Dynamic Systems and Applications 22 (2013) 325-336

A DIMENSION-DEPENDING MULTIPLICITY RESULT FOR A PERTURBED SCHRÖDINGER EQUATION

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ABSTRACT. We consider the Schrödinger equation

$$\begin{cases} -\Delta u + V(x)u = \lambda K(x)f(u) + \mu L(x)g(u) & \text{in} \quad \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{$P_{\lambda,\mu}$}$$

where $N \geq 2$, $\lambda, \mu \geq 0$ are parameters, $V, K, L : \mathbb{R}^N \to \mathbb{R}$ are radially symmetric potentials, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function with sublinear growth at infinity, and $g : \mathbb{R} \to \mathbb{R}$ is a continuous sub-critical function. We first prove that for λ small enough no non-zero solution exists for $(P_{\lambda,0})$, while for λ large and μ small enough at least two distinct non-zero radially symmetric solutions do exist for $(P_{\lambda,\mu})$. By exploiting a Ricceri-type three-critical points theorem, the principle of symmetric criticality and a group-theoretical approach, the existence of at least $N - 3\,(N \bmod 2)$ distinct pairs of non-zero solutions is guaranteed for $(P_{\lambda,\mu})$ whenever λ is large and μ is small enough, $N \neq 3$, and f, g are odd.

Keywords: Schrödinger equation, sublinear, three-critical points theorem, principle of symmetric criticality

1. INTRODUCTION

In this paper we consider the perturbed Schrödinger equation

$$\begin{cases} -\Delta u + V(x)u = \lambda K(x)f(u) + \mu L(x)g(u) & \text{in} \quad \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{$P_{\lambda,\mu}$}$$

where $N \geq 2$, $V, K, L : \mathbb{R}^N \to \mathbb{R}$ are some non-negative potentials, $\lambda, \mu \geq 0$ are parameters, while $f, g : \mathbb{R} \to \mathbb{R}$ are nonlinear continuous functions with different behavior. The interest in this problem comes from mathematical physics; for instance, certain kinds of solitary waves in the nonlinear Klein-Gordon or Schrödinger equations appear as solutions of problem $(P_{\lambda,\mu})$.

The first author was supported by Grant CNCSIS PCCE-55/2008 "Sisteme diferenţiale în analiza neliniară şi aplicaţii"

Received January 16, 2013

The non-perturbed problem $(P_\lambda) = (P_{\lambda,0})$ or its related form has been studied by many authors during the last two decades under various assumptions on the potentials V, K and on the nonlinear function f. Most of these papers address the case when V and K have suitable sign- and growth-properties, and f has a superlinear and subcritical growth. In these papers existence and multiplicity results for (P_λ) are established via various variational arguments, see Rabinowitz [8], Bartsch *et al.* [1, 2], and further subsequent papers. In particular, if f is odd, the existence of infinitely many solutions for (P_λ) is usually guaranteed. A particularly interesting paper is due to Clapp and Weth [3] where the existence of at least $\frac{N}{2}+1$ pairs of non-zero solutions for (P_λ) is proved for every $\lambda > 0$ by assuming certain one-sided asymptotic estimates for V and K when $f(s) = |s|^{p-2}s$, $p \in (2, 2^*)$. Problem (P_λ) has been also studied in the case when f is odd and has an asymptotically linear growth at infinity; more precisely, Liu, van Heerden and Wang [7] prove a multiplicity result for (P_λ) whenever $V(x) = \mu g(x) + 1$, $\mu > 0$, and the number of solutions for (P_λ) depends on the behavior of the dimension of the eigenspace of a specific Dirichlet eigenvalue problem defined on the bounded domain $\Omega = \text{int}(g^{-1}(0))$.

The aim of the present paper is to supplement the aforementioned contributions by requiring that the non-zero continuous function $f : \mathbb{R} \to \mathbb{R}$ has a *sublinear growth at infinity* and a *superlinear growth near zero*, and problem $(P_\lambda) = (P_{\lambda,0})$ is perturbed by an arbitrarily nonlinear term. More precisely, we assume that

(f_1) $f(s) = o(|s|)$ as $|s| \to \infty$;

(f_2) $f(s) = o(|s|)$ as $s \to 0$;

(f_3) there exists $s_0 \in \mathbb{R}$ such that $F(s_0) > 0$, where $F(s) = \int_0^s f(t)dt$.

In order to avoid technicalities, we assume in the sequel that potentials $V, K, L : \mathbb{R}^N \to \mathbb{R}$ satisfy

(H_V) $V \in C(\mathbb{R}^N)$ is radially symmetric and $\inf_{\mathbb{R}^N} V > 0$;

$(H_{K,L})$ $K, L \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ are radially symmetric and $K \geq 0$, $K \not\equiv 0$.

