered in the Banach space l^∞ under the below listed assumptions.

(i) $x_0 = (x_i^0) \in l^\infty$.

(ii) The mapping $g = (g_1, g_2, \dots)$ transformsthe set $I_T \times l^\infty$ into l^∞ and is uniformly continuous.

(iii) There exists a sequence (b_i) convergent to zero and such that $|g_i(t, x_1, x_2, \dots)| \leq b_i$ for all $t \in I_T$, $x = (x_i) \in l^\infty$ and for $i = 1, 2, \dots$.

(iv) For all natural numbers i, j the function $a_{ij}(t) = a_{ij} : I_T \to \mathbb{R}_+$ is nondecreasing on the interval I_T.

(v) For each $i \in \mathbb{N}$ the function series $\sum_{j=1}^{\infty} a_{ij}(t)$ is uniformly convergent on I_T.

Taking into account assumptions (iv), (v), for an arbitrarily fixed $i \in \mathbb{N}$ we can consider the functions $A_i(t)$, $\overline{A_i}(t)$ and $\overline{\overline{A_i}}(t)$ defined on the interval I_T in the following way:

$$A_i(t) = \sum_{j=1}^{\infty} a_{ij}(t),$$

$$\overline{A_i}(t) = \sum_{j=1}^{i-1} a_{ij}(t),$$

$$\overline{\overline{A_i}}(t) = \sum_{j=i}^{\infty} a_{ij}(t).$$

Obviously the above formula defines the function $\overline{A_i}(t)$ for $i \geq 2$. We can extend this definition putting $\overline{A_1}(t) = 0$ for $t \in I_T$.

Moreover, let us observe that the functions $A_i(t)$, $\overline{A_i}(t)$ and $\overline{\overline{A_i}}(t)$ are nonnegative and nondecreasing on the interval I_T.

In what follows we will additionally impose the following assumptions.

(vi) The sequence $(\overline{A_i}(t))$ converges uniformly to zero on I_T.

(vii) The sequence $(A_i(t))$ is equicontinuous and equibounded on the interval I_T.

Remark 3.1. Let us observe that in assumption (ii) it is sufficient to require that the mapping $g = (g_1, g_2, \dots)$ is uniformly continuous on the set $I_T \times B(x_0, r)$ for arbitrarily fixed $r > 0$. This observation is a consequence of Theorem 2.2 which is used immediately in the proof of the below presented result.

For our further purposes let us define the following constants.

$$A = \sup\{A_i(t) : t \in I_T,\ i = 1, 2, \dots\},$$
$$B = \sup\{b_i : i = 1, 2, \dots\}.$$

Notice that in view of the imposed assumptions we have that $A < \infty$ and $B < \infty$.

Now, we formulate our main result.

Theorem 3.2. *Let assumptions (i)–(vii) be satisfied and let $AT < 1$. Then problem* (3.1)–(3.2) *has at least one solution $x = x(t) = (x_i(t))$ defined on the interval I_T such that $x(t) \in l^\infty$ for $t \in I_T$.*

Proof. For an arbitrarily fixed $x = (x_i) \in l^\infty$ and $t \in I_T$ let us denote

$$f_i(t, x) = \sum_{j=1}^{\infty} a_{ij}(t) x_j + g_i(t, x_1, x_2, \dots),$$
$$f(t, x) = (f_1(t, x),\ f_2(t, x), \dots) = (f_i(t, x)).$$

Further, let us fix $i \in \mathbb{N}$. Then, applying the imposed assumptions, we obtain:

$$|f_i(t, x)| \le \sum_{j=1}^{\infty} a_{ij}(x)|x_j| + |g_i(t, x_1, x_2, \dots)|$$
$$\left(\sum_{j=1}^{\infty} a_{ij}(t)\right) \sup\{|x_j| : j = 1, 2, \dots\} + b_i \le A_i(t)\|x\| + b_i.$$

This yields the following estimate

$$\|f(t, x)\| \le A\|x\| + B, \tag{3.3}$$

where the symbol $\|\cdot\|$ stands for the norm in the space l^∞. From the above estimate we deduce that the operator $f = f(t, x)$ transforms the set $I_T \times l^\infty$ into l^∞.

Now, let us take the number $r = (BT + AT\|x_0\|)/(1 - AT)$ (cf. Theorem 2.2). We will consider the operator f on the set $I_T \times B(x_0, r)$.

