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CRITICAL POINT LOCALIZATION THEOREMS VIA EKELAND'S VARIATIONAL PRINCIPLE

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ABSTRACT. In this paper some existence results for critical points of extremum in conical annular regions are established by Ekeland's variational principle. An application to two-point boundary value problems is included to illustrate the theory.

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1. INTRODUCTION AND PRELIMINARIES

The well-known Ekeland's variational principle [3], [4] is one of the fundamental results of nonlinear analysis.

Theorem 1.1 (Ekeland). *Let (M,d) be a complete metric space and $F:M\to\mathbf{R}$ lower semicontinuous and bounded from below. For every $\varepsilon>0$ and $x_0\in M$ such that $F(x_0)\le\inf F(M)+\varepsilon$, and every $\delta>0$, there exists $x\in M$ such that*

(i) $F(x)\le F(x_0)$;
(ii) $d(x_0,x)\le\delta$;
(iii) $F(x)<F(y)+\frac{\varepsilon}{\delta}d(y,x)$ *for all* $y\ne x$.

An immediate consequence of Ekeland's variational principle is concerning with the existence of a critical point of minimum for a C^1-functional on a Banach space.

Corollary 1.1. *Let X be a Banach space with norm $|\cdot|$, and $F:X\to\mathbf{R}$ a C^1-functional, bounded from below. There exists a sequence (x_n) such that*

$$F(x_n)\to\inf F(X),\quad F'(x_n)\to 0.$$

If in addition F satisfies the Palais-Smale condition (i.e. any sequence as above has a convergent subsequence), then there exists $x\in X$ with

$$F(x)=\inf F(X),\quad F'(x)=0.$$

For its proof it is sufficient to apply Theorem 1.1 with $\delta=1$ and $\varepsilon=\frac{1}{n}$, to obtain a sequence (x_n) satisfying the following conditions:

$$F(x_n)\le\inf F(X)+\frac{1}{n},\quad F(x_n)<F(y)+\frac{1}{n}|y-x_n|\ \text{ for all } y\ne x_n.$$

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The first inequality gives us $F(x_n) \to \inf F(X)$, while the second inequality, for $y = x_n - tz$, $t > 0$, $|z| = 1$, yields

$$-t \langle F'(x_n), z\rangle + o(t) + \frac{t}{n} > 0.$$

Here and throughout the paper, by $\langle \cdot, \cdot \rangle$ we understand the duality between a space X and its dual X', i.e. $\langle x^*, x\rangle = x^*(x)$ for $x^* \in X'$ and $x \in X$. Dividing by t and letting t tend to zero, we obtain $\langle F'(x_n), z\rangle \leq \frac{1}{n}$, whence $|F'(x_n)|_{X'} \leq \frac{1}{n}$, that is $F'(x_n) \to 0$. We stress on the fact that, in this case, when the domain of F is the whole space, the choice $y = x_n - tz$ is possible for every $z \in X$ with $|z| = 1$ and $t > 0$. As we are going to see, such a choice is not possible for every z, and every t in a right vicinity of 0, in case that the domain of F is a proper subset D of X and x_n is not interior to D.

In this paper, similar results to Corollary 1.1, in a subset of X, are presented as consequences of Ekeland's principle. Thus, by a simple and direct proof, we obtain variants and extensions of some results of Schechter [14], [15], and a compression critical point theorem established in [11], in a conical annular domain. Finally, we present an application to two-point boundary value problems, as a variational alternative to the fixed point approach (see e.g. [1], [5], [6], [8], [10], [12]).

2. CRITICAL POINT LOCALIZATION THEOREMS

Let X be a Hilbert space with scalar product and norm $(\cdot,\cdot)$, $|\cdot|_X$, and let Y, Z be two linear normed spaces with norms $|\cdot|_Y$ and $|\cdot|_Z$, respectively. We shall assume that $X \subset Y$ and $X \subset Z$ with continuous embeddings. Let c_Y, c_Z be the embedding constants, i.e. $|x|_Y \leq c_Y |x|_X$ and $|x|_Z \leq c_Z |x|_X$ for every $x \in X$. Clearly the dual spaces are in relations $Y' \subset X'$ and $Z' \subset X'$ with continuous embeddings too. Denote by P, Q the duality mappings of Y and Z and assume that they are single-valued, i.e.

$$P : Y \to Y', \quad Q : Z \to Z',$$

$$\begin{aligned} |Px|_{Y'} &= |x|_Y, \quad \langle Px, x\rangle = |x|_Y^2 \quad \text{for all } x \in Y; \\ |Qx|_{Z'} &= |x|_Z, \quad \langle Qx, x\rangle = |x|_Z^2 \quad \text{for all } x \in Z. \end{aligned}$$

We shall assume in addition that P and Q are continuous. Let L be the continuous linear operator from X to X' (the canonical isomorphism of X onto X'), given by

$$(x, y) = \langle Lx, y\rangle \quad \text{for all } x, y \in X$$

and let J be its inverse. Then $J : X' \to X$ is a continuous linear operator and

$$(Jx^*, x) = \langle x^*, x\rangle \quad \text{for all } x^* \in X', \ x \in X.$$

Let K be a wedge of X, i.e. a closed convex subset of X such that $K \neq \{0\}$ and $\lambda K \subset K$ for every $\lambda \in \mathbf{R}_+$. Notice that K can be a cone, i.e. may have the property

$K \cap (-K) = \{0\}$, and also can be the whole space X. For two positive numbers r, R, denote

$$K_R = \{x \in K : |x|_Z \leq R\}, \quad D_r = \{x \in K : |x|_Y \geq r\},$$

$$K_{r,R} = \{x \in K : r \leq |x|_Y \text{ and } |x|_Z \leq R\},$$

i.e. $K_{r,R} = K_R \cap D_r$. Also by ∂K_R and ∂D_r we shall understand the sets $\{x \in K : |x|_Z = R\}$, $\{x \in K : |x|_Y = r\}$. Obviously, since the embeddings $X \subset Y$ and $X \subset Z$ are continuous, the sets K_R, D_r and $K_{r,R}$ are closed in X. In this section we are interested in critical points of extremum for a real functional $F \in C^1(X)$, which are located in K_R, D_r, or $K_{r,R}$.

