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VARIATIONAL VERSUS PSEUDOMONOTONE OPERATOR APPROACH IN PARAMETER-DEPENDENT NONLINEAR ELLIPTIC PROBLEMS

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ABSTRACT. We study the existence of nontrivial solutions of parameter-dependent quasilinear elliptic Dirichlet problems of the form

$$-\Delta_p u = \lambda f(u) \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega,$$

in a bounded domain $\Omega \subset \mathbb{R}^N$ with sufficiently smooth boundary, where λ is a real parameter and Δ_p denotes the p-Laplacian. Recently the authors obtained multiplicity results by employing an abstract localization principle of critical points of functionals of the form $\mathbb{E} = \Phi - \lambda\Psi$ on open sublevels of Φ, i.e., of sets of the form $\Phi^{-1}(-\infty, r)$, combined with differential inequality techniques and topological arguments. Unlike in those recent papers by the authors, the approach in this paper is based on pseudomonotone operator theory and fixed point techniques. The obtained results are compared with those obtained via the abstract variational principle. Moreover, by applying truncation techniques and regularity results we are able to deal with elliptic problems that involve discontinuous nonlinearities without making use of nonsmooth analysis methods.

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1. INTRODUCTION

In this paper, we investigate the existence of a nontrivial solution for the following quasilinear elliptic problem

$$\begin{cases} -\Delta_p u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplace operator with $1 < p < +\infty$, $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded domain with a sufficiently smooth boundary, λ is a real parameter and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. We denote by $W_0^{1,p}(\Omega)$ the

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Sobolev space of functions with generalized homogeneous boundary values endowed with the norm

$$\|u\| := \left(\int_\Omega |\nabla u(x)|^p \, dx \right)^{1/p}. \tag{1.2}$$

It is well known that, if $1 \leq p < +\infty$,

$$\|u\|_{L^q(\Omega)} \leq c_q \|u\|, \qquad \forall u \in W_0^{1,p}(\Omega). \tag{1.3}$$

for every $q \in [1, p^*)$, where, as usual, p^* is the critical Sobolev exponent given by

$$p^* = \begin{cases} +\infty & \text{if } N \leq p < \infty, \\ \frac{pN}{N-p} & \text{if } 1 \leq p < N, \end{cases}$$

and the embedding $W_0^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ is compact. A function $u \in W_0^{1,p}(\Omega)$ is called a solution of (1.1) if the following holds true:

$$\int_\Omega |\nabla u(x)|^{p-2} \nabla u(x) \nabla \varphi(x) \, dx = \lambda \int_\Omega f(u(x)) \varphi(x) \, dx, \quad \forall \varphi \in W_0^{1,p}(\Omega). \tag{1.4}$$

A great deal of work has already been done on the existence of solutions of the problem (1.1), see e.g. [1, 4, 5, 12, 15, 16, 18, 20] and [24]. However, as is well known the assumptions on the nonlinearity f makes the difference between the individual papers. In all of the above cited papers an asymptotic behaviour of the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is required either at zero or at infinity or at both. In many other cases, such type of asymptotic conditions are imposed also on the primitive of f, i.e., on the function $F(s) := \int_0^s f(t) \, dt$, see for instance [17], [19], [21] and the references therein.

Unlike in the above mentioned papers, in a very recent paper [8] the authors established existence and multiplicity results of problem (1.1) without imposing any asymptotic conditions at zero or at infinity. The approach used in [8] is based on an abstract localization principle of critical points of functionals of the form $\mathbb{E} = \Phi - \lambda\Psi$ on open sublevels of Φ, i.e., of sets of the form $\Phi^{-1}(-\infty, r)$, which reads as follows, see [2, 4].

Theorem 1.1. *Let X be a reflexive Banach space, $\Phi : X \to \mathbb{R}$ and $\Psi : X \to \mathbb{R}$ two continuously Gâteaux differentiable functionals such that Φ is coercive and sequentially weakly lower semicontinuous, while Ψ is sequentially weakly upper semicontinuous. Let $r > \inf_X \Phi$ and put*

$$\varphi(r) := \inf_{v \in \Phi^{-1}(-\infty, r)} \frac{\sup_{u \in \Phi^{-1}(-\infty,r)} \Psi(u) - \Psi(v)}{r - \Phi(v)}.$$

Then, for every $\lambda \in \left(0, \frac{1}{\varphi(r)}\right)$ the functional $\mathbb{E} = \Phi - \lambda\Psi$ has a critical point $u_\lambda \in \Phi^{-1}(-\infty, r)$ such that $\mathbb{E}(u_\lambda) \leq \mathbb{E}(v)$ for every $v \in \Phi^{-1}(-\infty, r)$.