Note that solutions of $(P_{\lambda,\mu})$ are being sought in weak form in the space

$$W = \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)dx < \infty \right\}.$$

In fact, under the above conditions on f, V and K which will be assumed throughout in the sequel, every weak solution u of $(P_{\lambda,0})$ is a classical one. Indeed, we have $\Delta u =: h \in L^2_{\text{loc}}(\mathbb{R}^N)$, thus $u \in H^2_{\text{loc}}(\mathbb{R}^N)$ (cf. Evans [4, §8.3]) and u satisfies (P_λ) for a.a. $x \in \mathbb{R}^N$.

The hypotheses (f_1), (f_2) and (f_3) guarantee that the number

$$c_f = \max_{s \neq 0} \left| \frac{f(s)}{s} \right|$$

is well-defined, positive and finite. Now, we are in a position to state our main result.

Theorem 1.1. *Assume that* $N \geq 2$. *Let* $V, K, L : \mathbb{R}^N \to \mathbb{R}$ *be two potentials such that both* (H_V) *and* $(H_{K,L})$ *hold, and let* $f : \mathbb{R} \to \mathbb{R}$ *be a continuous function verifying* $(f_1) - (f_3)$. *Then, the following assertions hold:*

(i) *For every* $\lambda \in [0, c_f^{-1}\|K\|_{L^\infty}^{-1}\inf_{\mathbb{R}^N} V)$, *problem* $(P_\lambda) = (P_{\lambda,0})$ *has only the zero solution;*

(ii) *There exists* $\Lambda_0 > 0$ *such that for every* $\lambda > \Lambda_0$ *and every subcritical nonlinearity* $g : \mathbb{R} \to \mathbb{R}$, *there exists* $\delta_0 > 0$ *such that for every* $\mu \in [0, \delta_0]$, *problem* $(P_{\lambda,\mu})$ *has at least two distinct non-zero radially symmetric weak solutions in* W;

(iii) *If* f *is odd, there exists* $\Lambda_1 > 0$ *such that for every* $\lambda > \Lambda_1$ *and every odd subcritical nonlinearity* $g : \mathbb{R} \to \mathbb{R}$, *there exists* $\delta_1 > 0$ *such that for every* $\mu \in [0, \delta_1]$, *problem* $(P_{\lambda,\mu})$ *has at least* $s_N = \max\{2, N - 3 \cdot (N \bmod 2)\}$ *distinct pairs of non-zero weak solutions* $\{\pm u_i^\lambda\} \subset W$, $i = 1, \ldots, s_N$.

The function $g : \mathbb{R} \to \mathbb{R}$ is said to be subcritical is for some $c > 0$ and $2 < p < 2^*$,

$$|g(s)| \leq c(|s| + |s|^{p-1}) \text{ for all } s \in \mathbb{R}.$$

The proof of Theorem 1.1 (i) is direct. In order to prove Theorem 1.1 (ii)–(iii), we find critical points of the energy functional associated with problem $(P_{\lambda,\mu})$ by means of a Ricceri-type three-critical points theorem and the well-known Palais' principle of symmetric criticality. In particular, the proof of the multiplicity in Theorem 1.1 (iii) requires special treatment. Our strategy is to apply Ricceri's result to some particular *subspaces* of W which have two main properties:

- they can be compactly embedded into $L^p(\mathbb{R}^N)$, $p \in (2, 2^*)$;
- they cannot be compared from a symmetrical point of view, i.e., their pairwise intersections contain only the 0 element.

After a careful group-theoretical analysis inspired from Bartsch and Willem [2], we construct $s'_N = \left[\frac{N-1}{2}\right] + (-1)^N$ such subspaces of W whenever $N \neq 3$. Further energy-level analysis together with Ricceri's multiplicity result provides at least two pairs of distinct non-zero solutions for $(P_{\lambda,\mu})$ belonging to these subspaces separately whenever λ is large enough and μ is small. Thus, the minimal number of distinct pairs of non-zero solutions for $(P_{\lambda,\mu})$ is $s_N = 2s'_N = N - 3\,(N \bmod 2)$. One can also observe that $s_N \geq 2$ for every $N \neq 3$. Furthermore, in each dimension $N \geq 2$, two pairs of solutions are radially symmetric, while if $s_N > 2$ (which occurs for $N = 4$ or $N \geq 6$), the rest of the $(s_N - 2)$ pairs of solutions are sign-changing and non-radially symmetric functions. This statement is based on the aforementioned group-theoretical argument which is described in Section 2.

In Section 2 we recall Ricceri's three-critical point theorem and display the group-theoretical arguments needed for the proof of Theorem 1.1 (iii). In Section 3 we prove our main theorem.