We close this introductory section by a basic result (see [2, p. 96]) concerning single-valued duality mappings: For every $x, y \in Y$,

$$\langle Px, y\rangle = |x|_Y \lim_{t \to 0+} t^{-1} \left(|x|_Y - |x - ty|_Y\right). \tag{2.1}$$

According to this formula, if $\langle Px, y\rangle < 0$, then $|x - ty|_Y > |x|_Y$ for every sufficiently small $t > 0$, while if $\langle Px, y\rangle > 0$, then $|x - ty|_Y < |x|_Y$ for small enough $t > 0$. The same is true for Q.

2.1. **Critical points of minimum in a ball.**

Theorem 2.1. *Let $F : X \to \mathbf{R}$ be a C^1-functional, bounded from below on K_R. Assume that*

$$F(x) \geq \inf F(K_R) + c \quad \text{for all } x \in K_R \text{ with } |x|_X \geq R_0 \tag{2.2}$$

and some $c, R_0 > 0$. In addition assume that for some $\nu > 0$,

$$\langle Qx, JF'(x)\rangle \geq -\nu > -\infty \quad \text{for all } x \in \partial K_R \text{ with } |x|_X \leq R_0, \tag{2.3}$$

and

$$\text{for each } x \in K_R \text{ with } |x|_X \leq R_0 \text{ and each } \mu \in \left[-\frac{\nu R_0^2}{R^4}, 0\right], \text{ there is } \eta > 0 \text{ with } \eta x - JF'(x) + \mu JQx \in K. \tag{2.4}$$

Then there exists a sequence (x_n), $x_n \in K_R$, such that $F(x_n) \to \inf F(K_R)$, and one of the following two properties hold:

(i) $F'(x_n) \to 0$;

(ii) $|x_n|_Z = R$, $\langle Qx_n, JF'(x_n)\rangle \leq 0$ *for all n, and*

$$JF'(x_n) - \frac{\langle Qx_n, JF'(x_n)\rangle}{|JQx_n|_X^2} JQx_n \to 0. \tag{2.5}$$

If in addition F satisfies a Palais-Smale type compactness condition guaranteeing that any sequence as above has a convergent subsequence, and the boundary condition

$$JF'(x) + \mu JQx \neq 0 \quad \textit{for all } x \in \partial K_R \textit{ with } |x|_X \leq R_0 \textit{ and } \mu > 0, \tag{2.6}$$

then there exists $x \in K_R$ with

$$F(x) = \inf F(K_R), \quad F'(x) = 0.$$

Proof. From Theorem 1.1 applied to the complete metric space K_R endowed with the metric induced by $|\cdot|_X$, it follows that there exists a sequence (x_n) in K_R such that $F(x_n) \leq \inf F(K_R) + \frac{1}{n}$, whence $F(x_n) \to \inf F(K_R)$, and

$$F(x_n) < F(y) + \frac{1}{n} |y - x_n|_X \quad \text{for every } y \in K_R \text{ with } y \neq x_n. \tag{2.7}$$

In view of (2.2), we may assume that $|x_n|_X \leq R_0$. Three cases are possible: (a) There is a subsequence with $x_n = 0$ for all n; (b) There is a subsequence with $0 < |x_n|_Z < R$; (c) The terms of the sequence (x_n), except possibly a finite number, belong to ∂K_R.

In the first case we have $F(0) = \inf F(K_R)$. Also $F(0) < F(y) + \frac{1}{n} |y|_X$ for every $y \in K_R \setminus \{0\}$ and all n. We claim that $F'(0) = 0$. Indeed, otherwise, from $\eta 0 - JF'(0) \in K$ and $F'(0) \neq 0$, if in (2.7) we take $y = -tJF'(0)$ for $t > 0$, we obtain

$$-t |JF'(0)|_X^2 + o(t) + \frac{t}{n} |JF'(0)|_X \geq 0,$$

whence we derive $|JF'(0)|_X = 0$, a contradiction.

In the second case, we may suppose that $0 < |x_n|_Z < R$ for all n. For a fixed n, apply (2.7) with $y = (1 + t) x_n$, $t \neq 0$. Clearly $y \in K_R$ for $|t|$ small enough. Then

$$t \langle F'(x_n), x_n \rangle + o(|t|) + \frac{|t|}{n} |x_n|_X \geq 0.$$

Dividing by t in each of the cases $t > 0$, $t < 0$ and letting $t \to 0$ we obtain

$$-\frac{1}{n} |x_n|_X \leq \langle F'(x_n), x_n \rangle \leq \frac{1}{n} |x_n|_X .$$

Hence $\langle F'(x_n), x_n \rangle \to 0$ as $n \to \infty$. Next in (2.7) we take $y = x_n + t(x_n - JF'(x_n))$ with $t > 0$ small enough. We have

$$t \langle F'(x_n), x_n - JF'(x_n) \rangle + o(t) + \frac{t}{n} |x_n - JF'(x_n)|_X \geq 0.$$

Divide by t and let t go to zero to obtain

$$\langle F'(x_n), x_n - JF'(x_n) \rangle + \frac{1}{n} |x_n - JF'(x_n)|_X \geq 0.$$

It follows that

$$|JF'(x_n)|_X^2 \leq \langle F'(x_n), x_n \rangle + \frac{1}{n} |x_n - JF'(x_n)|_X ,$$

whence $|JF'(x_n)|_X \to 0$ as $n \to \infty$. Thus, $F'(x_n) \to 0$ in X' and so, property (i) holds in case (b).