The variational functional of (1.1) is of the structure $\mathbb{E}$ of Theorem 1.1 with $X = W_0^{1,p}(\Omega)$, and

$$\Phi(u) := \frac{1}{p}\|u\|^p, \quad \Psi(u) := \int_\Omega F(u(x))\,dx, \quad \mathbb{E}(u) := \Phi(u) - \lambda\Psi(u).$$

Based on Theorem 1.1, in ([8, Theorem 2.1]) among others the following result has been obtained.

Theorem 1.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with*

$$f(0) \neq 0, \tag{1.5}$$

and assume that

(f$_*$) *There exist two positive constants M_1 and M_2 and $q \in [1, p^*)$ such that*

$$|f(s)| \leq M_1 + M_2|s|^{q-1}, \quad \forall s \in \mathbb{R}.$$

Put,

$$\lambda^* := \begin{cases} +\infty, & 1 \leq q < p; \\ \dfrac{1}{c_p^p M_2}, & q = p; \\ \dfrac{q^{\frac{p-1}{q-1}}}{p(q-1)} \left(\dfrac{q-p}{c_1 M_1}\right)^{\frac{q-p}{q-1}} \left(\dfrac{p-1}{c_q^q M_2}\right)^{\frac{p-1}{q-1}}, & p < q < p^*, \end{cases} \tag{1.6}$$

Then, for every $\lambda \in (0, \lambda^*)$, problem (1.1) *admits a nontrivial solution $u_\lambda \in C_0^1(\overline{\Omega})$.*

The variational approach adopted in [8] not only allows for the existence of at least one nontrivial solution u_λ, but also provides a localization of u_λ, namely

$$\|u_\lambda\| < (p\bar{r})^{1/p}, \tag{1.7}$$

where $\bar{r} = \bar{r}(\lambda)$ can a priori be estimated from below in the case $1 \leq q \leq p$, while in the case $p < q < p^*$ we have a uniform bound given by

$$\bar{r} = \frac{1}{p}\left[\frac{qc_1 M_1(p-1)}{c_q^q M_2(q-p)}\right]^{p/(q-1)}. \tag{1.8}$$

Moreover, the following variational characterization holds: *u_λ is a local minimum of the energy functional $\mathbb{E}$ related to problem* (1.1).

It is worth noticing that this last variational property of u_λ plays a crucial role in [7] and [8] whenever the existence and multiplicity of solutions are investigated in the case $f(0) = 0$.

An important novelty in studying problem (1.1), introduced in [3] for $p = 2$ and [8] for $p \neq 2$, is that the existence of at least one solution has been established under the natural subcritical growth condition (f$_*$) only and without requiring any asymptotic condition for f neither at zero nor at infinity. In case that $N < p$, without any growth condition on f and only assuming some asymptotic condition at zero, in [7] the following multiplicity result has been obtained: Assume

(f_1) $\lim_{s\to 0} \frac{f(s)}{|s|^{p-2}s} = L \in (0,+\infty)$;

$(f_2^{\lambda_k})$ *There exists* $\rho_0 > 0$ *such that*

$$\frac{\max_{|s|\le\rho_0} F(s)}{\rho_0^p} < \frac{1}{c^p\lambda_k|\Omega|}\lim_{s\to 0}\frac{F(s)}{|s|^p},$$

where c is the constant of the embedding $W_0^{1,p}(\Omega) \hookrightarrow C^0(\bar\Omega)$, $\lambda_k,\ k=1,2$, are the first and second eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$ and $|\Omega|$ stands for the Lebesgue measure of Ω, then there are two intervals given by

$$\Lambda_k = \left(\frac{\lambda_k}{L}, \frac{\rho_0^p}{pc^p|\Omega|\max_{|s|\le\rho_0} F(s)}\right), \qquad k=1,2;$$

such that

- *for every* $\lambda \in \Lambda_1$ *problem* (D_λ) *admits at least two constant-sign solutions*;
- *for every* $\lambda \in \Lambda_2$ *problem* (D_λ) *admits, in addition, a third sign-changing solution.*

The aim of the present note is to prove the existence of at least one nontrivial solution of (1.1) by applying an alternative, nonvariational method. The approach in this paper is based on pseudomonotone operator theory and Schauder's fixed point theorem, see Theorem 2.1. The obtained results are then compared with those obtained via the abstract variational principle. It turns out that the nonvariational approach used here improves the result of Theorem 1.2, because, under exactly the same assumptions, a relevant bigger range of parameter can be obtained. The price we pay for the improvement of the parameter range is that we may loose the variational feature of the solutions.