Next, fix arbitrarily $t, s \in I_T$ and $x, y \in B(x_0, r)$. Without loss of generality we may assume that $s < t$ (cf. assumption (iv)). Then, in virtue of our assumptions, for a fixed $i \in \mathbb{N}$ we get:

$$= \left| \sum_{j=1}^{\infty} a_{ij}(t) x_j + g_i(t, x_1, x_2, \dots) - \sum_{j=1}^{\infty} a_{ij}(s) y_j - g_i(s, y_1, y_2, \dots) \right|$$

$$
\begin{aligned}
&\leq \left|\sum_{j=1}^{\infty} a_{ij}(t)x_j - \sum_{j=1}^{\infty} a_{ij}(s)y_j\right| + |g_i(t, x_1, x_2, \dots) - g_i(s, y_1, y_2, \dots)| \\
&\leq \left|\sum_{j=1}^{\infty} a_{ij}(t)x_j - \sum_{j=1}^{\infty} a_{ij}(s)x_j\right| + \left|\sum_{j=1}^{\infty} a_{ij}(s)x_j - \sum_{j=1}^{\infty} a_{ij}(s)y_j\right| \\
&\quad + |g_i(t, x_1, x_2, \dots) - g_i(s, y_1, y_2, \dots)| \\
&= \left|\sum_{j=1}^{\infty}[a_{ij}(t) - a_{ij}(s)]x_j\right| + \left|\sum_{j=1}^{\infty} a_{ij}(s)(x_j - y_j)\right| \\
&\quad + |g_j(t, x_1, x_2, \dots) - g_i(s, y_1, y_2, \dots)| \\
&\leq \sum_{j=1}^{\infty}[a_{ij}(t) - a_{ij}(s)]|x_j| + \sum_{j=1}^{\infty} a_{ij}(s)|x_j - y_j| \\
&\quad + |g_i(t, x) - g_i(s, y)| \\
&\leq \|x\| \left[\sum_{j=1}^{\infty} a_{ij}(t) - \sum_{j=1}^{\infty} a_{ij}(s)\right] + \|x - y\| \sum_{j=1}^{\infty} a_{ij}(s) \\
&\quad + |g_i(t, x) - g_i(s, y)| \\
&\leq \|x\||A_i(t) - A_i(s)| + A\|x - y\| + \|g(t, x) - g(s, y)\| \\
&\leq (\|x_0\| + r) \sup\{|A_i(t) - A_i(s)| :\ i = 1, 2, \dots\} + A\|x - y\| \\
&\quad + \|g(t, x) - g(s, y)\|
\end{aligned}
$$

Hence, keeping in mind assumptions (ii) and (vii) we infer that the operator $f(t, x)$ is uniformly continuous on the set $I_T \times B(x_0, r)$.

Further, let us take a nonempty subset X of the ball $B(x_0, r)$ and fix $x, y \in X$, $t \in I_T$. Then, for an arbitrarily fixed natural number i, $i \geq 2$, we obtain:

$$
\begin{aligned}
&|f_i(t, x) - f_i(t, y)| \leq \left|\sum_{j=1}^{\infty} a_{ij}(t)x_j - \sum_{j=1}^{\infty} a_{ij}(t)y_j\right| \\
&\quad + |g_i(t, x_1, x_2, \dots) - g_i(t, y_1, y_2, \dots)| \\
&\leq \left|\sum_{j=1}^{i-1} a_{ij}(t)x_j + \sum_{j=i}^{\infty} a_{ij}(t)x_j - \sum_{j=1}^{i-1} a_{ij}(t)y_j - \sum_{j=i}^{\infty} a_{ij}(t)y_j\right| \\
&\quad + |g_i(t, x_1, x_2, \dots)| + |g_i(t, y_1, y_2, \dots)| \\
&\leq \left|\sum_{j=1}^{i-1} a_{ij}(t)(x_j - y_j)\right| + \left|\sum_{j=i}^{\infty} a_{ij}(t)(x_j - y_j)\right| + 2b_i \\
&\leq \sum_{j=1}^{i-1} a_{ij}(t)|x_j - y_j| + \sum_{j=i}^{\infty} a_{ij}(t)|x_j - y_j| + 2b_i
\end{aligned}
$$

$$\leq \|x-y\| \sum_{j=1}^{i-1} a_{ij}(t) + \left(\sum_{j=i}^{\infty} a_{ij}(t)\right) \sup\{|x_j - y_j| : j \geq i\} + 2b_i$$
$$\leq \overline{A_i}(t) \mathrm{diam} X + \overline{\overline{A_i}}(t) \sup\{\mathrm{diam} X_j : j \geq i\} + 2bi.$$