Assume now case (c), i.e. $|x_n|_Z = R$ for all n. Two subcases are now possible: (1) $\langle Qx_n, JF'(x_n)\rangle > 0$ for a subsequence. Then we apply (2.7) to $y = x_n - tJF'(x_n)$. Clearly $y = t(\eta_n x_n - JF'(x_n)) + (1 - t\eta_n)x_n \in K$. Also, according to (2.1), from $\langle Qx_n, JF'(x_n)\rangle > 0$, we have that $|y|_Z \le R$ for $t > 0$ small enough. Hence $y \in K_R$. Now (2.7) gives us

$$\langle F'(x_n), JF'(x_n)\rangle = |JF'(x_n)|_X^2 \le \frac{1}{n}|JF'(x_n)|_X,$$

whence $JF'(x_n) \to 0$, equivalently $F'(x_n) \to 0$. Thus (i) holds.

(2) Assume $\langle Qx_n, JF'(x_n)\rangle \le 0$ for all n except possibly a finite number of indices. Then in (2.7) we take $y = x_n - t(\varepsilon x_n + z_n)$, where $\varepsilon > 0$, $z_n = JF'(x_n) - \mu_n JQx_n$, $\mu_n = \frac{\langle Qx_n, JF'(x_n)\rangle}{|JQx_n|_X^2}$ and $t > 0$. Notice that

$$\langle Qx_n, z_n\rangle = 0 \quad \text{and} \quad \langle Qx_n, \varepsilon x_n + z_n\rangle = \varepsilon R^2 > 0,$$

which in view of (2.1), guarantees

$$|y|_Z \le |x_n|_Z = R$$

if $t > 0$ is small enough. Also from

$$R^2 = |x_n|_Z^2 = \langle Qx_n, x_n\rangle = (JQx_n, x_n) \le |JQx_n|_X |x_n|_X \le R_0 |JQx_n|_X,$$

we have $|JQx_n|_X \ge \frac{R^2}{R_0}$, and so

$$-\frac{\nu R_0^2}{R^4} \le \mu_n \le 0. \tag{2.8}$$

Then, from (2.4), there is $\eta_n > 0$ with $\eta_n x_n - JF'(x_n) + \mu_n JQx_n \in K$. Consequently

$$y = t[\eta_n x_n - JF'(x_n) + \mu_n JQx_n] + (1 - t\varepsilon - t\eta_n)x_n \in K$$

since $1 - t\varepsilon - t\eta_n$ is positive for small $t > 0$. Applying (2.7) we obtain

$$-t\langle F'(x_n), \varepsilon x_n + z_n\rangle + o(t) + \frac{t}{n}|\varepsilon x_n + z_n|_X \ge 0,$$

whence

$$-\langle F'(x_n), \varepsilon x_n + z_n\rangle + \frac{1}{n}|\varepsilon x_n + z_n|_X \ge 0,$$

and letting $\varepsilon \to 0$,

$$-\langle F'(x_n), z_n\rangle + \frac{1}{n}|z_n|_X \ge 0,$$

or equivalently

$$-(JF'(x_n), z_n) + \frac{1}{n}|z_n|_X \ge 0.$$

Since $\langle Qx_n, z_n\rangle = 0$, this is equivalent to the inequality

$$-|z_n|_X^2 + \frac{1}{n}|z_n|_X \ge 0.$$

Thus $|z_n|_X \le \frac{1}{n}$ and so $z_n \to 0$ as desired.

For the last part of the theorem, assume via the Palais-Smale condition, that $x_n \to x$. In case (i) we immediately obtain $F'(x) = 0$. In case (ii), (2.8) guarantees, at least for a subsequence, that $\mu_n \to -\mu \leq 0$, and so $JF'(x) + \mu JQx = 0$, where $x \in \partial K_R$ and $\mu \geq 0$. The case $\mu > 0$ being excluded by the boundary condition (2.6), it remains that $JF'(x) = 0$, that is $F'(x) = 0$. □

Remark 2.1. (a) Condition (2.2) is trivial if $X = Z$ (in this case, take R_0 any number greater than R to see that there are no elements $x \in K_R$ with $|x|_X \geq R_0$).

(b) In particular, condition (2.4) holds if $x - JF'(x) \in K$ for every $x \in K$ and JQ is the identity map.

(c) Condition (2.4) trivially holds when $K = X$; the case $X = Z = K$ was considered in [13].

2.2. **Critical points of minimum outside the ball.** We now look for critical points of extremum in the set D_r, where $r > 0$ is a given number.

Theorem 2.2. *Let $F : X \to \mathbf{R}$ be a C^1-functional, bounded from below on D_r. Assume that*

$$F(x) \geq \inf F(D_r) + c \quad \text{for all } x \in D_r \text{ with } |x|_X \geq R_0 \tag{2.9}$$

and some $c, R_0 > 0$. In addition assume that for some $\nu > 0$,

$$\langle Px, JF'(x)\rangle \leq \nu < \infty \quad \text{for all } x \in \partial D_r \text{ with } |x|_X \leq R_0, \tag{2.10}$$

and

$$\begin{aligned} &\text{for each } x \in D_r \text{ with } |x|_X \leq R_0 \text{ and each } \mu \in [0, \frac{\nu R_0^2}{r^4}], \\ &\text{there is } \eta > 0 \text{ with } \eta x - JF'(x) + \mu JPx \in K. \end{aligned} \tag{2.11}$$

Then there exists a sequence (x_n), $x_n \in D_r$, such that $F(x_n) \to \inf F(D_r)$, and one of the following two properties holds:

(i) $F'(x_n) \to 0$;

(ii) $|x_n|_Y = r$, $\langle Px_n, JF'(x_n)\rangle \geq 0$ *for all n, and*

$$JF'(x_n) - \frac{\langle Px_n, JF'(x_n)\rangle}{|JPx_n|_X^2} JPx_n \to 0. \tag{2.12}$$

If in addition F satisfies a Palais-Smale type compactness condition guaranteeing that any sequence as above has a convergent subsequence, and the following boundary condition holds

$$JF'(x) + \mu JPx \neq 0 \text{ for all } x \in \partial D_r \text{ with } |x|_X \leq R_0 \text{ and } \mu < 0,$$

then there exists $x \in D_r$ with

$$F(x) = \inf F(D_r), \quad F'(x) = 0.$$

Proof. We apply Theorem 1.1 in D_r. The resulting sequence (x_n) clearly satisfies $|x_n|_X \leq R_0$ and is now in one of the cases: (a) There is a subsequence of (x_n) with $|x_n|_Y > r$; (b) The terms of the sequence (x_n), except possibly a finite number, belong to ∂D_r. In the first case one proceeds like in case (b) of the proof of Theorem 2.1. In the second case, we are in one of the following situations: (1) $\langle Px_n, JF'(x_n)\rangle < 0$ for a subsequence. Then we apply (2.7) to $y = x_n - tJF'(x_n)$. Clearly $y = t(\eta_n x_n - JF'(x_n)) + (1 - t\eta_n)x_n \in K$. Also, according to (2.1), from $\langle Px_n, JF'(x_n)\rangle < 0$, we have that $|y|_Y \geq r$ for $t > 0$ small enough. Hence $y \in D_r$. Now (2.7) gives us

$$\langle F'(x_n), JF'(x_n)\rangle = |JF'(x_n)|_X^2 \leq \frac{1}{n}|JF'(x_n)|_X,$$

whence $JF'(x_n) \to 0$, that is (i) holds; (2) otherwise, $\langle Px_n, JF'(x_n)\rangle \geq 0$ for all n, except possibly a finite number of indices. Then in (2.7), we take $y = x_n - t(-\varepsilon x_n + z_n)$, where $t, \varepsilon > 0$, $z_n = JF'(x_n) - \mu_n JPx_n$ and $\mu_n = \frac{\langle Px_n, JF'(x_n)\rangle}{|JPx_n|_X^2}$. This choice of y is correct. Indeed,

$$y = t(\eta_n x_n - JF'(x_n) + \mu_n JPx_n) + (1 - t\eta_n + t\varepsilon)x_n \in K$$

since $1 - t\eta_n + t\varepsilon$ is positive for small $t > 0$; in addition, since $\langle Px_n, -\varepsilon x_n + z_n\rangle = -\varepsilon r^2 < 0$, again in view of (2.1), we have that $|y|_Y \geq r$ for $t > 0$ small enough. Then (2.7) gives

$$\langle F'(x_n), -\varepsilon x_n + z_n\rangle \leq \frac{1}{n}|-\varepsilon x_n + z_n|_X,$$

whence after passing to limit with $\varepsilon \to 0$, we deduce that $\langle F'(x_n), z_n\rangle \leq \frac{1}{n}|z_n|_X$. Now observe that

$$\begin{aligned}\langle F'(x_n), z_n\rangle &= \langle F'(x_n) - \mu_n Px_n,\ z_n\rangle = (JF'(x_n) - \mu_n JPx_n,\ z_n) \\ &= |z_n|_X^2.\end{aligned}$$

As a result $|z_n|_X \leq \frac{1}{n}$, which shows that $z_n \to 0$ as desired. The proof of the last part of the theorem is similar to that of Theorem 2.1. □

Remark 2.2. In particular, condition (2.11) holds if $x - JF'(x) \in K$ for every $x \in K$ and $JP(K) \subset K$.

2.3. **Critical points of minimum in annular domains.** Combining the ideas of Theorems 2.1 and 2.2 we obtain critical point theorems in the annular domain

$$K_{r,R} := \{x \in K : r \leq |x|_Y,\ |x|_Z \leq R\}.$$

We shall assume that there exist elements x in $K_{r,R}$ with $r < |x|_Y$ and $|x|_Z < R$; for instance, if $x_0 \in K$, $|x_0|_Z = 1$ and numbers r, R are such that $r < |x_0|_Y R$, then μx_0 is such an element, i.e. $r < |\mu x_0|_Y$ and $|\mu x_0|_Z < R$, for each μ satisfying $\frac{r}{|x_0|_Y} < \mu < R$.

Theorem 2.3. *Let $F : X \to \mathbf{R}$ be a C^1-functional, bounded from below on $K_{r,R}$. Assume that for some $c, R_0 > 0$,*

$$F(x) \geq \inf F(K_{r,R}) + c \quad \text{for all } x \in K_{r,R} \text{ satisfying} \tag{2.13}$$
$$\text{either} \quad |x|_X \geq R_0, \ \text{or both } |x|_Y = r, \ |x|_Z = R.$$

In addition assume that conditions (2.3), (2.4), (2.10) *and* (2.11) *hold. Then there exists a sequence (x_n), $x_n \in K_{r,R}$ such that $F(x_n) \to \inf F(K_{r,R})$, and one of the following three situations holds:*

(i) $F'(x_n) \to 0$;

(ii) $|x_n|_Y = r$, $\langle Px_n, JF'(x_n)\rangle \geq 0$ *for all n, and*

$$JF'(x_n) - \frac{\langle Px_n, JF'(x_n)\rangle}{|JPx_n|_X^2} JPx_n \to 0; \tag{2.14}$$