Finally, by applying truncation techniques and regularity results, in Theorem 3.2 the existence of constant-sign solutions of problem (1.1) is shown when f is discontinuous at the origin without making use of nonsmooth analysis methods.

2. MAIN RESULT

The main result of this section is the following theorem.

Theorem 2.1. *Let* $f:\mathbb{R}\to\mathbb{R}$ *be a continuous function satisfying* (1.5) *and* (f_*). *Put*

$$\mu^* := \begin{cases} +\infty, & 1\le q<p; \\ \frac{1}{c_p^p M_2}, & q=p; \\ \frac{1}{q-1}\left(\frac{q-p}{c_1M_1}\right)^{\frac{q-p}{q-1}}\left(\frac{p-1}{c_q^q M_2}\right)^{\frac{p-1}{q-1}}, & p<q<p^*, \end{cases} \tag{2.1}$$

Then, for every $0 < |\lambda| < \mu^*$ *problem* (1.1) *admits a nontrivial solution* $v \in C_0^1(\overline{\Omega})$.

Proof. Let $X := W_0^{1,p}(\Omega)$ and X^* be its dual space, and $\langle \cdot, \cdot \rangle$ the duality pairing. The norm in X is given in (1.2). Define the operators $A : X \to X^*$ and $G : X \to X^*$ as follows:

$$\langle Au, \varphi \rangle := \int_\Omega |\nabla u(x)|^{p-2} \nabla u(x) \nabla \varphi(x)\, dx, \quad u, \varphi \in X,$$

$$\langle G(u), \varphi \rangle := \int_\Omega f(u(x))\varphi(x)\, dx, \quad u, \varphi \in X.$$

Thus, we may rewrite problem (1.1) in the form of the following operator equation:

$$u \in X : \langle Au - \lambda G(u), \varphi \rangle = 0, \quad \forall \varphi \in X. \tag{2.2}$$

The operator A is monotone (even strictly monotone), bounded and continuous. To avoid too much notations, let us denote the Nemytskij operator generated by $f : \mathbb{R} \to \mathbb{R}$ again denote by f, i.e., $f(u)(x) = f(u(x))$. Because of $q \in [1, p^*)$, the embedding $X \hookrightarrow L^q(\Omega)$ is compact. Moreover, if q' denotes the Hölder conjugate to q, then, due to the growth condition (f_*), the Nemytskij operator $f : L^q(\Omega) \to L^{q'}(\Omega)$ is continuous and bounded. Taking into account that the embedding $L^{q'}(\Omega) \hookrightarrow X^*$ is continuous, we see that the operator $G : X \to X^*$ is bounded, continuous and compact.

A monotone, bounded and continuous operator is pseudomonotone, and a compact, continuous and bounded operator is pseudomonotone as well, which implies that

$$A - \lambda G : X \to X^*$$

is bounded, continuous and pseudomonotone for all $\lambda \in \mathbb{R}$. By the main theorem on pseudomonotone operators due to Brezis (see, e.g., [25]), the operator equation (2.2) has a solution, provided that $A - \lambda G : X \to X^*$ is coercive, i.e., provided that the following holds:

$$\lim_{\|u\| \to +\infty} \frac{\langle Au - \lambda G(u), u \rangle}{\|u\|} = +\infty. \tag{2.3}$$

Now let us distinguish three cases.

