Dynamic Systems and Applications 22 (2013) 411-418

CRITICAL NONLINEARITIES FOR ELLIPTIC DIRICHLET PROBLEMS

GABRIELE BONANNO AND GIUSEPPINA D'AGUÌ

Department of Civil, Information Technology, Construction, Environmental Engineering and Applied Mathematics, University of Messina, Italy

ABSTRACT. In this paper the existence of one positive weak solution for Dirichlet problems with a critical growth of the nonlinearity is established. To this end, it is previously proved that the associated energy functional satisfies a suitable type of Palais Smale condition in order to apply a very recent local minimum theorem.

AMS (MOS) Subject Classification. 35J60, 35J20, 49J35, 49J52

1. INTRODUCTION

Consider the following Dirichlet problem

$$(D_\lambda^f) \qquad \begin{cases} -\Delta u = \lambda f(u) \quad \text{in} \quad \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

where Ω is a non-empty bounded open subset of the Euclidean space $(\mathbb{R}^N, |\cdot|)$, $N \geq 3$, with boundary of class C^1, λ is a positive parameter and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function with a critical growth, that is,

$$f(t) = |t|^{2^*-2}t + g(t)$$

for all $t \in \mathbb{R}$, being g a nonnegative continuous function satisfying

(h) *there exist* $a > 0$ *and* $q \in]1, 2N/(N-2)[$ *such that*

$$g(t) \leq a|t|^{q-1}$$

for every $t \in \mathbb{R}$,

and $2^* = \frac{2N}{N-2}$.

The aim of this paper is to prove that the energy functional associated to Problem (D_λ^f) satisfies a suitable type of Palais Smale condition previously introduced in [1, Section 2]. As a consequence, applying a local minimum theorem established in [1], the following existence result is here pointed out.

Received October 24, 2012

Theorem 1.1. *Put* $G(\xi) = \int_0^\xi g(t)dt$ *for every* $\xi \in \mathbb{R}$ *and assume that*

$$\limsup_{t\to 0^+} \frac{G(t)}{t^2} = +\infty.$$

Then, there is $\lambda^* > 0$ *such that, for each* $\lambda \in]0, \lambda^*[$*, the problem*

$$\begin{cases} -\Delta u = \lambda \left(|u|^{2^*-2}u + g(u) \right) & \text{in} \quad \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

admits at least one positive weak solution.

To be precise, in this paper we give a detailed proof of a suitable type of Palais Smale condition for the energy functional associated to problem (D_λ^f) (see Lemma 3.1) and, as a consequence, the existence theorem for the problem (D_λ^f) (that is, Theorem 1.1) is presented, exclusively in the critical case (see Remark 3.4).

The paper is arranged as follows. In Section 2, the local minimum theorem is recalled (Theorem 2.1) as well as the definition of this type of Palais Smale condition is pointed out. Section 3 is devoted to our main results. This type of Palais Smale condition for the energy functional associated to elliptic Dirichlet problem (Lemma 3.1) is proved. Moreover, a proof of Theorem 1.1 is given. Finally, in the same section, an example of application, for which a classical result of Brezis and Nirenberg ([3, Theorem 2.1]) cannot be applied, is pointed out (see Example 3.2 and Remark 3.3). Clearly, these types of results are mutually independent because of different position of the parameter, beyond the fact that in the Brezis-Nirenberg Theorem the solution is a mountain pass point, while, on the contrary, in our result it is a local minimum.

2. PRELIMINARIES

Let $(X, \|\cdot\|)$ be a real Banach space, let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals and put

$$I = \Phi - \Psi.$$

Fixed $r_1, r_2 \in [-\infty, +\infty]$, with $r_1 < r_2$ we say that the functional I verifies the *Palais-Smale condition cut off lower at* r_1 *and upper at* r_2 (in short ${}^{[r_1]}(PS)^{[r_2]}$-condition) if any sequence $\{u_n\}$ such that

(α) $\{I(u_n)\}$ is bounded,
(β) $\lim_{n\to+\infty} \|I'(u_n)\|_{X^*} = 0$,
(γ) $r_1 < \Phi(u_n) < r_2 \quad \forall n \in \mathbb{N}$,

has a convergent subsequence.