(iii) $|x_n|_Z = R$, $\langle Qx_n, JF'(x_n)\rangle \leq 0$ *for all n, and*

$$JF'(x_n) - \frac{\langle Qx_n, JF'(x_n)\rangle}{|JQx_n|_X^2} JQx_n \to 0. \tag{2.15}$$

If in addition F satisfies a Palais-Smale type compactness condition guaranteeing that any sequence as above has a convergent subsequence, and the boundary conditions

$$JF'(x) + \mu JQx \neq 0 \text{ for all } x \in \partial K_R \text{ with } |x|_X \leq R_0 \text{ and } \mu > 0; \tag{2.16}$$
$$JF'(x) + \mu JPx \neq 0 \text{ for all } x \in \partial D_r \text{ with } |x|_X \leq R_0 \text{ and } \mu < 0,$$

then there exists $x \in K_{r,R}$ with

$$F(x) = \inf F(K_{r,R}), \quad F'(x) = 0.$$

Proof. Obviously, the result is a joint consequence of Theorems 2.1 and 2.2. The additional hypothesis that $F(x) \geq \inf F(K_{r,R}) + c$ for all $x \in K_{r,R}$ satisfying both equalities $|x|_Y = r$, $|x|_Z = R$ is needed to guarantee that whenever $|x_n|_Y = r$, we have $|x_n|_Z < R$, making possible the choices $y = x_n - tJF'(x_n)$ and $y = x_n - t(-\varepsilon x_n + z_n)$ as in the proof of Theorem 2.2, i.e. $|y|_Z \leq R$ for $t > 0$ small enough. Similarly, if $|x_n|_Z = R$, we will have $|x_n|_Y > r$, and thus for $y = x_n - tJF'(x_n)$ and $y = x_n - t(\varepsilon x_n + z_n)$ as in the proof of Theorem 2.1, the necessary inequality $|y|_Y \geq r$ holds for $t > 0$ small enough. □

2.4. **Dual results for maxima.** The dual result of Theorem 2.1 for maxima is the following theorem.

Theorem 2.4. *Let $F : X \to \mathbf{R}$ be a C^1-functional, bounded from above on K_R. Assume that*

$$F(x) \leq \sup F(K_R) - c \quad \text{for all } x \in K_R \text{ with } |x|_X \geq R_0 \tag{2.17}$$

and some $c, R_0 > 0$. *In addition assume that for some* $\nu > 0$,

$$\langle Qx, JF'(x)\rangle \leq \nu < \infty \quad \text{for all } x \in \partial K_R \text{ with } |x|_X \leq R_0, \tag{2.18}$$

and

(2.19) *for each* $x \in K_R$ *with* $|x|_X \leq R_0$ *and each* $\mu \in \left[-\frac{\nu R_0^2}{R^4}, 0\right]$,

there is $\eta > 0$ *with* $\eta x + JF'(x) + \mu JQx \in K$.

Then there exists a sequence (x_n), $x_n \in K_R$, *such that* $F(x_n) \to \sup F(K_R)$, *and and one of the following two properties holds:*

(i)$F'(x_n) \to 0$;

(ii) $|x_n|_Z = R$, $\langle Qx_n, JF'(x_n)\rangle \geq 0$ *for all* n, *and*

$$JF'(x_n) - \frac{\langle Qx_n, JF'(x_n)\rangle}{|JQx_n|_X^2} JQx_n \to 0. \tag{2.20}$$

If in addition F *satisfies a Palais-Smale type compactness condition guaranteeing that any sequence as above has a convergent subsequence, and the boundary condition*

$$JF'(x) + \mu JQx \neq 0 \quad \text{for all } x \in \partial K_R \text{ with } |x|_X \leq R_0 \text{ and } \mu < 0,$$

then there exists $x \in K_R$ *with*

$$F(x) = \sup F(K_R), \quad F'(x) = 0.$$

Proof. Apply Theorem 2.1 to the functional $-F$. □

The dual result of Theorem 2.2 for maxima is the following theorem.

Theorem 2.5. *Let* $F : X \to \mathbf{R}$ *be a* C^1*-functional, bounded from above on* D_r. *Assume that*

$$F(x) \leq \sup F(D_r) - c \quad \text{for all } x \in D_r \text{ with } |x|_X \geq R_0 \tag{2.21}$$

and some $c, R_0 > 0$. *In addition assume that for some* $\nu > 0$,

$$\langle Px, JF'(x)\rangle \geq -\nu > -\infty \quad \text{for all } x \in \partial D_r \text{ with } |x|_X \leq R_0, \tag{2.22}$$

and

(2.23) *for each* $x \in D_r$ *with* $|x|_X \leq R_0$ *and each* $\mu \in \left[0, \frac{\nu R_0^2}{r^4}\right]$,

there is $\eta > 0$ *with* $\eta x + JF'(x) + \mu JPx \in K$.

Then there exists a sequence (x_n), $x_n \in D_r$, *such that* $F(x_n) \to \sup F(D_r)$, *and and one of the following two properties holds:*

(i) $F'(x_n) \to 0$;

(ii) $|x_n|_Y = r$, $\langle Px_n, JF'(x_n)\rangle \le 0$ *for all* n, *and*

$$JF'(x_n) - \frac{\langle Px_n, JF'(x_n)\rangle}{|JPx_n|_X^2} JPx_n \to 0. \tag{2.24}$$

If in addition F *satisfies a Palais-Smale type compactness condition guaranteeing that any sequence as above has a convergent subsequence, and the boundary condition*

$$JF'(x) + \mu JPx \neq 0 \text{ for all } x \in \partial D_r \text{ with } |x|_X \le R_0 \text{ and } \mu > 0,$$

then there exists $x \in D_r$ *with*

$$F(x) = \sup F(D_r), \quad F'(x) = 0.$$

Obviously, a dual result of Theorem 2.3, in the annulus can also be stated.