Case (i): $1 \le q < p$. In this case we have by Young's inequality, for any $\varepsilon > 0$

$$|s|^{q-1} \le c(\varepsilon) + \varepsilon |s|^{p-1}, \forall s \in \mathbb{R},$$

where $c(\varepsilon)$ is some constant depending on ε only. Thus, from (f_*), it follows:

$$|f(s)| \le C(\varepsilon) + \varepsilon |s|^{p-1}, \forall s \in \mathbb{R}. \tag{2.4}$$

Using (2.4) and

$$\|u\|_{L^p(\Omega)} \le c_p \|u\|, \quad \forall u \in W_0^{1,p}(\Omega), \tag{2.5}$$

we get the following estimate:

$$\langle Au - \lambda G(u), u\rangle \geq \|u\|^p - |\lambda| \left(C(\varepsilon)\|u\|_{L^1(\Omega)} + \varepsilon \|u\|^p_{L^p(\Omega)} \right) \geq \left(1 - |\lambda| c_p^p \varepsilon\right) \|u\|^p - |\lambda| C(\varepsilon)\|u\|_{L^1(\Omega)}, \tag{2.6}$$

which yields the coercivity for any $\lambda \in \mathbb{R}$ when choosing ε sufficiently small. This proves assertion (i).

Case (ii): $q = p$. By using (f_*) we get

$$\langle Au - \lambda G(u), u\rangle \geq \|u\|^p - |\lambda| \left(M_1\|u\|_{L^1(\Omega)} + M_2 \|u\|^p_{L^p(\Omega)} \right) \geq \left(1 - |\lambda| c_p^p M_2\right) \|u\|^p - |\lambda| M_1\|u\|_{L^1(\Omega)}, \tag{2.7}$$

which shows that $A - \lambda G : X \to X^*$ is coercive provided that

$$1 - |\lambda| c_p^p M_2 > 0 \Leftrightarrow |\lambda| < \frac{1}{c_p^p M_2}.$$

This proves assertion (ii).

Case (iii): $p < q < p^*$. In this case we apply Schauder's fixed point theorem. First, since $A : X \to X^*$ is strictly monotone, continuous, bounded and coercive, from the main theorem on monotone operators it follows that for any $b \in X^*$ there exists a uniquely defined solution of

$$u \in X : \; Au = b, \tag{2.8}$$

which implies that $A : X \to X^*$ is a bijection. Moreover, $A^{-1} : X^* \to X$ is monotone as well, demicontinuous and bounded. One can then shows that $A^{-1} : X^* \to X$ is even continuous. The latter makes use of the specific property of the p-Laplacian. Now we can rewrite our problem as a fixed point equation:

$$u \in X : \; u = A^{-1} \circ (\lambda G)(u), \tag{2.9}$$

where the fixed point operator $T_\lambda = A^{-1} \circ (\lambda G) : X \to X$ is compact, continuous and bounded. Any fixed point of T is a solution of problem (1.1) and vice versa. Thus, it remains to verify the existence of fixed points. By Schauder's Theorem the existence of a fixed point is proved provided T_λ can be shown to be a selfmapping of some closed, bounded and convex set. Let us show that for some parameter range for λ, there is a closed ball $B(0, R) := \{v \in X : \|v\| \leq R\}$ such that $T_\lambda : B(0, R) \to B(0, R)$. Let $u = T_\lambda(v)$, i.e., u is the unique solution of

$$u \in X : Au = \lambda G(v), \tag{2.10}$$

which implies by using in (2.10) as special test function $\varphi = u$ the following:

$$\|u\|^p = \lambda \int_\Omega f(v) u dx \leq |\lambda| \int_\Omega (M_1 + M_2 |v|^{q-1})|u| \leq |\lambda| \left(M_1 c_1 \|u\| + M_2 c_q^q \|v\|^{q-1} \|u\|\right), \tag{2.11}$$

which yields

$$\|u\|^{p-1} \leq |\lambda| \left(M_1 c_1 + M_2 c_q^q \|v\|^{q-1}\right). \tag{2.12}$$

Therefore, from (2.12), the condition for T_λ being a mapping of a ball $B(0, R)$ into itself is

$$|\lambda| \left(M_1 c_1 + M_2 c_q^q \|v\|^{q-1}\right) \leq R^{p-1} \tag{2.13}$$

which gives the range for λ as

$$|\lambda| \leq \frac{R^{p-1}}{M_1 c_1 + M_2 c_q^q R^{q-1}}. \tag{2.14}$$

A simple computation shows that μ^*, as introduced in (2.1), is the maximum of the map $\varphi(R) = \frac{R^{p-1}}{M_1 c_1 + M_2 c_q^q R^{q-1}}$, for every $R > 0$, and, in particular, it is attained at $R_{\max}$ given by

$$R_{\max} = \left[\frac{c_1 M_1 (p-1)}{c_q^q M_2 (q-p)}\right]^{1/q-1}. \tag{2.15}$$