When we fix $r_2 = -\infty$, that is, $\Phi(u_n) < r_2\ \forall n \in \mathbb{N}$, we denote this type of Palais Smale condition with $(PS)^{[r_2]}$. When, in addition, $r_2 = +\infty$, it is the classical Palais Smale condition.

Now, put

$$\beta(r_1,r_2)=\inf_{v\in\Phi^{-1}(]r_1,r_2[)}\frac{\sup_{u\in\Phi^{-1}(]r_1,r_2[)}\Psi(u)-\Psi(v)}{r_2-\Phi(v)} \tag{2.1}$$

and

$$\rho_2(r_1,r_2)=\sup_{v\in\Phi^{-1}(]r_1,r_2[)}\frac{\Psi(v)-\sup_{u\in\Phi^{-1}(]-\infty,r_1])}\Psi(u)}{\Phi(v)-r_1} \tag{2.2}$$

for all $r_1, r_2 \in \mathbb{R}$, with $r_1 < r_2$.

Our main tool to prove Theorem 1.1 is the local minimum theorem established in [1], which is recalled below.

Theorem 2.1 (See [1, Theorem 5.1])**.** *Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functions. Assume that there are r_1, $r_2 \in \mathbb{R}$, with $r_1 < r_2$, such that*

$$\beta(r_1,r_2)<\rho_2(r_1,r_2),$$

where β and ρ_2 are given by (2.1) and (2.2), and for each $\lambda \in \left]\frac{1}{\rho_2(r_1,r_2)},\frac{1}{\beta(r_1,r_2)}\right[$ the function $I_\lambda = \Phi - \lambda\Psi$ satisfies ${}^{[r_1]}(PS)^{[r_2]}$-condition.

Then, for each $\lambda \in \left]\frac{1}{\rho_2(r_1,r_2)},\frac{1}{\beta(r_1,r_2)}\right[$ there is $u_{0,\lambda} \in \Phi^{-1}(]r_1,r_2[)$ such that $I_\lambda(u_{0,\lambda}) \le I_\lambda(u)$ for all $u \in \Phi^{-1}(]r_1,r_2[)$ and $I'_\lambda(u_{0,\lambda}) = 0$.

Now, we recall that

$$\|u\|_{L^q(\Omega)}\le c_q\|u\|\,,\quad u\in H_0^1(\Omega)\,,\quad q\in[1,2^*]$$

$$c_{2^*}=\frac{1}{\sqrt{N(N-2)\pi}}\Big(\frac{N!}{2\Gamma(1+N/2)}\Big)^{1/N}, \tag{2.3}$$

$$c_q\le\frac{meas(\Omega)^{\frac{2^*-q}{2^*q}}}{\sqrt{N(N-2)\pi}}\Big(\frac{N!}{2\Gamma(N/2+1)}\Big)^{1/N} \tag{2.4}$$

and that the embedding $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ is not compact if $q = 2^*$.

3. MAIN RESULTS

In this section we present our main results.

Let f, g be as defined in Introduction. Without loss of generality we can assume $g(t) = 0$ for all $t < 0$. Moreover, put $h(t) = |t|^{2^*-2}t$ for all $t \in \mathbb{R}$. Clearly, one has $f(t) \le (1 + a) + (1 + a)|t|^{2^*-1}$ for all $t \in \mathbb{R}$. As usual, put $X = H_0^1(\Omega)$ endowed with the norm $\|u\| = \left(\int_\Omega |\nabla u(x)|^2dx\right)^{\frac{1}{2}}$ and $\Phi(u) = \frac{\|u\|^2}{2}$, $\Psi(u) = \int_\Omega F(u(x))dx$ for all $u \in X$, where $F(\xi) = \int_0^\xi f(t)dt$ for every $\xi \in \mathbb{R}$, that is, $F(\xi) = \int_0^\xi h(t)dt+\int_0^\xi g(t)dt = H(\xi) + G(\xi) = \frac{1}{2^*}|\xi|^{2^*} + G(\xi)$ for all $\xi \in \mathbb{R}$. We observe that one has $F(\xi) \ge 0$ for all $\xi \in \mathbb{R}$.