2. PRELIMINARIES

2.1. **A Ricceri-type three-critical point theorem.** The functional space

$$W = \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)dx < \infty \right\}$$

is endowed with its natural inner product $\langle u, v\rangle_W = \int_{\mathbb{R}^N} (\nabla u \nabla v + V(x)uv)\,dx$ and norm $\|\cdot\|_W = \sqrt{\langle\cdot,\cdot\rangle_W}$. Due to hypothesis (H_V), it is clear that the embeddings $W \subset H^1(\mathbb{R}^N) \subset L^p(\mathbb{R}^N)$ are continuous, $p \in [2, 2^*)$. Here, $2^* = \infty$ if $N = 2$, and $2^* = 2N/(N-2)$ for $N \geq 3$. Once (f_1) and (f_2) hold, the functional $\mathcal{F} : W \to \mathbb{R}$ defined by

$$\mathcal{F}(u) = \int_{\mathbb{R}^N} K(x)F(u)dx$$

is well-defined and is of class C^1. If g is subcritical and L verifies (H_L), the functional $J : W \to \mathbb{R}$ defined by

$$\mathcal{J}(u) = \int_{\mathbb{R}^N} L(x)G(u)dx,$$

is of class C^1. Moreover, the critical points of the functional $E_{\lambda,\mu} : W \to \mathbb{R}$ defined by

$$E_{\lambda,\mu}(u) = \frac{1}{2}\|u\|_W^2 - \lambda\mathcal{F}(u) - \mu\mathcal{J}(u)$$

are precisely the weak solutions for problem $(P_{\lambda,\mu})$. In order to find critical points for $E_{\lambda,\mu}$, we will apply the principle of symmetric criticality together with a recent critical point theorem due to Ricceri [9]. In order to recall Ricceri's result, we need the following definition: if X is a Banach space, we denote by $\mathcal{W}_X$ the class of those functionals $E : X \to \mathbb{R}$ having the property that if $\{u_n\}$ is a sequence in X converging weakly to $u \in X$ and $\liminf_n E(u_n) \leq E(u)$ then $\{u_n\}$ has a subsequence converging strongly to u.

Theorem 2.1 ([9, Theorem 2]). *Let $(X, \|\cdot\|)$ be a separable, reflexive, real Banach space, let $E_1 : X \to \mathbb{R}$ be a coercive, sequentially weakly lower semicontinuous C^1 functional belonging to $\mathcal{W}_X$, bounded on each bounded subset of X and whose derivative admits a continuous inverse on X^*; and let $E_2 : X \to \mathbb{R}$ be a C^1 functional with compact derivative. Assume that E_1 has a strict local minimum point u_0 with $E_1(u_0) = E_2(u_0) = 0$. Assume that $\tau < \chi$, where*

$$\tau := \max\left\{0, \limsup_{\|u\|\to\infty} \frac{E_2(u)}{E_1(u)}, \limsup_{u\to u_0} \frac{E_2(u)}{E_1(u)}\right\}, \tag{2.1}$$

$$\chi = \sup_{E_1(u)>0} \frac{E_2(u)}{E_1(u)}. \tag{2.2}$$

Then, for each compact interval $[a, b] \subset (1/\chi, 1/\tau)$ (*with the conventions* $1/0 = \infty$ *and* $1/\infty = 0$) *there exists* $\kappa > 0$ *with the following property: for every* $\lambda \in [a, b]$ *and every* C^1 *functional* $E_3 : X \to \mathbb{R}$ *with compact derivative, there exists* $\delta > 0$ *such that for each* $\mu \in [0, \delta]$*, the equation*

$$E_1'(u) - \lambda E_2'(u) - \mu E_3'(u) = 0$$

admits at least three solutions in X *having norm less than* κ*.*

2.2. **Special symmetries.** Let $N \geq 2$ be fixed and assume that a closed subgroup of the orthogonal group $\mathbf{O}(N)$ acts on the space W, i.e., $(\phi, u) \mapsto \phi * u \in W$, $\phi \in G$, $u \in W$. We define the set of fixed points of W with respect to the group G which contains the G-invariant functions, i.e.,

$$W_G = \{u \in W : \phi * u = u \text{ for every } \phi \in G\}.$$

In particular, if $G = \mathbf{O}(N)$ and $'*'$ is the standard linear and isometric action defined as

$$(\phi * u)(x) = u(\phi^{-1}x) \text{ for } x \in \mathbb{R}^N, \phi \in \mathbf{O}(N), \tag{2.3}$$

the set $W_{\mathbf{O}(N)}$ is exactly the subspace of radially symmetric functions of W. Standard arguments show that $W_{\mathbf{O}(N)} \subset L^p(\mathbb{R}^N)$ is compact for every $p \in (2, 2^*)$, see Lions [6].