In any case (1.1) admits a solution u that is nontrivial because $f(0) \neq 0$. Moreover, by [13, Theorem 7.1, pag. 286], u belongs to $L^\infty(\Omega)$ and by the nonlinear regularity theory, see [10, 14], we conclude that $u \in C_0^1(\overline{\Omega})$. □

Remark 2.2. From the proof of Theorem 2.1 one can deduce that, in the case (iii) the nontrivial solution satisfies an a priori estimation of type (2.12). In particular,

$$\|u\| \leq \left[\frac{c_1 M_1 (p-1)}{c_q^q M_2 (q-p)}\right]^{1/q-1}$$

uniformly with respect to λ.

Remark 2.3. Comparing the results of Theorem 2.1 with Theorem 1.2, it turns out that Theorem 2.1 improves Theorem 1.2, because an easy computation results in $\lambda^* = \mu^*$ if $1 \leq q \leq p$ and

$$\frac{\lambda^*}{\mu^*} = \frac{q^{\frac{p-1}{q-1}}}{p} \tag{2.16}$$

if $p < q < p^*$. Hence, observing that the function $t \mapsto t^{\frac{1}{t-1}}$ for every $t > 1$ is decreasing, from (2.16) one can obtain that

$$\lambda^* < \mu^*.$$

Finally, we observe that the range of parameters assured in Theorem 2.1 is bigger, as it involves also negative values of the parameter.

Remark 2.4. We note that in condition (f_*) the constants M_1 and M_2 are assumed to be strictly positive. However, while M_1 cannot be equal to zero, because this would be in contradiction with (1.5), M_2 is allowed to vanish. In this case, the function f is assumed to be bounded and one readily sees that μ^* does not depend anymore on q, being always $\mu^* = +\infty$. In particular, following the proof of Theorem 2.1 in the case $p < q < p^*$ one obtains:

for every $\lambda \neq 0$, (1.1) admits at least one nontrivial solution such that

$$\|u\| \leq R \tag{2.17}$$

whenever $R > 0$ satisfies $0 < |\lambda| < \frac{R^{p-1}}{c_1 M_1}$.

3. DISCONTINUOUS NONLINEARITY

In this section we are going to show the existence of nontrivial solutions of (1.1) when the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ may be discontinuous. The tools used here to deal with discontinuous elliptic problems is the main result of the preceding section along with truncation techniques and regularity results.

Let us first recall some preliminary, technical lemma which will be useful later, see [8, Lemma 3.1].

Lemma 3.1. *Let $g, h : \mathbb{R} \to \mathbb{R}$ be two continuous functions and assume that*

$$\lim_{t\to 0^+} \frac{g(t)}{|t|^{p-2}t} = L_+ > 0 \quad \textit{and} \quad \lim_{t\to 0^-} \frac{h(t)}{|t|^{p-2}t} = L_- > 0.$$

Then, for every $M > 0$ there exist two positive constants c, $\bar{c}$ such that

$$-g(t) \leq ct^{p-1} \quad \forall t \in [0, M],$$

$$h(t) \leq \bar{c}|t|^{p-1} \quad \forall t \in [-M, 0].$$

Proof. Fix $M > 0$ and put $\alpha := \max\{\max_{0\leq t\leq M} |g(t)|, \max_{-M\leq t\leq 0} |h(t)|\}$. If $0 < \beta < \min\{L_-, L_+\}$ there exists $\delta > 0$ such that

$$\frac{g(t)}{|t|^{p-2}t} > \beta \quad \forall t \in (0, \delta), \tag{3.1}$$

$$\frac{h(t)}{|t|^{p-2}t} > \beta \quad \forall t \in (-\delta, 0). \tag{3.2}$$

Let us put $c = \max\left\{1, \frac{\alpha}{\delta^{p-1}}\right\}$ and $\bar{c} = \frac{\alpha}{\delta^{p-1}}$. Hence, for every $t \in [0, M]$ one has that

- if $0 \leq t < \delta$, then, in view of (3.1), $g(t) \geq \beta|t|^{p-2}t \geq -|t|^{p-2}t$, that is

$$-g(t) \leq |t|^{p-2}t \leq ct^{p-1}.$$

- If $\delta \leq t \leq M$, then

$$-g(t) \leq \alpha = \delta^{p-1}\frac{\alpha}{\delta^{p-1}} \leq c\delta^{p-1} \leq ct^{p-1}.$$