Now, fix $r > 0$ and put

$$\lambda_r^* = \frac{r}{\left(\sqrt{2r}c_1(1+a) + \frac{(2r)^{2^*/2}}{2^*}c_{2^*}^{2^*}(1+a)\right)}, \qquad \tilde{\lambda}_r = \frac{1}{c_{2^*}^{2^*}(2rN)^{\frac{2}{N-2}}},$$

$$\overline{\lambda}_r = \min\left\{\lambda_r^*, \tilde{\lambda}_r\right\},$$

where c_1, c_{2^*} are given by (2.4) and (2.3), while a is given by (h).

Our main result is the following.

Lemma 3.1. *Let Φ and Ψ be the functional defined as above and fix $r > 0$. Then, for each $\lambda \in]0, \overline{\lambda}_r[$ the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies the $(PS)^{[r]}$-condition.*

Proof. Fix λ as in the conclusion and let $\{u_n\} \subseteq X$ be a sequence such that

(α) $\{I_\lambda(u_n)\}$ is bounded,
(β) $\lim_{n\to+\infty} \|I'_\lambda(u_n)\|_{X^*} = 0$,
(γ) $\Phi(u_n) < r \ \forall n \in \mathbb{N}$.

In particular, from $\Phi(u_n) < r \ \forall n \in \mathbb{N}$ we obtain that $\{u_n\}$ is bounded in X. So, going to a subsequence if necessary, we can assume $u_n \rightharpoonup u_0$ in X, $u_n \to u_0$ in $L^q(\Omega)$ if $q < 2^*$, $u_n \to u_0$ a.e. on Ω and, taking (α) into account, $\lim_{n\to\infty} I_\lambda(u_n) = c$. Moreover, $\{u_n\}$ is bounded in $L^{2^*}(\Omega)$.

First step. We prove that u_0 is a weak solution of problem (D_λ^f).

Since $\{u_n\}$ is bounded in $L^{2^*}(\Omega)$, it follows that $\{h(u_n)\}$ is bounded in $L^{\frac{2^*}{2^*-1}}(\Omega)$. Indeed, one has $\int_\Omega |h(u_n)|^{\frac{2^*}{2^*-1}}dx = \int_\Omega |u_n|^{2^*}dx$. Therefore, it follows that $h(u_n) \rightharpoonup h(u_0)$ in $L^{\frac{2^*}{2^*-1}}(\Omega)$. In fact, since h is continuous and $u_n \to u_0$ a.e. $x \in \Omega$, we obtain $h(u_n) \to h(u_0)$ a.e. $x \in \Omega$, that, together with boundedness of $\{h(u_n)\}$ in $L^{\frac{2^*}{2^*-1}}(\Omega)$, ensures the weak convergence of $h(u_n)$ to $h(u_0)$ in $L^{\frac{2^*}{2^*-1}}(\Omega)$ (see [2, Remark (iii)]).

Moreover, since $u_n \to u_0$ in $L^q(\Omega)$, taking into account [5, Theorem A.2], one has that $g(u_n) \to g(u_0)$ in $L^{\frac{q}{q-1}}(\Omega)$. So, in particular, $g(u_n) \rightharpoonup g(u_0)$ in $L^{\frac{q}{q-1}}(\Omega)$.