In order to prove Theorem 1.1 (iii), more specific groups and actions are needed whose origin can be found in Bartsch and Willem [2]. Let us fix $N = 4$ or $N \geq 6$ and define the number

$$t_N = \left[\frac{N-3}{2}\right] + (-1)^N.$$

Note that $t_N \geq 1$ and for every $i \in \{1, \ldots, t_N\}$, we may introduce the following subgroups of the orthogonal group $\mathbf{O}(N)$:

$$G_{N,i} = \begin{cases} \mathbf{O}(\frac{N}{2}) \times \mathbf{O}(\frac{N}{2}), & \text{if } \ i = \frac{N-2}{2}, \\ \mathbf{O}(i+1) \times \mathbf{O}(N-2i-2) \times \mathbf{O}(i+1), & \text{if } \ i \neq \frac{N-2}{2}. \end{cases}$$

We introduce the involution function $\tau_i : \mathbb{R}^N \to \mathbb{R}^N$ associated with $G_{N,i}$ by

$$\tau_i(x) = \begin{cases} (x_3, x_1), & \text{if } \ i = \frac{N-2}{2}, \text{ and } x = (x_1, x_3) \text{ with } x_1, x_3 \in \mathbb{R}^{\frac{N}{2}}; \\ (x_3, x_2, x_1), & \text{if } \ i \neq \frac{N-2}{2}, \text{ and } x = (x_1, x_2, x_3) \text{ with } x_1, x_3 \in \mathbb{R}^{i+1}, \ x_2 \in \mathbb{R}^{N-2i-2}. \end{cases}$$

By definition, we clearly have that $\tau_i \notin G_{N,i}$, $\tau_i G_{N,i} \tau_i^{-1} = G_{N,i}$ and $\tau_i^2 = \mathrm{id}_{\mathbb{R}^N}$.

Now, let $G_{N,i}^{\tau_i} = \langle G_{N,i}, \tau_i \rangle = G_{N,i} \cup \tau_i G_{N,i}$. We know from the properties of τ_i that only two types of elements in $G_{N,i}^{\tau_i}$ can be distinguished; namely, $\phi = g \in G_{N,i}$, and $\phi = \tau_i g \in G_{N,i}^{\tau_i} \setminus G_{N,i}$ (with $g \in G_{N,i}$). The action of the compact group $G_{N,i}^{\tau_i}$ on W is defined by

$$(g * u)(x) = u(g^{-1}x), \qquad ((\tau_i g) * u)(x) = -u(g^{-1}\tau_i^{-1}x), \tag{2.4}$$

for $g \in G_{N,i}$, $u \in W$ and $x \in \mathbb{R}^N$. Now from Bartsch and Willem [2, pp. 455-457], the embedding $W_{G_{N,i}^{\tau_i}} \subset L^p(\mathbb{R}^N)$ is compact for every $p \in (2, 2^*)$.

The next result is of crucial importance in Theorem 1.1 (iii); a similar statement can be found in Kristály, Rădulescu and Varga [5], thus we omit its proof.

Theorem 2.2. *The following statements hold true:*

(i) *If $N = 4$ or $N \geq 6$, then $W_{G_{N,i}^{\tau_i}} \cap W_{\mathbf{O}(N)} = \{0\}$ for all $i \in \{1, \dots, t_N\}$;*
(ii) *If $N = 6$ or $N \geq 8$, then $W_{G_{N,i}^{\tau_i}} \cap W_{G_{N,j}^{\tau_j}} = \{0\}$ for every $i, j \in \{1, \dots, t_N\}$ with $i \neq j$.*

3. PROOF OF THEOREM 1.1

In the sequel we assume that all the assumptions of Theorem 1.1 are fulfilled.

Proof of Theorem 1.1 (i). Assume that $u \in W$ is a solution of $(P_\lambda) = (P_{\lambda,0})$. Multiplying (P_λ) by the test function u and using the definition of the number $c_f > 0$, we obtain

$$\begin{aligned} \|u\|_W^2 &= \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)dx \\ &= \lambda \int_{\mathbb{R}^N} K(x)f(u)u \\ &\leq \lambda \frac{\|K\|_{L^\infty}}{\inf_{\mathbb{R}^N} V} c_f \int_{\mathbb{R}^N} V(x)u^2 \\ &\leq \lambda \frac{\|K\|_{L^\infty}}{\inf_{\mathbb{R}^N} V} c_f \|u\|_W^2. \end{aligned}$$

Now, if $0 \leq \lambda < c_f^{-1}\|K\|_{L^\infty}^{-1} \inf_{\mathbb{R}^N} V$, the above estimate implies $u = 0$, which concludes the proof of (i). □

As we pointed out in the Introduction, the solutions of $(P_{\lambda,\mu})$ are exactly the critical points for the functional $E_{\lambda,\mu} = \mathcal{E}_1 - \lambda\mathcal{E}_2 - \mu\mathcal{E}_3 : W \to \mathbb{R}$, where

$$\mathcal{E}_1(u) = \frac{1}{2}\|u\|_W^2, \ \mathcal{E}_2(u) = \mathcal{F}(u) \text{ and } \mathcal{E}_3(u) = \mathcal{J}(u), \ u \in W.$$