Analogously, for every $t \in [-M, 0]$ one has that

- if $-\delta < t \leq 0$, then, in view of (3.2)

$$h(s) \leq \beta |t|^{p-2} t \leq \bar{c}|t|^{p-1}.$$

- If $-M \leq t \leq -\delta$, then

$$h(t) \leq \alpha = \delta^{p-1} \frac{\alpha}{\delta^{p-1}} \leq \bar{c}\delta^{p-1} \leq \bar{c}|t|^{p-1},$$

and the proof is complete. □

As a consequence of Theorem 2.1 we can obtain the following result.

Theorem 3.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying condition* (f_*) *and such that f is continuous in $\mathbb{R} \setminus \{0\}$. Moreover, assume that*

(f_0) *zero is a discontinuity point of first kind such that*

$$f(0^-) \cdot f(0^+) \neq 0, \tag{3.3}$$

where $f(0^+) = \lim_{t \to 0^+} f(t)$ and $f(0^-) = \lim_{t \to 0^-} f(t)$. Then, the following conclusions hold:

(j_1) *If $f(0^-) < 0 < f(0^+)$, then for every $\lambda \in (0, \mu^*)$ problem* (1.1) *admits at least two solutions u_-, u_+ such that $u_+ \in \mathrm{int}(C_0^1(\bar{\Omega})_+)$ and $u_- \in -\mathrm{int}(C_0^1(\bar{\Omega})_+)$.*

(j_2) *If $\min\{f(0^+), f(0^-)\} > 0$, then for every $\lambda \in (0, \mu^*)$ problem* (1.1) *admits at least one solution $u_+ \in \mathrm{int}(C_0^1(\bar{\Omega})_+)$ and for every $\lambda \in (-\mu^*, 0)$ problem* (1.1) *admits at least one solution $u_- \in -\mathrm{int}(C_0^1(\bar{\Omega})_+)$.*

(j_3) *If $f(0^+) < 0 < f(0^-)$, then for every $\lambda \in (-\mu^*, 0)$ problem* (1.1) *admits at least two solutions u_-, u_+ such that $u_+ \in \mathrm{int}(C_0^1(\bar{\Omega})_+)$ and $u_- \in -\mathrm{int}(C_0^1(\bar{\Omega})_+)$.*

(j_4) *If $\max\{f(0^+), f(0^-)\} < 0$, then for every $\lambda \in (0, \mu^*)$ problem* (1.1) *admits at least one solution $u_- \in -\mathrm{int}(C_0^1(\bar{\Omega})_+)$ and for every $\lambda \in (-\mu^*, 0)$ problem* (1.1) *admits at least one solution $u_+ \in -\mathrm{int}(C_0^1(\bar{\Omega})_+)$.*

Proof. We are going to prove cases (j_1) and (j_2) only, because the other cases can be treated in a similar way.

Case (j_1): $f(0^-) < 0 < f(0^+)$. Put

$$g(t) = \begin{cases} f(t) & \text{if } t > 0 \\ f(0^+) & \text{if } t \leq 0. \end{cases}$$

Clearly g satisfies all the assumptions of Theorem 2.1. Hence, for every $\lambda \in (0, \mu^*)$ there exists a function $u_+ \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega |\nabla u_+(x)|^{p-2} \nabla u_+(x) \nabla \varphi(x)\, dx = \lambda \int_\Omega g(u_+(x))\varphi(x)\, dx,\ \forall \varphi \in W_0^{1,p}(\Omega). \tag{3.4}$$

Moreover, thanks to [13, Theorem 7.1, pag. 286], $u_+ \in L^\infty(\Omega)$ and applying the classical regularity theory, see [10, 14], $u_+ \in C_0^1(\bar{\Omega})$.

Let $u_+^-(x) = \max\{-u_+(x), 0\}$, then $u_+^- \in W_0^{1,p}(\Omega)$, and hence, testing (3.4) with u_+^-, one gets

$$-\|u_+^-\|^p = \lambda f(0^+) \int_\Omega u_+^-(x)\, dx.$$

Because $f(0^+) > 0$ we can conclude that $u_+^- = 0$ which implies that $u_+(x) \geq 0$ a.e. in Ω, that is u_+ must be nonnegative.