Due to what seen before, that is, $u_n \rightharpoonup u_0$ in X, $h(u_n) \rightharpoonup h(u_0)$ in $L^{\frac{2^*}{2^*-1}}(\Omega)$ and $g(u_n) \rightharpoonup g(u_0)$ in $L^{\frac{q}{q-1}}(\Omega)$, one has

$$\lim_{n\to+\infty}\left(\int_\Omega \nabla u_n(x)\nabla v(x)dx - \lambda\int_\Omega h(u_n(x))v(x)dx - \lambda\int_\Omega g(u_n(x))v(x)dx\right)$$
$$= \int_\Omega \nabla u_0(x)\nabla v(x)dx - \lambda\int_\Omega h(u_0(x))v(x)dx - \lambda\int_\Omega g(u_0(x))v(x)dx$$

for all $v \in H_0^1(\Omega)$. Therefore, owing to (β) we obtain that $0 = \int_\Omega \nabla u_0(x)\nabla v(x)dx - \lambda\int_\Omega h(u_0(x))v(x)dx - \lambda\int_\Omega g(u_0(x))v(x)dx$ for all $v \in H_0^1(\Omega)$, that is, u_0 is a weak solution of (D_λ^f).

Second step. We prove that

$$\text{(A)} \qquad I_\lambda(u_0) > -r.$$

In fact, $\Psi(u) = \int_\Omega F(u(x))dx \leq (1+a)\|u\|_{L^1(\Omega)} + \frac{1+a}{2^*}\|u\|^{2^*}_{L^{2^*}(\Omega)} \leq (1+a)c_1\|u\| + \frac{1+a}{2^*}c_{2^*}^{2^*}\|u\|^{2^*}$. Hence,

$$\Psi(u) \leq (1+a)c_1\|u\| + \frac{1+a}{2^*}c_{2^*}^{2^*}\|u\|^{2^*}, \quad \forall u \in X.$$

Therefore, for all $u \in X$ such that $\|u\| \leq (2r)^{1/2}$ one has $I_\lambda(u) = \Phi(u) - \lambda\Psi(u) \geq \frac{\|u\|^2}{2} - \lambda\Big((1+a)c_1\|u\| + \frac{1+a}{2^*}c_{2^*}^{2^*}\|u\|^{2^*}\Big) \geq -\lambda\Big((1+a)c_1(2r)^{1/2} + \frac{1+a}{2^*}c_{2^*}^{2^*}(2r)^{2^*/2}\Big) = -\lambda\frac{r}{\lambda_r^*} > -r$. So, taking into account (γ) and that Φ is sequentially weakly lower semicontinuous, we have $\|u_0\| \leq \liminf_{n\to\infty}\|u_n\| \leq \sqrt{2r}$ and, hence, $I_\lambda(u_0) > -r$.

Third step. Put $v_n = u_n - u_0$. We prove that one has

$$c = \Phi(u_0) - \lambda\Psi(u_0) + \lim_{n\to\infty}\left(\frac{1}{2}\|v_n\|^2 - \frac{\lambda}{2^*}\int_\Omega |v_n|^{2^*}dx\right). \tag{B}$$

In fact, one has $\|u_n\|^2 = \|v_n + u_0\|^2 = \|v_n\|^2 + \|u_0\|^2 + 2 < v_n, u_0 >$. So, it follows

$$\|u_n\|^2 = \|v_n\|^2 + \|u_0\|^2 + o(1).$$

Moreover, the Brezis-Lieb Lemma (see [2, Theorem 1]) leads to

$$\int_\Omega |u_n|^{2^*}dx = \int_\Omega |v_n|^{2^*}dx + \int_\Omega |u_0|^{2^*}dx + o(1).$$

Finally, since $u \to \int_\Omega G(u)dx$ is locally Lipschitz in $L^q(\Omega)$ (see, for instance, [4, Theorem 7.2.1]) and $u_n \to u_0$ in $L^q(\Omega)$, one has

$$\int_\Omega G(u_n)dx = \int_\Omega G(u_0)dx + o(1).$$

Hence, by starting from $c = \lim_{n\to\infty}(\Phi(u_n) - \lambda\Psi(u_n))$, one has $c = \Phi(u_n) - \lambda\Psi(u_n) + o(1) = \frac{1}{2}\|u_n\|^2 - \frac{\lambda}{2^*}\int_\Omega |u_n|^{2^*}dx - \lambda\int_\Omega G(u_n)dx + o(1) = \frac{1}{2}\|v_n\|^2 + \frac{1}{2}\|u_0\|^2 - \frac{\lambda}{2^*}\int_\Omega |v_n|^{2^*}dx - \frac{\lambda}{2^*}\int_\Omega |u_0|^{2^*}dx - \lambda\int_\Omega G(u_0)dx + o(1) = \Phi(u_0) - \lambda\Psi(u_0) + \frac{1}{2}\|v_n\|^2 - \frac{\lambda}{2^*}\int_\Omega |v_n|^{2^*}dx + o(1)$. Hence, (B) is proved.

