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CONVERGENCE OF INEXACT ITERATES OF NONEXPANSIVE MAPPINGS IN METRIC SPACES

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ABSTRACT. We study the influence of computational errors on the convergence of iterates of a nonexpansive mapping in an arbitrary metric space.

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1. INTRODUCTION

Convergence analysis of iterations of nonlinear operators on Banach and metric spaces is a central topic in Nonlinear Functional Analysis. It began with the classical Banach theorem [1] on the existence of a unique fixed point for a strict contraction. Banach's celebrated result also yields convergence of iterates to the unique fixed point. There are several generalizations of the Banach theorem which show that the convergence of iterates holds for larger classes of nonexpansive (that is, 1-Lipschitz) mappings. Note that this situation is in some sense typical [5, 6, 9] because it turns out that most (in the sense of Baire's categories) nonexpansive mappings possess a unique fixed point which attracts all their powers.

In view of the above discussion, it is natural to ask if convergence either to limit points or to attractor sets of the iterates of nonexpansive mappings will be preserved in the presence of computational errors. In [10] we present and discuss several affirmative answers to this question. These answers were obtained in [2–4, 7, 8]. In all these papers the convergence of exact and inexact orbits was studied for all iterates with arbitrary initial points. In contrast, in the present paper we only assume that one sequence of iterates with a certain initial point x_0 converges to a fixed point x_* and that the mapping under consideration is nonexpansive in a neighborhood of x_*. Under these assumptions, we show that, given $\epsilon > 0$, all inexact orbits with this initial point x_0 and with sufficiently small summable computational errors will still remain in an ϵ-ball about x_* from a certain iterate on.

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2. A STABLE CONVERGENCE THEOREM

Let (X, ρ) be a metric space. For any $x \in X$ and any $r > 0$, set

$$B(x,r) = \{y \in X : \ \rho(x,y) \le r\}.$$

Let $T : X \to X$ and $x_* \in X$.

Assume that for a certain point $x_0 \in X$, the mapping T is continuous at the point x_0 and at the iterates $T^j(x_0)$ for all natural numbers j, and that

$$\lim_{j\to\infty} \rho(T^j(x_0), x_*) = 0. \tag{2.1}$$

Assume further that there is a number $r_* > 0$ such that

$$\rho(T(x), T(y)) \le \rho(x,y) \text{ for all } x, y \in B(x_*, r_*). \tag{2.2}$$

Put $T^0 x = x$, $x \in X$.

Theorem 2.1. *Let a sequence* $\{\epsilon_i\}_{i=0}^\infty \subset (0,\infty)$, *with*

$$\sum_{i=0}^\infty \epsilon_i < \infty, \tag{2.3}$$

and a number $\epsilon > 0$ *be given. Then there exist a real number* $\delta > 0$ *and a natural number* n_1 *such that for each sequence* $\{x_i\}_{i=0}^\infty \subset X$ *satisfying*

$$\rho(x_{i+1}, T(x_i)) \le \min\{\epsilon_i, \delta\} \tag{2.4}$$

for all integers $i \ge 0$, *the following inequality holds:*

$$\rho(x_i, x_*) \le \epsilon \textit{ for all integers } i \ge n_1 + 1.$$

3. PROOF OF THE THEOREM

We may assume without any loss of generality that

$$\epsilon < r_*/4. \tag{3.1}$$

By (2.3), there is a natural number $n_0 > 2$ such that

$$\sum_{i=n_0}^\infty \epsilon_i < \epsilon/4. \tag{3.2}$$

By (2.1), there exists an integer $n_1 > n_0 + 2$ such that

$$\rho(T^{n_1}(x_0), x_*) < \epsilon/8. \tag{3.3}$$

Put

$$\delta_{n_1} = \epsilon/8. \tag{3.4}$$

By the continuity of T at $T^{n_1-1}(x_0)$, there is a number

$$\delta_{n_1-1} \in (0, 2^{-1}\epsilon/8) \tag{3.5}$$

such that

$$T(B(T^{n_1-1}(x_0), 2\delta_{n_1-1})) \subset B(T^{n_1}(x_0), \epsilon/8). \tag{3.6}$$