We are going to show next that the solution u_+ of (3.4) is in fact a solution of (1.1). To this end we only need to verify that $u_+ \in \mathrm{int}(C_0^1(\bar{\Omega})_+)$, because then it holds $f(u_+(x)) = g(u_+(x))$ for every $x \in \Omega$, which proves u_+ is a solution of (1.1). First, observe that

$$\lim_{t \to 0^+} \frac{g(t)}{t^{p-1}} = +\infty.$$

Since u_+ is bounded, by Lemma 3.1 with $M = \|u_+\|_\infty$, there exists a positive constant c such that

$$-g(t) \leq c|t|^{p-1}$$

for every $t \in [0, \|u_+\|_\infty]$. Hence,

$$\Delta_p u_+(x) = -\lambda g(u_+(x)) \leq \lambda c u_+(x)^{p-1}$$

a.e. in Ω. Thus, we can apply the Vazquez's maximum principle [22, Theorem 5] and conclude that $u_+ \in \mathrm{int}(C_0^1(\bar{\Omega})_+)$. This completes the proof of the existence of a positive solution.

A negative solution u_- can be shown to exist in a similar way by replacing the function g by the function h as follows:

$$h(t) = \begin{cases} f(t) & \text{if } t < 0 \\ f(0^-) & \text{if } t \geq 0, \end{cases}$$

and observing that

$$\Delta_p(-u_-(x)) = -\Delta_p u_-(x) = \lambda h(u_-(x)) \leq \lambda \bar{c}| - u_-(x)|^{p-1}$$

a.e. in Ω, so that the conclusion follows again by Vasquez's maximum principle.

Case (j_2): $\min\{f(0^+), f(0^-)\} > 0$. Fix $\lambda \in (0, \mu^*)$, then the existence of u_+ can be obtained exactly as in the previous case. Now let $\lambda \in (-\mu^*, 0)$ and define the following function

$$h(t) = \begin{cases} f(t) & \text{if } t < 0 \\ f(0^-) & \text{if } t \geq 0. \end{cases}$$

Theorem 2.1 assures that there exists $u_- \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega |\nabla u_-(x)|^{p-2} \nabla u_-(x) \nabla \varphi(x)\, dx = \lambda \int_\Omega h(u_-(x)) \varphi(x)\, dx, \ \forall \varphi \in W_0^{1,p}(\Omega). \tag{3.5}$$

Moreover, arguing as in the previous case, $u_- \in C_0^1(\bar{\Omega})$. Set $u_-^+(x) = \max\{0, u_-(x)\}$, then it is well known that $u_-^+ \in W_0^{1,p}(\Omega)$ and, putting $\varphi = u_-^+$ in (3.5), one has

$$\|u_-^+\|^p = \lambda f(0^-) \int_\Omega u_+(x)\ dx.$$

Because $f(0^-) > 0$ and $\lambda < 0$, one concludes that $u_-^+ = 0$ which shows that $u_-(x) \leq 0$ a.e. in Ω, i.e., u_- must be nonpositive in Ω. Moreover, since h is a continuous function, a simple computation shows that there exists a positive constant c

$$h(t) \geq -c|t|^{p-1} \quad \forall t \in (-\|u_-\|_\infty, 0).$$

Indeed, since $h(0) > 0$, there exists $0 < \delta < \|u_-\|_\infty$ such that

$$h(t) > 0 \quad \forall t \in (-\delta, 0).$$

Put $m = \min\left\{-1, \min_{t\in[-\|u_-\|_\infty, 0]} h(t)\right\}$ and $c = -\frac{m}{\delta^{p-1}}$, if $t \in (-\delta, 0)$ one has

$$h(t) > 0 > -c|t|^{p-1}.$$

Otherwise, if $t \in (-\|u_-\|_\infty, -\delta)$, one has

$$h(t) \geq m = \frac{m}{|t|^{p-2}t}|t|^{p-2}t \geq c|t|^{p-2}t = -c|t|^{p-1}.$$

Hence,

$$\Delta_p(-u_-(x)) = -\Delta_p u_-(x) = \lambda h(u_-(x)) \leq -\lambda c|-u_-(x)|^{p-1}$$

a.e. in Ω. Finally, again by the Vasquez's maximum principle, $u_- \in -\mathrm{int}(C_0^1(\bar{\Omega})_+)$, and thus u_- is a solution of problem (1.1). □