From the above estimate we conclude the following inequality

$$\mathrm{diam} f_i(t,X) \leq \overline{A_i}(t) \mathrm{diam} X + \overline{\overline{A_i}}(t) \sup\{\mathrm{diam} X_j : j \geq i\} + 2b_i,$$

which holds for any $i \in \mathbb{N}$. Hence, we obtain

$$\sup\{\mathrm{diam} f_j(t,X) : j \geq i\} \leq \sup\{\overline{A_j}(t) : j \geq i\} \mathrm{diam} X + [\sup\{\overline{\overline{A_j}}(t) :\ j \geq i] \sup\{\mathrm{diam} X_j : j \geq i\} + 2 \sup\{b_j : j \geq i\}.$$

The above estimate and assumptions (iii) and (v)–(vii) allows us to deduce the following inequality

$$\mu(f(t,X)) \leq p(t)\mu(X), \tag{3.4}$$

where the function $p(t)$ is defined on the interval I_T in the following way

$$p(t) = \limsup_{i\to\infty} \overline{\overline{A_i}}(t)\ .$$

Finally, taking into account (3.3), (3.4) and other facts established in the above conducted proof, in view of Theorem 2.2 we infer that there exists a solution $x(t) = (x_i(t))$ of problem (3.1)–(3.2) such that $x(t) \in l^\infty$ for each $t \in I_T$. The proof is complete. □

Remark 3.3. Notice that on the basis of Theorem 2.2 it can be shown [2, 5] that all solutions $x = x(t) = (x_i(t))$ of problem (3.1)–(3.2) belonging to the ball $B(x_0, r)$ i.e., $x(t) \in B(x_0, r)$ for $t \in I_T$, are such that $x(t) \in \ker \mu$ for $t \in I_T$, where μ is the measure of noncompactness in l^∞ defined by formula (2.1).

Keeping in mind the description of the kernel $\ker \mu$ given in Section 2 we conclude that all solutions of the infinite system of differential equations (3.1) satisfying the initial conditions (3.2) and belonging to the ball $B(x_0, r)$ for all $t \in I_T$, are asymptotically coordinable stable according to twe following definition accepted in [5]:

We say that solutions of problem (3.1)–(3.2) are *asymptotically coordinable stable* if for any $\varepsilon > 0$ and $t \in I_T$ there exists $i_0 \in \mathbb{N}$ such that for arbitrary solutions $x(t)$, $y(t)$ of (3.1)–(3.2) with $x, y \in B(x_0, r)$ we have that $|x_i(t) - y_i(t)| \leq \varepsilon$ for $i \geq i_0$.

In what follows we present the corrected version of the main result obtained in the paper [5], where the following perturbed semilinear upper diagonal infinite system of differential equations was investigated

$$x_i' = \sum_{j=i}^{\infty} a_{ij}(t)x_j + g_i(t, x_1, x_2, \dots) \tag{3.5}$$

together with initial conditions (3.2) i.e.,

$$x_i(0) = x_i^0, \tag{3.6}$$

for $i = 1, 2, \dots$ and $t \in I_T = [0, T]$.

Observe that system (3.5) is a particular case of system (3.1). Indeed, if we put in (3.1) $a_{ij}(t) = 0$ for $t \in I_T$ and for $j = 1, 2, \dots, i-1$ ($i \geq 2$), then we obtain the above written infinite system (3.5). This observation allows us easily formulate an existence result concerning problem (3.5)–(3.6). To this end it is sufficient to adapt suitably Theorem 3.2.

First of all notice that for system (3.5) assumption (vi) of Theorem 3.2 is automatically satisfied since $\overline{A_i} \equiv 0$ on the interval I_T for $i = 2, 3, \dots$. On the other hand observe that the remaining assumptions of Theorem 3.2 should be only modified.

Now, we formulate the corrected version of the main result of [5].