Fourth step. We prove the following

$$\lim_{n\to\infty}\left(\|v_n\|^2 - \lambda\int_\Omega |v_n|^{2^*}dx\right) = 0. \tag{C}$$

In fact, from (β) we have $\lim_{n\to\infty} I'(u_n)(u_n) = 0$. So, $\int_\Omega \nabla u_n \nabla u_n dx - \lambda\int_\Omega |u_n|^{2^*-1}u_n dx - \lambda\int_\Omega g(u_n)u_n dx = o(1)$, for which $\|u_n\|^2 - \lambda\int_\Omega |u_n|^{2^*}dx - \lambda\int_\Omega g(u_n)u_n dx = o(1)$. Therefore, as seen in the proof of (B) and taking into account that $\int_\Omega g(u_n)u_n dx = \int_\Omega g(u_0)u_0 dx + o(1)$ owing to the fact that $g(u_n) \to g(u_0)$ in $L^{\frac{q}{q-1}}(\Omega)$ (see the first step) and $u_n \to u_0$ in $L^q(\Omega)$, one has $\|v_n\|^2 + \|u_0\|^2 - \lambda\int_\Omega |v_n|^{2^*}dx - \lambda\int_\Omega |u_0|^{2^*}dx - \lambda\int_\Omega g(u_0)u_0 dx = o(1)$, that is,

$$\|v_n\|^2 - \lambda\int_\Omega |v_n|^{2^*}dx = -\|u_0\|^2 + \lambda\int_\Omega |u_0|^{2^*}dx + \lambda\int_\Omega g(u_0)u_0 dx + o(1).$$

Since u_0 is a weak solution of (D_λ^f), one has $\|u_0\|^2 - \lambda \int_\Omega |u_0|^{2^*} dx - \lambda \int_\Omega g(u_0) u_0 dx = 0$. Therefore,

$$\|v_n\|^2 - \lambda \int_\Omega |v_n|^{2^*} dx = o(1),$$

that is, (C) is proved.

Conclusion. Finally, we observe that $\|v_n\|^2$ is bounded in $\mathbb{R}$ being u_n bounded in X. So, there is a subsequence, called again $\|v_n\|^2$, which converges to $b \in \mathbb{R}$. Hence,

$$\lim_{n\to\infty} \|v_n\|^2 = b.$$

If $b = 0$ we have proved the lemma. In fact, we have that $\lim_{n\to\infty} \|u_n - u_0\| = 0$, that is, u_n strongly converges to u_0 in X. So, arguing by contradiction, we assume that $b \neq 0$. From (C) we obtain $\lim_{n\to\infty} \lambda \int_\Omega |v_n|^{2^*} dx = b$. Now, taking into account that $\|v_n\|_{L^{2^*}(\Omega)} \leq c_{2^*} \|v_n\|$, for which $\int_\Omega |v_n|^{2^*} dx \leq \int_\Omega |v_n|^{2^*} dx \leq c_{2^*}^{2^*} \|v_n\|^{2^*}$, and passing to the limit, one has $\frac{b}{\lambda} \leq c_{2^*}^{2^*} b^{2^*/2}$ and then, since $b \neq 0$, one has

$$b \geq \left(\frac{1}{\lambda}\right)^{\frac{N-2}{2}} \left(\frac{1}{c_{2^*}}\right)^N.$$