Before proving (ii) and (iii) of Theorem 1.1, we need the following

Lemma 3.1. (i) $\limsup_{\|u\|_W \to \infty} \frac{\mathcal{F}(u)}{\|u\|_W^2} \leq 0$;
(ii) $\limsup_{u \to 0} \frac{\mathcal{F}(u)}{\|u\|_W^2} \leq 0$;
(iii) *Let X be a closed subspace of W which is compactly embedded into $L^r(\mathbb{R}^N)$, $r \in (2, 2^*)$. Then $\mathcal{F}|_X$ and $\mathcal{J}|_X$ have compact derivatives.*

Proof. Due to (f_1) and (f_2), for every fixed $\varepsilon > 0$ there is a $\delta_\varepsilon \in (0,1)$ such that

$$|f(s)| < \varepsilon \frac{\inf_{\mathbb{R}^N} V}{\|K\|_{L^\infty}} |s| \text{ for all } |s| \leq \delta_\varepsilon \text{ and } |s| \geq \delta_\varepsilon^{-1}.$$

Since $f \in C(\mathbb{R}, \mathbb{R})$, there also exist two constants $M_\varepsilon^1, M_\varepsilon^2 > 0$ such that

$$\frac{|f(s)|}{|s|^{q-1}} \leq M_\varepsilon^1 \text{ and } \frac{|f(s)|}{|s|^{p-1}} \leq M_\varepsilon^2 \text{ for all } |s| \in [\delta_\varepsilon, \delta_\varepsilon^{-1}],$$

where $1 < q < 2 < p < 2^*$. Combining the above two relations, we obtain that

$$|f(s)| \leq \varepsilon \frac{\inf_{\mathbb{R}^N} V}{\|K\|_{L^\infty}} |s| + M_\varepsilon^1 |s|^{q-1} \quad \text{for all } s \in \mathbb{R}; \tag{3.1}$$

$$|f(s)| \leq \varepsilon \frac{\inf_{\mathbb{R}^N} V}{\|K\|_{L^\infty}} |s| + M_\varepsilon^2 |s|^{p-1} \quad \text{for all } s \in \mathbb{R}. \tag{3.2}$$

On account of (3.1), since $K \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ and the embedding $W \subset L^r(\mathbb{R}^N)$ is continuous for $r \in (2, 2^*)$, by the Hölder inequality one can find $C_\varepsilon^1 > 0$ such that

$$\begin{aligned} \mathcal{F}(u) &\leq \int_{\mathbb{R}^N} K(x)|F(u)| \\ &\leq \int_{\mathbb{R}^N} K(x) \left[\varepsilon \frac{\inf_{\mathbb{R}^N} V}{2\|K\|_{L^\infty}} u^2 + \frac{M_\varepsilon^1}{q} |u|^q \right] \\ &\leq \frac{\varepsilon}{2} \|u\|_W^2 + C_\varepsilon^1 \|u\|_W^q. \end{aligned}$$

Consequently, for every $u \in W \setminus \{0\}$, we have that

$$\frac{\mathcal{F}(u)}{\|u\|_W^2} \leq \frac{\varepsilon}{2} + C_\varepsilon^1 \|u\|_W^{q-2}.$$

Since $q < 2$, the arbitrariness of $\varepsilon > 0$ yields (i).

A similar argument based on (3.2) gives the existence of a $C_\varepsilon^2 > 0$ such that for every $u \in W \setminus \{0\}$,

$$\frac{\mathcal{F}(u)}{\|u\|_W^2} \leq \frac{\varepsilon}{2} + C_\varepsilon^2 \|u\|_W^{p-2}.$$

Since $\varepsilon > 0$ is arbitrary and $p > 2$, (ii) follows readily.

The proof of (iii) is standard. □

For any $0 < r_1 < r_2$, let $A[r_1, r_2] = \{x \in \mathbb{R}^N : r_1 \leq |x| \leq r_2\}$ be the closed annulus with radii r_1 and r_2 . Since $K \in L^\infty(\mathbb{R}^N)$ is a radially symmetric function with $K \geq 0$ and $K \not\equiv 0$ (cf. hypothesis (H_K)), one can find real numbers $R > r > 0$ and $K_0 > 0$ such that

$$\operatorname{ess\,inf}_{x \in A[r,R]} K(x) \geq K_0. \tag{3.3}$$

Proof of Theorem 1.1 (ii). Let $s_0 \in \mathbb{R}$ from (f_3). For a fixed $\sigma \in (0, (R-r)/2)$ with r, R from (3.3), we can define a radially symmetric truncation function $u_\sigma \in W_{\mathbf{O}(N)}$ such that

(a) $\operatorname{supp} u_\sigma \subseteq A[r,R]$;
(b) $\|u_\sigma\|_{L^\infty} \leq |s_0|$;
(c) $u_\sigma(x) = s_0$ for every $x \in A[r+\sigma, R-\sigma]$.