Using induction and the continuity of T at the points $T^j(x_0)$, $j = 0, 1, \ldots$, we now define a sequence $\{\delta_i\}_{i=0}^{n_1} \subset (0, \epsilon/8]$ such that for each $i = 0, \ldots, n_1 - 1$,

$$\delta_i < \delta_{i+1}/2 \tag{3.7}$$

and

$$T(B(T^i(x_0), 2\delta_i)) \subset B(T^{i+1}(x_0), \delta_{i+1}). \tag{3.8}$$

Put

$$\delta := \delta_0. \tag{3.9}$$

Assume that the sequence $\{x_i\}_{i=0}^{\infty} \subset X$ satisfies (2.4) for all integers $i \geq 0$. By the choice of $\{\delta_i\}_{i=0}^{n_1}$ (see (3.7) and (3.8)), (3.9) and (2.4), we have

$$x_1 \in B(T(x_0), \delta) \subset B(T(x_0), 2\delta_1). \tag{3.10}$$

Next we show by induction that for $i = 1, \ldots, n_1$,

$$x_i \in B(T^i(x_0), 2\delta_i). \tag{3.11}$$

In view of (3.10), (3.11) indeed holds when $i = 1$. Assume that $j \in \{1, \ldots, n_1 - 1\}$ and that (3.11) holds with $i = j$. Then

$$x_j \in B(T^j(x_0), 2\delta_j). \tag{3.12}$$

By (3.8) and (3.12),

$$T(x_j) \in B(T^{j+1}(x_0), \delta_{j+1}). \tag{3.13}$$

By (2.4), (3.13), (3.7), (3.8) and (3.9),

$$\rho(x_{j+1}, T^{j+1}(x_0)) \leq \rho(x_{j+1}, T(x_j)) + \rho(T(x_j), T^{j+1}(x_0)) \leq \delta + \delta_{j+1} \leq 2\delta_{j+1}$$

and so

$$x_{j+1} \in B(T^{j+1}(x_0), 2\delta_{j+1}).$$

Thus (3.11) indeed holds for all integers $i = 1, \ldots, n_1$ and in view of (3.4),

$$x_{n_1} \in B(T^{n_1}(x_0), 2\delta_{n_1}) = B(T^{n_1}(x_0), \epsilon/4). \tag{3.14}$$

By (3.3) and (3.14),

$$\rho(x_{n_1}, x_*) \leq \rho(x_*, T^{n_1}(x_0)) + \rho(T^{n_1}(x_0), x_{n_1}) < \epsilon/8 + \epsilon/4 = (3/8)\epsilon. \tag{3.15}$$

By (2.4), (3.15), (2.2) and (3.1),

$$\rho(x_{n_1+1}, x_*) \leq \rho(x_{n_1+1}, T(x_{n_1})) + \rho(T(x_{n_1}), T(x_*)) \leq \epsilon_{n_1} + (3/8)\epsilon. \tag{3.16}$$

Now we show that for all integers $i \geq n_1 + 1$,

$$\rho(x_i, x_*) \leq (3/8)\epsilon + \sum_{p=n_1}^{i-1} \epsilon_p. \tag{3.17}$$

By (3.16), for $i = n_1 + 1$ (3.17) is true.

Assume that $j \geq n_1 + 1$ in an integer and that (3.17) holds with $i = j$. Then

$$\rho(x_j, x_*) \leq (3/8)\epsilon + \sum_{p=n_1}^{j-1} \epsilon_p. \tag{3.18}$$

By (3.18), (2.4), the inequality $n_1 > n_0 + 2$, (3.2), (3.1) and (2.2), we have

$$\rho(x_{j+1}, x_*) \leq \rho(x_{j+1}, T(x_j)) + \rho(T(x_j), T(x_*))$$
$$\leq \epsilon_j + \rho(x_j, x_*) \leq (3/8)\epsilon + \sum_{p=n_1}^{j} \epsilon_p.$$

Thus (3.17) holds with $i = j + 1$. Therefore (3.17) indeed holds for all integers $i \geq n_1 + 1$. In view of (3.2) and (3.17), we also have, for all integers $i \geq n_1 + 1$,

$$\rho(x_i, x_*) \leq (3/8)\epsilon + \sum_{p=n_0}^{\infty} \epsilon_p < (3/8)\epsilon + \epsilon/4 < \epsilon.$$

This completes the proof of our theorem.

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