Theorem 3.4. *Assume that there are satisfied hypotheses (i)–(iii) of Theorem* 3.2 *and additionally, the following ones:*

(iv') *For all pairs of natural numbers (i, j) such that $j \geq i$ the function $a_{ij} : I_T \to \mathbb{R}_+$ is nondecreasing on the interval I_T.*

(v') *For any $i \in \mathbb{N}$ the function series $\sum_{j=i}^{\infty} a_{ij}(t)$ is uniformly convergent on the interval I_T.*

(vii') *The sequence $(A_i(t))$, where $A_i(t) = \sum_{j=i}^{\infty} a_{ij}(t)$, is equicontinuous and equibounded on I_T.*

If $AT < 1$ then problem (3.5)–(3.6) *has at least one solution $x = x(t) = (x_i(t))$ defined on the interval I_T such that $x(t) \in l^\infty$ for any $t \in I_T$.*

Further on, we illustrate the result contained in Theorem 3.2 by an example.

Example 3.5. Consider the semilinear infinite system of differential equations of form (3.1), where the functions $a_{ij}(t)$ and $g_i(t, x_1, x_2, \dots)$ are defined in the following way:

$$a_{ij}(t) = \frac{t^j}{ij},$$

$$g_i(t, x_1, x_2, \dots) = \frac{t \arctan(x_i + x_{i+1})}{i + x_i^2 + x_{i+1}^2} \ ,$$

where $i, j = 1, 2, \dots$ and $t \in I_T$, and $T < 1$. Moreover, we assume that the above indicated infinite system of differential equations is investigated together with initial conditions (3.2).

Using the classical tools of mathematical analysis it is easily seen that functions $a_{ij}(t)$ satisfy assumptions (iv) and (v).

Moreover, for an arbitrarily fixed natural number i we have

$$A_i(t) = \frac{1}{i}\sum_{j=1}^{\infty}\frac{t^j}{j} = -\frac{1}{i}\ln(1-t) \tag{3.7}$$

for $t \in I_T$. This implies that the sequence $(\overline{A_i}(t))$ appearing in assumption (vi) is uniformly convergent to zero on the interval I_T. Indeed, in view of the inequality

$$\overline{A_i}(t) \le A_i(t) \le -\frac{1}{i}\ln(1-T)$$

we derive our assertion.

Further observe that in view of equality (3.7) we infer that there is satisfied assumption (vii). Moreover, we have

$$A = \sup\{A_i(t) : t \in I_T, i = 1, 2, \dots\} = -\ln(1-T). \tag{3.8}$$

Next, let us notice that the following inequality holds for an arbitrary $i \in \mathbb{N}$ and for $x = (x_i) \in l^\infty$:

$$|g_i(t, x_1, x_2, \dots)| \le \frac{T\pi/2}{i + x_i^2 + x_{i+1}^2} \le \frac{T\pi}{2i}.$$

This yields immediately that assumption (iii) is satisfied with $b_i = T\pi/2i$ for $i = 1, 2, \dots$. Obviously, assumption (i) is satisfied provided we impose that $x_0 = (x_i^0) \in l^\infty$.

Finally, let us fix arbitrarily a number $r > 0$ and consider the ball $B(x_0, r)$ in the space l^∞. Then, it is easy to verify that the function $g_i(t, x_1, x_2, \dots)$ is uniformly continuous on the set $I_T \times B(x_0, r)$. This statement is a simple consequence of the fact that the mapping $g_1 = g_1(t, x_1, x_2, \dots)$ has the largest modulus of continuity among the functions g_i $(i = 1, 2, \dots)$ on the set $I_T \times B(x_0, r)$. On the other hand all functions g_i are uniformly continuous on the set $I_T \times B(x_0, r)$ since the function $g_i(t, x_1, x_2, \dots)$ depends upon three variables only. Thus, keeping in mind Remark 3.1 we conclude that there is satisfied assumption (ii) of Theorem 3.2.

Now, we deduce that the semilinear infinite system of differential equations considered here has at least one solution $x = x(t) = (x_i(t))$ defined on the interval $I_T = [0, T]$, where $T < 1$ and T satisfies the following inequality (cf. (3.8))

$$-T\ln(1-T) < 1 .$$

Apart from this we have that $(x_i(t)) \in l^\infty$ for each $t \in I_T$ and solutions of the studied infinite system are asymptotically coordinable stable (cf. Remark 3.3).

4. SOME FURTHER EXISTENCE RESULTS

In this section we will consider some special cases of the previously investigated semilinear infinite system of differential equations (3.1) with initial conditions (3.2).

More precisely, we will consider some particular cases of the perturbation term $g = g(t, x) = (g_1(t, x), g_2(t, x), \dots)$ appearing in infinite system (3.1).