Here is an example of such a function $u_\sigma : \mathbb{R}^N \to \mathbb{R}$

$$u_\sigma(x) = \begin{cases} \frac{s_0}{\sigma}(|x|-r)_+ & \text{if } \ |x| \leq r+\sigma; \\ s_0 & \text{if } \ r+\sigma < |x| \leq R-\sigma; \\ \frac{s_0}{\sigma}(R-|x|)_+ & \text{if } \ |x| \geq R-\sigma, \end{cases}$$

where $z_+ = \max(z,0)$. Denoting by ω_N the volume of the unit ball in $\mathbb{R}^N$, we clearly have from the properties (a)–(c) and relation (3.3) that

$$\|u_\sigma\|_W^2 \geq s_0^2 \omega_N \inf_{\mathbb{R}^N} V\left((R-\sigma)^N - (r+\sigma)^N\right),$$

and

$$\begin{aligned} \mathcal{F}(u_\sigma) \ \geq \ & \omega_N [K_0 F(s_0)\left((R-\sigma)^N - (r+\sigma)^N\right) - \|K\|_{L^\infty} \max_{|t|\leq |s_0|} |F(t)| \times \\ & \times \left((r+\sigma)^N - r^N + R^N - (R-\sigma)^N\right)]. \end{aligned}$$

If σ is close enough to 0, the right-hand sides of both inequalities are strictly positive. Therefore, we can define the number

$$\lambda_0 = \inf\left\{\frac{\|u\|_W^2}{2\mathcal{F}(u)} : u \in W_{\mathbf{O}(N)},\ \mathcal{F}(u) > 0\right\}. \tag{3.4}$$

Moreover, it is also clear (cf. Lemma 3.1 and the above estimates) that

$$\chi_0 = \sup\left\{\frac{2\mathcal{F}(u)}{\|u\|_W^2} : u \in W_{\mathbf{O}(N)} \setminus \{0\}\right\} \in (0,\infty)$$

and $\chi_0^{-1} = \lambda_0$.

Now, we are in a position to apply Theorem 2.1 with $X = W_{\mathbf{O}(N)}$ and $E_1, E_2, E_3 : W_{\mathbf{O}(N)} \to \mathbb{R}$ defined by $E_1 = \mathcal{E}_1|_{W_{\mathbf{O}(N)}}$, $E_2 = \mathcal{E}_2|_{W_{\mathbf{O}(N)}}$ and $E_3 = \mathcal{E}_3|_{W_{\mathbf{O}(N)}}$. On account of Lemma 3.1, the assumptions of Theorem 2.1 are fulfilled with $u_0 = 0 \in W_{\mathbf{O}(N)}$ and $\tau = 0$.

Thus, for every $\lambda > \Lambda_0 := \lambda_0 = \chi_0^{-1} > 0$ and every subcritical nonlinearity $g : \mathbb{R} \to \mathbb{R}$, there exists $\delta_0 > 0$ such that for every $\mu \in [0,\delta_0]$ the functional $E_{\lambda,\mu}|_{W_{\mathbf{O}(N)}}$ has at least three distinct critical points in $W_{\mathbf{O}(N)}$. Since $E_{\lambda,\mu}$ is $\mathbf{O}(N)-$invariant, i.e., $E_{\lambda,\mu}(\phi * u) = E_{\lambda,\mu}(u)$ for every $\phi \in \mathbf{O}(N)$ and $u \in W$ (cf. relation (2.3) and hypotheses (H_V) and $(H_{K,L})$), the principle of symmetric criticality implies that the critical points of $E_{\lambda,\mu}|_{W_{\mathbf{O}(N)}}$ are also critical points for $E_{\lambda,\mu}$. This concludes the proof. $\square$

Proof of Theorem 1.1 (iii). Let $N \neq 3$. Since f and g are odd, the energy functional $E_{\lambda,\mu}$ is even, and its critical points (hence solutions for $(P_{\lambda,\mu})$) appear in symmetric pairs. Consequently, a similar argument as in (ii) shows that there exists $\lambda_0 > 0$ such that for every $\lambda > \lambda_0$ and every odd subcritical nonlinearity $g : \mathbb{R} \to \mathbb{R}$, there exists $\delta_0^\lambda > 0$ such that for every $\mu \in [0,\delta_0^\lambda]$ problem $(P_{\lambda,\mu})$ has at least two pairs of solutions $\{\pm u_{0,1}^{\lambda,\mu}\}$ and $\{\pm u_{0,2}^{\lambda,\mu}\}$ which are non-zero distinct functions belonging to $W_{\mathbf{O}(N)}$. In the case when $N = 2$ or $N = 5$ we have $s_N = 2$, i.e., the conclusion of (iii) follows from the latter arguments.