Remark 2.3. (a) In case that $JF'(x) = x - N(x)$, which happens in most applications, the boundary conditions (2.16) read as follows

$$N(x) \neq x + \mu JPx \quad \text{for } |x|_Y = r,\ |x|_X \le R_0 \text{ and } \mu < 0; \tag{2.25}$$

$$N(x) \neq x + \mu JQx \quad \text{for } |x|_Z = R,\ |x|_X \le R_0 \text{ and } \mu > 0 \tag{2.26}$$

and can be seen as a compression type property of the operator N, on the annular domain $K_{r,R}$.

(b) For $X = Z = K$, Theorems 2.1 and 2.4 reduce to some results by Schechter [14], [15, Theorems 5.3.3, 5.5.5] in a ball of a Hilbert space.

(c) Theorem 2.3 is a version of a theorem first established in [11] by a completely different technique. Thus, Ekeland's principle gives us a direct and simple alternative proof to such type of theorems.

3. APPLICATION

Consider the two-point boundary value

$$\begin{cases} -u''(t) = f(t, u(t)), & t \in I, \\ u(0) = u(1) = 0, \end{cases} \tag{3.1}$$

where $I = [0,1]$ and $f : I \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory function, i.e. $f(\cdot, u)$ is measurable for each $u \in \mathbf{R}$ and $f(t, \cdot)$ is continuous for almost every $t \in I$. In addition it is assumed that for each $b > 0$ there exists $h_b \in L^1(I)$ such that $|f(t,\tau)| \le h_b(t)$ for $|\tau| \le b$ and almost every $t \in I$; in this case we say that f is a L^1-Carathéodory function.

We shall apply the abstract results from Section 2 to spaces: $X = Z = H_0^1(I)$ endowed with the scalar product and norm

$$(u, v) = \int_I u'(t)\, v'(t)\, dt, \quad |u|_{H_0^1(I)} = \left(\int_I u'(t)^2\, dt\right)^{1/2},$$

$Y = L^p(I)$, where $p > 1$, and to the functional $F : H_0^1(I) \to \mathbf{R}$,

$$F(u) = \int_I \left(\frac{1}{2} u'(t)^2 - g(t, u(t)) \right) dt,$$

where $g : I \times \mathbf{R} \to \mathbf{R}$ is defined by

$$g(t, \tau) = \int_0^\tau f(t, s)\, ds.$$

Here $L : H_0^1(I) \to H^{-1}(I)$, $Lu = -u''$; $J : H^{-1}(I) \to H_0^1(I)$, $Jv = u$, where u is the unique weak solution of the problem

$$\begin{cases} -u'' = v, & t \in I, \\ u(0) = u(1) = 0, \end{cases}$$

that is $u \in H_0^1(I)$ and $(u, w) = \langle v, w \rangle$ for all $w \in H_0^1(I)$. Also, $Q = L$ and $P : L^p(I) \to L^q(I)$, $\frac{1}{p} + \frac{1}{q} = 1$ is defined by

$$(Pu)(t) = |u|_{L^p(I)}^{2-p} |u(t)|^{p-2} u(t).$$

We have $F'(u) = -u'' - f(\cdot, u)$ and $JF'(u) = u - N(u)$, where

$$N(u) = Jf(\cdot, u) = \int_I G(\cdot, s) f(s, u(s))\, ds,$$

with the Green function $G(t, s) = s(1 - t)$ for $0 \le s \le t \le 1$, $G(t, s) = t(1 - s)$ for $0 \le t < s \le 1$. Notice that since the embedding of $H_0^1(I)$ into $C(I)$ is compact, the operator N is completely continuous from $H_0^1(I)$ to itself guaranteeing the Palais-Smale condition.

We shall be interested into positive solutions and therefore we assume that

$$f(I \times \mathbf{R}_+) \subset \mathbf{R}_+. \tag{3.2}$$

Then N maps positive functions into positive functions. Moreover, if we fix a subinterval $[t_0, t_1] \subset (0, 1)$, then we have $G(t, s) \le G(s, s)$ for all $t, s \in [0, 1]$ and $G(t, s) \ge MG(s, s)$ for every $t \in I_0 := [t_0, t_1]$, $s \in [0, 1]$, where

$$M = \min\{t_0, 1 - t_1\}.$$

Then, for each $h \in L^1(I, \mathbf{R}_+)$, $t \in I_0$ and $t' \in I$,

$$\begin{aligned} (Jh)(t) &= \int_I G(t, s) h(s)\, ds \ge M \int_I G(s, s) h(s)\, ds \\ &\ge M \int_I G(t', s) h(s)\, ds = M (Jh)(t'), \end{aligned}$$

and consequently, $(Jh)(t) \ge M |Jh|_{L^p(I)}$ $(t \in I_0)$. Hence if we consider the cone

$$K = \left\{ u \in H_0^1(I) : u \ge 0 \text{ on } I;\ u(t) \ge M |u|_{L^p(I)} \text{ for } t \in I_0 \right\},$$

then $N(K) \subset K$, equivalently $u - JF'(u) \in K$ for every $u \in K$. In addition, JQ is the identity map and since P maps positive functions into positive functions,

$(JP)(K) \subset K$. Consequently, according to Remarks 2.1 (b) and 2.2, conditions (2.4) and (2.11) hold.