At the beginning let us take into account the following semilinear infinite system of differential equations

$$x_i' = \sum_{j=1}^{\infty} a_{ij}(t)x_j + g_i(t, x_i, x_{i+1}, \dots) \tag{4.1}$$

with initial conditions

$$x_i(0) = x_i^0 \tag{4.2}$$

for $i = 1, 2, \dots$ and for $t \in I_T = [0, T]$.

In what follows we will study problem (4.1)–(4.2) under assumptions (i), (iv)–(vii) of Theorem 3.2. Moreover, assumptions (ii), (iii) will be replaced by the following ones:

(ii') The function $t \to g(t, x)$ acting from the interval I_T into the space l^∞ is uniformly continuous on I_T, uniformly with respect to x belonging to an arbitrary ball $B(x_0, r)$ in the space l^∞.

(iii') For each $i \in \mathbb{N}$ there exists a nonnegative constant k_i such that for all $x, y \in l^\infty$, $x = (x_i)$, $y = (y_i)$, the following inequality is satisfied

$$|g_i(t, x_i, x_{i+1}, \dots) - g_i(t, y_i, y_{i+1}, \dots)| \le k_i \sup\{|x_j - y_j| : j \ge i\}.$$

(iii") The sequence (k_i) of constants appearing in assumption (iii') is bounded.

Now, observe that keeping in mind assumptions (ii'), (iii') and (iii") we can define the following finite constants:

$$G = \sup\{|g_i(t, 0, 0, \dots)| : t \in I_T,\ i = 1, 2, \dots\},$$

$$k = \sup\{k_i : i = 1, 2, \dots\}.$$

Further on, let us notice that assumptions (iii'), (iii") imply that the function $g = g(t, x)$ satisfies the Lipschitz condition with the constant k with respect to the variable x. Indeed, for arbitrarily fixed $x = (x_i)$, $y = (y_i) \in l^\infty$ and for any fixed $t \in I_T$ we obtain:

$$\begin{aligned} \|g(t, x) - g(t, y)\| &= \sup\{|g_i(t, x_i, x_{i+1}, \dots) - g_i(t, y_i, y_{i+1}, \dots)| : i = 1, 2, \dots\} \\ &\le \sup\{k_i \sup\{|x_j - y_j| : j \ge i\}\ i = 1, 2, \dots\} \\ &\le \sup\{k_i\|x - y\| : i = 1, 2, \dots\} \le k\|x - y\|. \end{aligned} \tag{4.3}$$

Apart from this observe that the function $g = g(t, x)$ is uniformly continuous on the set $I_T \times B(x_0, r)$, where $r > 0$ is arbitrarily fixed.

To prove this assertion fix arbitrarily $t_1, t_2 \in I_T$ and $x_1, x_2 \in B(x_0, r)$. Then, in view of (4.3) we get:

$$\begin{aligned}\|g(t_2, x_2) - g(t_1, x_1)\| &\le \|g(t_2, x_2) - g(t_2, x_1)\| + \|g(t_2, x_1) - g(t_1, x_1)\| \\ &\le k\|x_2 - x_1\| + \|g(t_2, x_1) - g(t_1, x_1)\|.\end{aligned}$$

Hence, in view of assumption (ii') we obtain the desired uniform continuity.

Now, we can formulate an existence result concerning the initial value problem (4.1)–(4.2).

Theorem 4.1. *Suppose that assumptions (i), (ii'), (iii'), (iii") and (iv)–(vii) are satisfuied and* $T(A + k) < 1$. *Then problem* (4.1)–(4.2) *has at least one solution* $x = x(t) = (x_i(t))$ *defined on the interval* $I_T = [0, T]$ *and such that* $x(t) \in l^\infty$ *for* $t \in I_T$.

Proof. We proceed similarly as in the proof of Theorem 3.2. Thus, fix $i \in \mathbb{N}$. Then, for arbitrarily chosen $x = (x_i) \in l^\infty$ and $t \in I_T$, in view of assumptions and the facts established above, we get:

$$\begin{aligned}|f_i(t, x)| &\le \sum_{j=1}^{\infty} a_{ij}(t)|x_j| + |g_i(t, x_i, x_{i+1}, \dots)| \\ &\le A_i(t)\|x\| + |g_i(t, x_i, x_{i+1}, \dots) - g_i(t, 0, 0, \dots)| + |g_i(t, 0, 0, \dots)| \\ &\le A_i(t)\|x\| + k_i \sup\{|x_j| : \ j \ge i\} + G \le A\|x\| + k_i\|x\| + G.\end{aligned}$$

This yields the following estimate

$$\|f(t, x)\| \le (A + k)\|x\| + G.$$

The above estimate implies that the operator $f = f(t, x)$ transforms the set $I_T \times l^\infty$ into l^∞.