Consequently, it remains to consider $N = 4$ or $N \geq 6$. In this case $t_N \geq 1$, so we may fix $i \in \{1, \ldots, t_N\}$ arbitrarily. Without any loss of generality, we may assume for $0 < r < R$ in relation (3.3) that $r(5 + 4\sqrt{2}) \geq R$. Due to the latter choice, it is clear that the sets

$$Q_1 = \left\{ (x_1, x_3) \in \mathbb{R}^{i+1} \times \mathbb{R}^{i+1} : \sqrt{\left(|x_1| - \frac{R+3r}{4} \right)^2 + |x_3|^2} \leq \frac{R-r}{4} \right\};$$

$$Q_2 = \left\{ (x_1, x_3) \in \mathbb{R}^{i+1} \times \mathbb{R}^{i+1} : \sqrt{\left(|x_3| - \frac{R+3r}{4} \right)^2 + |x_1|^2} \leq \frac{R-r}{4} \right\}$$

are disjoint. For every $\sigma \in (0, 1]$, we introduce the set

$$\begin{aligned} D_\sigma^i = \Bigg\{ x \in \mathbb{R}^N : & \sqrt{\left(|x_1| - \frac{R+3r}{4} \right)^2 + |x_3|^2} \leq \sigma \frac{R-r}{4}, \\ & \sqrt{\left(|x_3| - \frac{R+3r}{4} \right)^2 + |x_1|^2} \leq \sigma \frac{R-r}{4}, \\ & |x_2| \leq \sigma \frac{R-r}{4} \Bigg\}, \end{aligned}$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^N$ with $x_1, x_3 \in \mathbb{R}^{i+1}$, $x_2 \in \mathbb{R}^{N-2i-2}$ whenever $i \neq \frac{N-2}{2}$, and $x = (x_1, x_3) \in \mathbb{R}^N$ with $x_1, x_3 \in \mathbb{R}^{\frac{N}{2}}$ whenever $i = \frac{N-2}{2}$ (and x_2 is considered formally 0). Note that the set $D_\sigma^i \subset \mathbb{R}^N$ is $G_{N,i}^{\tau_i}$-invariant, i.e., $\phi D_\sigma^i \subset D_\sigma^i$ for every $\phi \in G_{N,i}^{\tau_i}$. Moreover, $\operatorname{meas}(D_\sigma^i) > 0$ for every $\sigma \in (0, 1]$ and

$$\lim_{\sigma \to 1} \operatorname{meas}(D_1^i \setminus D_\sigma^i) = 0. \tag{3.5}$$

Let $s_0 \in \mathbb{R}$ from (f_3) and for a fixed number $\sigma \in (0, 1)$, we construct the following special truncation function

$$\begin{aligned} u_\sigma^i(x) \;=\; & \Bigg[\left(\frac{R-r}{4} - \max\left(\sqrt{\left(|x_1| - \frac{R+3r}{4} \right)^2 + |x_3|^2}, \sigma \frac{R-r}{4} \right) \right)_+ \\ & - \left(\frac{R-r}{4} - \max\left(\sqrt{\left(|x_3| - \frac{R+3r}{4} \right)^2 + |x_1|^2}, \sigma \frac{R-r}{4} \right) \right)_+ \Bigg] \times \\ & \times \left(\frac{R-r}{4} - \max\left(|x_2|, \sigma \frac{R-r}{4} \right) \right)_+ \frac{16 s_0}{(R-r)^2 (1-\sigma)^2}. \end{aligned}$$

The special shape of u_σ^i shows that $\phi * u_\sigma^i = u_\sigma^i$ for every $\phi \in G_{N,i}^{\tau_i}$ (see relation (2.4)), thus $u_\sigma^i \in W_{G_{N,i}^{\tau_i}}$. Moreover, the following useful properties hold:

(a') $\operatorname{supp} u^i_\sigma = D^i_1 \subseteq A[r,R]$;
(b') $\|u^i_\sigma\|_{L^\infty} \leq |s_0|$;
(c') $|u^i_\sigma(x)| = |s_0|$ for every $x \in D^i_\sigma$.