Notice that F is bounded from below in $K_{r,R}$ for every r, R. Indeed, for each $u \in K_{r,R}$, since $u(t) \leq |u|_{H_0^1(I)}$ for every $t \in I$, we have

$$\begin{aligned} F(u) &= \int_I \left(\frac{1}{2} u'(t)^2 - g(t, u(t)) \right) dt \\ &\geq - \int_I g(t, u(t))\, dt \geq - \int_I g(t, R)\, dt > -\infty. \end{aligned}$$

Hence

$$m := \inf F(K_{r,R}) > -\infty.$$

Let χ_{I_0} be the characteristic function of the interval I_0, i.e. $\chi_{I_0}(t) = 1$ if $t \in I_0$, $\chi_{I_0}(t) = 0$ for $t \in I \setminus I_0$, and let

$$\phi(t) = \frac{\sqrt{2}}{\pi} \sin \pi t$$

be the positive eigenfunction corresponding to the first eigenvalue $\lambda = \pi^2$, with $|\phi|_{H_0^1(I)} = 1$. Notice that if $0 < r < |\phi|_{L^p(I)} R$, then $\mu\phi \in K_{r,R}$ for $\frac{r}{|\phi|_{L^p(I)}} \leq \mu \leq R$.

Theorem 3.1. *Assume that $f : I \times \mathbf{R} \to \mathbf{R}$ is a L^1-Carathéodory function, $f(I \times \mathbf{R}_+) \subset \mathbf{R}_+$, and that there are two numbers r, R with $0 < r < |\phi|_{L^p(I)} R$ such that*

$$\left| J \left(\chi_{I_0} \inf_{\tau \in [Mr, \infty)} f(\cdot, \tau) \right) \right|_{L^p(I)} \geq r, \tag{3.3}$$

$$\left| \sup_{\tau \in [0,R]} f(\cdot, \tau) \right|_{L^1(I)} \leq R, \tag{3.4}$$

$$F(u) \geq m + c \quad \text{for every } u \in K_{r,R} \text{ with } |u|_{L^p(I)} = r \text{ and } |u|_{H_0^1(I)} = R. \tag{3.5}$$

Then (3.1) *has a solution u in $K_{r,R}$ which minimizes F on $K_{r,R}$.*

Proof. Clearly (3.5) means (2.13). Also, since N maps bounded sets into bounded sets, conditions (2.3) and (2.10) hold. Now we show condition (2.26). Assume the contrary. Then $N(u) = (1 + \mu) u$ for some $u \in K_{r,R}$ with $|u|_{H_0^1(I)} = R$ and $\mu > 0$. Since

$$0 \leq u(t) = \int_0^t u'(s)\, ds \leq \left(\int_I u'(s)^2\, ds \right)^{\frac{1}{2}} = |u|_{H_0^1(I)},$$

we deduce that

$$\begin{aligned} R^2 &= |u|_{H_0^1(I)}^2 = \frac{1}{1+\mu} (N(u), u) < (Jf(\cdot, u), u) = \langle f(\cdot, u), u \rangle \\ &= \int_I f(t, u(t))\, u(t)\, dt \leq R \int_I \sup_{\tau \in [0,R]} f(t, \tau)\, dt. \end{aligned}$$

Hence $R < \left|\sup_{\tau\in[0,R]} f(\cdot,\tau)\right|_{L^1(I)}$, a contradiction to (3.4). Next we check (2.25). Assume the contrary, i.e. $N(u) = u + \mu JPu$ for some $u \in K_{r,R}$ with $|u|_{L^p(I)} = r$ and $\mu < 0$. One has $u(t) \ge Mr$ in I_0, whence $f(t,u(t)) \ge \inf_{\tau\in[Mr,\infty)} f(t,\tau)$ for $t \in I_0$, that is

$$f(t,u(t)) \ge \chi_{I_0}(t) \inf_{\tau\in[Mr,\infty)} f(t,\tau) \quad \text{for all } t \in I.$$

Then, on $(0,1)$ we have

$$u > u + \mu JPu = N(u) = Jf(\cdot,u) \ge J\left(\chi_{I_0} \inf_{\tau\in[Mr,\infty)} f(\cdot,\tau)\right).$$

Consequently

$$|u|_{L^p(I)} > \left|J\left(\chi_{I_0} \inf_{\tau\in[Mr,\infty)} f(\cdot,\tau)\right)\right|_{L^p(I)},$$

which is excluded by (3.3). □

The next three remarks are concerning with condition (3.5).

Remark 3.1. If g is such that

$$g(t,u) \le a(t)\,u^{p-1} + b(t) \quad \text{for all } u \in \mathbf{R}_+, \text{ a.e. } t \in I, \tag{3.6}$$

where $p > 1$, $a \in L^p(I)$ and $b \in L^1(I)$, then a sufficient condition for (3.5) to hold is that

$$\int_I g(t,R\phi(t))\,dt > |a|_{L^p(I)}\,r^{p-1} + |b|_{L^1(I)}. \tag{3.7}$$

Indeed, for each $u \in K$ with $|u|_{L^p(I)} = r$ and $|u|_{H_0^1(I)} = R$ we have via Hölder's inequality

$$F(u) \ge \frac{R^2}{2} - \int_I \left[a(t)\,u(t)^{p-1} + b(t)\right] dt \ge \frac{R^2}{2} - |a|_{L^p(I)}\,r^{p-1} - |b|_{L^1(I)}.$$

On the other hand, since $|\phi|_{H_0^1(I)} = 1$,

$$F(R\phi) = \frac{R^2}{2} - \int_I g(t,R\phi(t))\,dt.$$

Hence

$$F(u) \ge F(R\phi) + c \ge m + c,$$

where

$$c = \int_I g(t,R\phi(t))\,dt - |a|_{L^p(I)}\,r^{p-1} - |b|_{L^1(I)} > 0.$$

We note that by (3.6), condition (3.7) implies $r < |\phi|_{L^p(I)}\,R$, and so $R\phi \in K_{r,R}$.