Further on, let us take the number

$$r = (GT + (A + k)T\|x_0\|)/(1 - (A + k)T).$$

Consider the operator $f(t, x)$ on the set $I_T \times B(x_0, r)$. In view of the earlier stated uniform continuity of the operator $g = g(t, x)$ on the set $I_T \times B(x_0, r)$ and the reasoning conducted in the proof of Theorem 3.2, we deduce that the operator $f(t, x)$ is uniformly continuous on the set $I_T \times B(x_0, r)$.

Now, fix a nonempty subset X of the ball $B(x_0, r)$ and take arbitrary $x, y \in X$ and $t \in I_T$. Then, for arbitrarily fixed $i \in \mathbb{N}$ we obtain

$$|f_i(t, x) - f_i(t, y)| \le \left| \sum_{j=1}^{\infty} a_{ij}(t)x_j - \sum_{j=1}^{\infty} a_{ij}(t)y_j \right|$$

$$
\begin{aligned}
&+ |g_i(t, x_i, x_{i+1}, \dots) - g_i(t, y_i, y_{i+1}, \dots)| \\
\le& \left| \sum_{j=1}^{i-1} a_{ij}(t)x_j + \sum_{j=i}^{\infty} a_{ij}(t)x_j - \sum_{j=1}^{i-1} a_{ij}(t)y_j - \sum_{j=i}^{\infty} a_{ij}(t)y_j \right| \\
&+ k_i \sup\{|x_j - y_j| : \ j \ge i\} \\
\le& \left| \sum_{j=1}^{i-1} a_{ij}(t)(x_j - y_j) \right| + \left| \sum_{j=i}^{\infty} a_{ij}(t)(x_j - y_j) \right| \\
&+ k_i \sup\{\mathrm{diam} X_j : j \ge i\} \\
\le& \sum_{j=1}^{i-1} a_{ij}(t)|x_j - y_j| + \sum_{j=i}^{\infty} a_{ij}(t)|x_i - x_j| \\
&+ \ k \sup\{\mathrm{diam} X_j : j \ge i\} \\
\le& \|x - y\| \sum_{j=1}^{i-1} a_{ij}(t) + \left(\sum_{j=i}^{\infty} a_{ij}(t) \right) \sup\{|x_j - y_j| : j \ge i\} \\
&+ k \sup\{\mathrm{diam} X_j : j \ge i\} \\
\le& \overline{A_i}(t) \mathrm{diam} X + \overline{\overline{A_i}}(t) \sup\{\mathrm{diam} X_j : j \ge i\} \\
&+ k \sup\{\mathrm{diam} X_j : \ j \ge i\}.
\end{aligned}
$$

The above estimate implies the following inequality

$$
\begin{aligned}
\mathrm{diam} f_i(t, X) \le& \overline{A_i}(t) \mathrm{diam} X + \overline{\overline{A_i}}(t) \sup\{\mathrm{diam} X_j : j \ge i\} \\
&+ \ k \sup\{\mathrm{diam} X_j : \ j \ge i\},
\end{aligned} \tag{4.4}
$$

which holds for any $i \in \mathbb{N}$.

Now, similarly as in the proof of Theorem 3.2 let us define the function $p : I_T \to \mathbb{R}_+$ by putting

$$p(t) = \limsup_{i \to \infty} \overline{\overline{A_i}}(t).$$

Then, from estimate (4.4) we derive the following inequality

$$\mu(f(t, X)) \le (p(t) + k)\mu(X),$$

where μ is the measure of noncompactness defined in Section 2 by formula (2.1).

Finally, gathering all above established facts and utilizing Theorem 3.2 we complete the proof. □

In what follows we draw our attention to the second special case of the semilinear infinite system of differential equations (3.1), which has the form

$$x_i' = \sum_{j=1}^{\infty} a_{ij}(t)x_j + g_i(t, x_1, x_2, \dots, x_i), \tag{4.5}$$

for $i = 1, 2, \ldots$ and for $t \in I_T$. Obviously system (4.5) will be considered with initial conditions (3.2) i.e.,

$$x_i(0) = x_i^0 \tag{4.6}$$

$(i = 1, 2, \ldots)$. Similarly as previously we will look for solutions of problem (4.5)–(4.6) in the space l^∞ and we will exploit as the main tool the measure of noncompactness μ defined by formula (2.1).