Remark 3.3. In [9] nonsmooth (variational-hemivariational inequalities) elliptic problems depending on a parameter λ in a nonlinear way have been considered, and the existence of nontrivial solutions have been proved by methods of nonsmooth analysis. As an application a discontinuous quasilinear elliptic problem of the form

$$-\Delta_p u = \lambda \chi_{\{u>0\}} \quad \text{in} \ \ \Omega, \quad u = g \quad \text{on} \ \ \partial\Omega,$$

has been treated via nonsmooth analysis methods, where χ_A denotes the characteristic function of the set A. Another example where classical variational approach is used to treat discontinuous problems is given in in [6]. Unlike in the above mentioned papers, here we obtain the existence of nontrivial solutions $u \in \mathrm{int}(C_0^1(\bar{\Omega})_+)$ of problem (1.1) with f discontinuous at the origin without using the classical results of nonsmooth analysis.

In the particular case $N < p$, we are able to show an existence result of (1.1) whose nonlinearity $f : \mathbb{R} \to \mathbb{R}$ may be discontinuous and is not subject to any growth condition. In this case we make use of the embedding $W_0^{1,p}(\Omega) \hookrightarrow C(\overline{\Omega})$, i.e,

$$\|u\|_\infty \leq c_\infty \|u\| \quad \forall\ u \in W_0^{1,p}(\Omega). \tag{3.6}$$

More precisely, the following theorem can be proved.

Theorem 3.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function and assume that there exists $d > 0$ such that*

(k_1) *f is continuous in $(0, d]$;*
(k_2) *$0 < f(0^+) < +\infty$.*

Then, if $N < p$, for every $\lambda \in \left(0, \frac{1}{c_1 c_\infty^{p-1}} \frac{d^{p-1}}{\sup_{t\in(0,d]} |f(t)|}\right)$, where c_1 and c_∞ are the constants of the embedding (1.3) *and* (3.6)*, respectively, problem* (1.1) *admits at least one nontrivial solution $u \in \operatorname{int}(C_0^1(\bar{\Omega})_+)$.*

Proof. Set

$$g(t) = \begin{cases} f(0^+) & \text{if } t \le 0 \\ f(t) & \text{if } 0 < t \le d \\ f(d) & \text{if } t > d. \end{cases}$$

It is clear that $g : \mathbb{R} \to \mathbb{R}$ is a continuous and bounded function such that $g(0) > 0$. In particular,

$$|g(t)| \le \sup_{t\in(0,d]} |f(t)|$$

for every $t \in \mathbb{R}$.

Hence, arguing as in Remark 2.4 with g in place of f, $M_1 = \sup_{t\in(0,d]} |f(t)|$ and $M_2 = 0$, fix $R = \frac{d}{c_\infty}$, for every $\lambda \in \left(0, \frac{R^{p-1}}{c_1 M_1}\right) = \left(0, \frac{1}{c_1 c_\infty^{p-1}} \frac{d^{p-1}}{\sup_{t\in(0,d]} |f(t)|}\right)$ there exists a function $u \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega |\nabla u(x)|^{p-2} \nabla u(x) \nabla \varphi(x)\, dx = \lambda \int_\Omega g(u(x))\varphi(x)\, dx, \quad \forall \varphi \in W_0^{1,p}(\Omega) \tag{3.7}$$

and (2.17) holds, namely, in view of (3.6),

$$\|u\|_\infty \le c_\infty \|u\| \le d.$$

Testing (3.7) with $\varphi = u^- = \max\{-u, 0\}$ we infer that u^- is a.e. zero in Ω, that is, because u is continuous, we get $0 \le u(x) \le d$ for every $x \in \Omega$. Finally, arguing as in the proof of Theorem 3.2 case (j_1), one can prove that $u \in \operatorname{int}(C_0^1(\bar{\Omega})_+)$, and thus the conclusion follows by observing that $f(u(x)) = g(u(x))$ for every $x \in \Omega$. ☐

Remark 3.5. We remark that the assumptions of Theorem 3.4 allow the nonlinearity to have at zero a first kind of discontinuity from the right and any kind of discontinuity from the left. Hence, thanks to Theorem 3.4, different situations with respect to Theorem 3.2 can be considered in case that $N < p$.

Remark 3.6. Taking into account the estimates of the Sobolev embedding constants contained in [23], a more concrete estimate for the interval of the parameter λ can be obtained, where the embedding constants are expressed in terms of the data of (1.1).

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