Remark 3.2. If f is such that (3.6) holds for $p \in (1,3)$, then condition (3.5) is satisfied if R is sufficiently large and $r < |\phi|_{L^p(I)}\,R$. Indeed,

$$\begin{aligned} F(u) &\ge \frac{1}{2}|u|^2_{H_0^1(I)} - |a|_{L^p(I)}\,|u|^{p-1}_{L^p(I)} - |b|_{L^1(I)} \\ &\ge \frac{1}{2}|u|^2_{H_0^1(I)} - |a|_{L^p(I)}\,|u|^{p-1}_{H_0^1(I)} - |b|_{L^1(I)}. \end{aligned}$$

Since $p<3$, for each $B\in(0,1)$, there exists $R_B\in\mathbf{R}_+$ with

$$F(u)\geq\frac{B^2}{2}|u|^2_{H^1_0(I)}\quad\text{for}\quad|u|_{H^1_0(I)}\geq R_B. \tag{3.8}$$

Let $B:=\frac{r}{|\phi|_{L^p(I)}R}$. Then for $|u|_{H^1_0(I)}=R\geq R_B,$ we obtain

$$F(u)\geq\frac{r^2}{2|\phi|^2_{L^p(I)}}=F(\mu\phi)+c,$$

where $\mu>0$ is such that $\mu|\phi|_{L^p(I)}=r$, and $c=\int_I g(t,\mu\phi(t))\,dt>0$. Hence, in particular, $F(u)\geq F(\mu\phi)+c\geq m+c$ for every $u\in K_{r,R}$ satisfying $|u|_{H^1_0(I)}=R$ and $|u|_{L^p(I)}=r$, as claimed.

Remark 3.3. If following [7], [9] we assume that a function $\alpha\in L^1(I)$ exists such that there is $\delta>0$ with $\int_I\left(u'(t)^2-\alpha(t)u(t)^2\right)dt\geq\delta|u|^2_{H^1_0(I)}$ for every $u\in H^1_0(I)$, and for every $\varepsilon>0$, there are $\beta_\varepsilon,\gamma_\varepsilon\in L^1(I)$ with

$$2g(t,u)\leq(\varepsilon+\alpha(t))u^2+\beta_\varepsilon(t)u+\gamma_\varepsilon(t)$$

for $(t,u)\in I\times\mathbf{R}_+$, then condition (3.5) holds if R is sufficiently large and $r<|\phi|_{L^p(I)}\delta R$. Indeed, for a fixed $\varepsilon<\delta$ and each $u\in H^1_0(I,\mathbf{R}_+)$ we have

$$\begin{aligned}F(u)&\geq\frac{1}{2}\int_I\left[u'(t)^2-\alpha(t)u(t)^2-\varepsilon u(t)^2-\beta_\varepsilon(t)u(t)-\gamma_\varepsilon(t)\right]dt\\&\geq\frac{1}{2}(\delta-\varepsilon)|u|^2_{H^1_0(I)}-|\beta_\varepsilon|_{L^1(I)}|u|_{H^1_0(I)}-|\gamma_\varepsilon|_{L^1(I)}.\end{aligned}$$

Thus for each $B<\delta-\varepsilon$ there exists $R_B\in\mathbf{R}_+$ such that (3.8) holds. Now if we take $B=\frac{r}{|\phi|_{L^p(I)}R}$, we have $B<\delta$ and we can choose ε any positive number less than $\delta-B$. The assertion now follows as in the previous remark.

Assuming now a monotonicity property of f with respect to the second variable and taking into account Theorem 3.1 and Remark 3.1, we can state the following multiplicity result.

Theorem 3.2. *Assume that $f:I\times\mathbf{R}_+\to\mathbf{R}_+$ is a Carathéodory function such that $f(t,\cdot)$ is nondecreasing on $\mathbf{R}_+$ and*

$$g(t,u)\leq a(t)u^{p-1}+b(t)\quad(u\in\mathbf{R}_+)$$

for almost every $t\in I$, where $p>1$, $a\in L^p(I)$, $b\in L^1(I)$.

1^0 *Let $(r_i)_{1\leq i\leq k},(R_i)_{1\leq i\leq k}$ $(k\leq\infty)$ be increasing finite or infinite sequences such that $R_i<r_{i+1}$ $(1\leq i\leq k-1)$,*

$$\int_I g(t,R_i\phi(t))\,dt>|a|_{L^p(I)}r_i^{p-1}+|b|_{L^1(I)}, \tag{3.9}$$

$$|J(\chi_{I_0}f(\cdot,Mr_i))|_{L^p(I)}\geq r_i, \tag{3.10}$$

$$|f(\cdot, R_i)|_{L^1(I)} \leq R_i \tag{3.11}$$

for all i. *Then* (3.1) *has* k *(respectively, when* $k = \infty$, *an infinite sequence of) distinct positive solutions* $u_i \in K_{r_i,R_i}$ *which minimizes* F *on* K_{r_i,R_i}.

2^0 *Let* $(r_i)_{i\geq 1}, (R_i)_{i\geq 1}$ *be decreasing infinite sequences such that* $R_{i+1} < r_i$ $(1 \leq i \leq k-1)$ *and conditions* (3.9)–(3.11) *hold for all* i. *Then* (3.1) *has an infinite sequence of distinct positive solutions* $u_i \in K_{r_i,R_i}$ *which minimizes* F *on* K_{r_i,R_i}.

Remark 3.4. In the autonomous case, when f does not depend on t, conditions (3.10) and (3.11) read as follows

$$\frac{f(Mr_i)}{Mr_i} \geq \frac{1}{M\,|J(\chi_{I_0})|_{L^p(I)}}, \quad \frac{f(R_i)}{R_i} \leq 1$$

showing that the function $f(\tau)/\tau$ oscillates above and below the values $\frac{1}{M|J(\chi_{I_0})|_{L^p(I)}}$ and 1, respectively.

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