Now, taking (A) into account, from (B) we have $c = \Phi(u_0) - \lambda\Psi(u_0) + \frac{1}{2}b - \frac{1}{2^*}b > -r + \left(\frac{1}{2} - \frac{1}{2^*}\right) b = -r + \frac{1}{N}b$, that is

$$c > -r + \frac{1}{N}b.$$

On the other hand, since $F(\xi) \geq 0$ for all $\xi \in \mathbb{R}$, one has $\Phi(u_n) - \lambda\Psi(u_n) < r$ for all $n \in \mathbb{N}$. Hence, we have

$$c \leq r.$$

So, $-r + \frac{1}{N}b < c \leq r$. It follows $\frac{1}{N}b < 2r$, that is,

$$b < 2rN.$$

Therefore, one has $\left(\frac{1}{\lambda}\right)^{\frac{N-2}{2}} \left(\frac{1}{c_{2^*}}\right)^N \leq b < 2rN$. So, it follows $\frac{1}{\lambda} < (2rNc_{2^*}^N)^{\frac{2}{N-2}}$. Hence, one has

$$\lambda > \frac{1}{(2rN)^{\frac{2}{N-2}}} \frac{1}{c_{2^*}^{2^*}} = \tilde{\lambda}_r,$$

and this is a contradiction. □

Next, by using the previous lemma, we prove Theorem 1.1.

Proof of Theorem 1.1. Fix $r > 0$ and $\lambda < \overline{\lambda}_r$. Our aim is to apply Theorem 2.1. To this end, first we observe that, owing to Lemma 3.1, the functional $\Phi - \lambda\Psi$ satisfies the $(PS)^{[r]}$–condition. Now, arguing as in the proof of [1, Theorem 8.1] and by choosing $r_1 = 0$ and $r_2 = r$, it is possible to prove that

$$\beta(r_1, r_2) < \frac{1}{\lambda} < \rho_2(r_1, r_2).$$

Hence, Theorem 2.1 ensures the conclusion. □

Finally, the following example of application is pointed out.

Example 3.2. Owing to Theorem 1.1, for each $\lambda < \lambda^1$, where

$$\lambda^1 = \min\left\{\frac{1}{2^{3/2}c_1 + \frac{2^{(2^*+2)/2}}{2^*}c_{2^*}^{2^*}}; \frac{1}{c_{2^*}^{2^*}(2N)^{\frac{2}{N-2}}}\right\},$$

the problem

$$(P) \qquad \begin{cases} -\Delta u = \lambda\left(|u|^{2^*-2}u + \sqrt{\log(1+u)}\right) & \text{in} \quad \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

admits at least one positive weak solution. In fact, it is enough to pick $g(t) = \sqrt{\log(1+t)}$ if $t \geq 0$, $g(t) = 0$ if $t \leq 0$ and $r = 1$.

Remark 3.3. We recall that in the classical and seminal paper of Brezis and Nirenberg [3], contrary to Theorem 1.1, the nonlinearity f is linear or super-linear at 0, that is, $\lim_{t\to 0^+} \frac{f(t)}{t} < +\infty$. So, in particular, Theorem 2.1 of [3] cannot be applied to the problem (P) in the Example 3.2.

Remark 3.4. We recall that in [1] it was proved a similar result when the nonlinearity f may be also only subcritical, beyond the same critical case, which is studied at the same time (see [1, Theorem 8.1]. Since, when f is sub-critical, the $(PS)^{[r]}-$condition immediately follows from [1, Proposition 2.1], the proof of the $(PS)^{[r]}$-condition can be limited only to the critical case. Moreover, the proof of $(PS)^{[r]}-$condition in the critical case deserves to be examined separately and it needs more details for greater clarity, as done in Lemma 3.1.

REFERENCES

[1] G. Bonanno, A critical point theorem via the Ekeland variational principle, Nonlinear Anal. **75** (2012), 2992–3007.

[2] H. Brezis and E. Lieb, A relation between pointwise convergence of functions and convergence of functionals, Comm. Pure Appl. Math. **36** (1983), 437–477.

[3] H. Brezis and L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, Proc. Amer. Math. Soc. **88** (1983), 486–490.

[4] J. Chabrowski, *Variational methods for potential operator equations*, de Gruyter, Berlin, 1997.

[5] M. Willem, *Minimax theorems*, Birkhäuser, Berlin, 1996.