Obviusly, we impose here assumptions (i), (iv)–(vii) formulated earlier, while assumptions (ii), (iii) are replaced by the following ones:

($\overline{\text{ii}}$) For each fixed natural number i the function $g_i : I_T \times \mathbb{R}^i \to \mathbb{R}$ is continuous and for any natural number $j (1 \leq j \leq i)$ there exists a nonnegative constant k_j^i such that for arbitrary $x = (x_i)$, $y = (y_i) \in l^\infty$ and for every $t \in I_T$ the following inequality is satisfied:

$$|g_i(t, x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_i) - g_i(t, x_1, \ldots, x_{j-1}, y_j, x_{j+1}, \ldots, x_i)| \\ \leq k_j^i |x_j - y_j|.$$

($\overline{\text{iii}}$) For each $i \in \mathbb{N}$ we have that $k_i = \sum_{j=1}^{i} k_j^i < 1$. Moreover, $k < 1$, where $k = \sup\{k_i : i = 1, 2, \ldots\}$.

Then, we are in a position to formulate our next existence result.

Theorem 4.2. *Under assumptions (i), ($\overline{ii}$), ($\overline{iii}$), (iv)–(vii), if additionally $T(A + k) < 1$, problem* (4.5)–(4.6) *has at least one solution $x = x(t) = (x_i(t))$ defined on the interval I_T and such that $x(t) \in l^\infty$ for each $t \in I_T$.*

The proof can be conducted similarly as the proof of Theorem 4.1 and is therefore omitted.

REFERENCES

[1] R. R. Akhmerov, M. I. Kamenski, A. S. Potapov, A. E. Rodkina, and B. N. Sadovskii, *Measures of Noncompactness and Condensing Operators*, Oper. Theory Adv. Appl. 55, Birkhäuser, Basel 1992.

[2] J. Banaś and K. Goebel, *Measures of Noncompactness in Banach Spaces*, Lect. Notes in Pure and Appl. Math., vol. 60 Marcel Dekker, New York, 1980.

[3] J. Banaś and M. Lecko, *Solvability of infinite systems of differential equations in Banach sequence spaces*, J. Comput. Appl. Math. 137 (2001) 363–375.

[4] J. Banaś and M. Lecko, *On solutions of an infinite system of differential equations*, Dyn. Syst. Appl. 11 (2002) 221–230.

[5] J. Banaś and B. Rzepka, *On existence and asymptotic behavior of solutions of infinite systems of differential equations*, Panamer. Math. J. 14 (2004) 105–115.

[6] E.A. Coddington and N. Levinson, *Theory of Differential Equations*, McGraw-Hill, New York 1955.

[7] K. Deimling, *Ordinary Differential Equations in Banach Spaces*, Lect. Notes in Math. 596, Springer Verlag, Berlin 1977.

[8] J. Dieudonné, *Deux examples singuliérs d'équation différentielles*, Acta Sci. Math. Szeged 12 (1950) 38–40.

[9] P. Hartman, *Ordinary Differential Equations*, J. Wiley, New York 1964.

[10] J. Kisyński, *Sur les équations différentielles dans les espaces de Banach*, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys. 7 (1959) 381–385.

[11] M. Mursaleen and S.A. Mohiuddine, *Applications of measures of noncompactness to the infinite system of differential equations in l_p spaces*, Nonlin. Anal. 75 (2012) 2111–2115.

[12] M. Mursaleen and A. Alotaibi, *Infinite system of differential equations in some BK spaces*, Abstact Appl. Anal., Volume 2012 (2012), Article ID 863484, 20 pages.

[13] C. Olech, *On the existence and uniqueness of solutions of an ordinary differential equation in the case of Banach space*, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys. 8 (1960) 667–675.

[14] B.N. Sadovskii, *Differential equations with uniformly continuous right side*, Tr. Nauk. Mat. Voronesh. Gos. Univ. 1 (1970) 128–136.

[15] T. Ważewski, *Sur l'existence et l'unicité des intégrales des équations différentielles ordinaries an cas de l'espace de Banach*, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys. 8 (1960) 301–305.