Since F is even (thus $F(s_0) = F(-s_0)$), by exploiting the properties (a')–(c'), we obtain that

$$\mathcal{F}(u^i_\sigma) \geq K_0 F(s_0)\text{meas}(D^i_\sigma) - \|K\|_{L^\infty} \max_{|t|\leq|s_0|} |F(t)|\text{meas}(D^i_1 \setminus D^i_\sigma).$$

If σ is close enough to 1, the right-hand side of the latter term is strictly positive, see (3.5). Consequently, we can introduce the number

$$\lambda_i = \inf\left\{\frac{\|u\|^2_W}{2\mathcal{F}(u)} : u \in W_{G^{\tau_i}_{N,i}},\ \mathcal{F}(u) > 0\right\}. \tag{3.6}$$

As before, one has that

$$\chi_i = \sup\left\{\frac{2\mathcal{F}(u)}{\|u\|^2_W} : u \in W_{G^{\tau_i}_{N,i}} \setminus \{0\}\right\} \in (0,\infty)$$

and $\chi_i^{-1} = \lambda_i$.

We can apply Theorem 2.1 with $X = W_{G^{\tau_i}_{N,i}}$ and $E_1, E_2, E_3 : W_{G^{\tau_i}_{N,i}} \to \mathbb{R}$ defined by $E_1 = \mathcal{E}_1|_{W_{G^{\tau_i}_{N,i}}}$, $E_2 = \mathcal{E}_2|_{W_{G^{\tau_i}_{N,i}}}$ and $E_3 = \mathcal{E}_3|_{W_{G^{\tau_i}_{N,i}}}$. Due to Lemma 3.1, the assumptions of Theorem 2.1 are satisfied with $u_0 = 0 \in W_{G^{\tau_i}_{N,i}}$ and $\tau = 0$. Consequently, for every $\lambda > \chi_i^{-1} = \lambda_i > 0$, there exists $\delta^\lambda_i > 0$ such that for each $\mu \in [-\delta^\lambda_i, \delta^\lambda_i]$, the functional $E_{\lambda,\mu}|_{W_{G^{\tau_i}_{N,i}}}$ has at least three critical points in $W_{G^{\tau_i}_{N,i}}$.

Due to the evenness of $E_{\lambda,\mu}$, relation (2.4), and hypotheses (H_V), $(H_{K,L})$, we have that $E_{\lambda,\mu}(\phi * u) = E_{\lambda,\mu}(u)$ for every $\phi \in G^{\tau_i}_{N,i}$ and $u \in W$, i.e., $E_{\lambda,\mu}$ is $G^{\tau_i}_{N,i}$–invariant on W. On account of the principle of symmetric criticality, the critical point pairs $\{\pm u^{\lambda,\mu}_{i,1}\}$ and $\{\pm u^{\lambda,\mu}_{i,2}\}$ of $E_{\lambda,\mu}|_{W_{G^{\tau_i}_{N,i}}}$ are also critical point pairs for $E_{\lambda,\mu}$ whenever $\lambda > \lambda_i$ and $\mu \in [-\delta^\lambda_i, \delta^\lambda_i]$, hence solutions for problem $(P_{\lambda,\mu})$.

Now, it remains to count the number of distinct solutions of the above type. Due to Theorem 2.2, there are at least $(1+t_N)$ subspaces of W whose mutual intersections contain only the 0 element:

(I) the subspace $W_{\mathbf{O}(N)}$ of radially symmetric functions of W, and
(II) t_N subspace(s) of W of the type $W_{G^{\tau_i}_{N,i}}$.

As we pointed out above, each of these subspaces contain two distinct pairs of non-zero solutions for $(P_{\lambda,\mu})$ whenever λ is large enough and μ is small enough. More precisely, if

$$\lambda > \Lambda_1 := \max\{\lambda_0, \lambda_1, \ldots, \lambda_{t_N}\} \text{ and } 0 \leq \mu \leq \min\{\delta^\lambda_0, \delta^\lambda_1, \ldots, \delta^\lambda_{t_N}\} =: \delta_1,$$

where λ_0 and δ^λ_0 come from the radial case (see (3.4)), while λ_i is from (3.6), $i \in \{1,\ldots,t_N\}$, problem $(P_{\lambda,\mu})$ has at least

$$s_N = 2(1+t_N) = N - 3(N \bmod 2)$$

distinct pairs of non-zero solutions. This concludes our proof. □

Remark 3.2. The statement of Theorem 1.1 (iii) is not relevant for $N = 3$ since $s_3 = 0$. However, Theorem 1.1 (ii) gives two distinct (pairs of) non-zero, radially symmetric solutions for $(P_{\lambda,\mu})$ whenever λ is large and μ is small enough (and f, g are odd).

Remark 3.3. The proof of Theorem 1.1 (iii) shows that in each dimension $N \geq 2$, two pairs of solutions are radially symmetric. Moreover, if $N = 4$ or $N \geq 6$, then $s_N \geq 4$ and the rest of the $(s_N - 2)$ pairs of solutions are sign-changing and non-radially symmetric functions in W.

Remark 3.4. From a Strauss-type estimate (see Lions [6]) we know that the elements $u \in W$ are homoclinic, i.e., $u(x) \to 0$ as $|x| \to \infty$. Thus, all solutions in Theorem 1.1 (ii)–(iii) have this property.

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