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TERMINAL VALUE PROBLEMS FOR CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, we obtain the unique solution of the Terminal Value Problem for Caputo Fractional Differential Equation by combining the technique of generalized quasilinearization and the bisection method under suitable conditions.

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1. INTRODUCTION

In the last few decades the study of fractional differential equations has become a major area of research. There is an extensive literature on the subject. Some of the major contributions are Diethelm [1], Hermann [3], Kilbas et al [4], Lakshmikantham et al [5], Miller et al [7], Oldham et al [8], and Podlubny [9].

The study of a terminal-value problem for ordinary differential equations using the method of lower and upper solutions can be found in [5]. In [10] the solution of an impulsive terminal-value problem for an ordinary differential equation was obtained constructively using the technique of Generalized Quasilinearization (GQL) and the secant method. In [1], Diethelm gave an example of a terminal-value problem for fractional differential equations for which he provided a numerical solution.

The study of terminal-value problems is intriguing, as the information is given at the end point of the interval and one has to work backwards to find the initial value at which the solution must start in order to reach the prescribed value at the end point of the interval. This problem becomes more interesting in the case of a fractional differential equation where it closely resembles a boundary-value problem, as suggested by Diethelm [1], in the sense that the initial value is inherently involved in

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the definition of the differential operator and the terminal value provides the condition at the right end point of the interval.

In this paper, we consider a terminal-value problem for fractional differential equations. We assume the existence of lower and upper solutions for the given problem such that the terminal value is in the sector defined by these solutions. Next, we construct a sequence of initial-value problems and find their solutions using the GQL method. Then, we successively use a combination of the GQL technique and the bisection method to obtain the unique solution of the given terminal-value problem of the Caputo fractional differential equation.

2. BASIC THEORY OF FRACTIONAL DIFFERENTIAL EQUATIONS

In this section we present some definitions and basic results that are needed in our subsequent work.

Let $0 < q < 1$ and $p = 1 - q$.

Definition 2.1. $m \in C_p[[t_0, T], \mathbb{R}]$ means that $m \in C[(t_0, T], \mathbb{R}]$ and $(t - t_0)^p m(t) \in C[[t_0, T], \mathbb{R}]$.

Definition 2.2. For $m \in C_p[[t_0, T], \mathbb{R}]$, the Riemann-Liouville fractional derivative of $m(t)$ is defined by

$$D^q m(t) = \frac{1}{\Gamma(p)} \frac{d}{dt} \int_{t_0}^{t} (t - s)^{p-1} m(s) ds.$$

Next, we state some known results which are taken from [11], where the hypothesis of Hölder continuity in Lemma 2.3.1 in [6] has been weakened to continuity. This lemma is a basic tool used to prove comparison theorems in the fractional differential equations context.

Lemma 2.3. *Let $m \in C_p[[t_0, T], \mathbb{R}]$. Suppose that for any $t_1 \in (t_0, T]$, we have $m(t_1) = 0$ and $m(t) < 0$ for $t_0 \leq t < t_1$, then it follows that $D^q m(t_1) \geq 0$.*

We now state the following theorems which are proved using this lemma.

Theorem 2.4. *Let $v, w \in C_p[[t_0, T], \mathbb{R}]$, $f \in C[[t_0, T] \times \mathbb{R}, \mathbb{R}]$*

$$D^q v(t) \leq f(t, v(t)) \tag{2.1}$$

and

$$D^q w(t) \geq f(t, w(t)), \tag{2.2}$$

$t_0 \leq t \leq T$, with one of the above inequalities being strict. Then $v^0 < w^0$, where $v^0 = v(t)(t-t_0)^{1-q}|_{t=t_0}$ and $w^0 = w(t)(t-t_0)^{1-q}|_{t=t_0}$, implies $v(t) < w(t)$, $t_0 \leq t \leq T$.

The next theorem deals with a result involving nonstrict inequalities.

Theorem 2.5. *Let* $v, w \in C_p[[t_0, T], \mathbb{R}]$, $f \in C[[t_0, T] \times \mathbb{R}, \mathbb{R}]$ *and* $D^q v(t) \leq f(t, v(t))$ *and* $D^q w(t) \geq f(t, w(t))$, $t_0 \leq t \leq T$. *Assume that* f *satisfies the Lipschitz condition*

$$f(t,x) - f(t,y) \leq L(x-y), \quad x \geq y, \quad L > 0.$$

Then, $v^0 \leq w^0$ *implies* $v(t) \leq w(t)$, $t \in [t_0, T]$.

Next, we define the Caputo derivative and state some related results.

Definition 2.6. $u \in C^q[[t_0, T], \mathbb{R}]$ if and only if the Caputo derivative denoted by ${}^cD^q u$ exists, that is, u satisfies

$${}^cD^q u(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q} u'(s) ds.$$

We observe that the Caputo and the Riemann-Liouville derivatives are related by the relation

$$(2.3) \qquad {}^cD^q x(t) = D^q[x(t) - x_0].$$

We now proceed to describe the Caputo and Riemann-Liouville initial-value problems and the corresponding Volterra fractional integral equations.

The initial-value problem of the Caputo fractional differential equation is given by

$$(2.4) \qquad {}^cD^q x = f(t,x), \quad x(t_0) = x_0,$$

and the corresponding Volterra fractional integral equation is given by

$$(2.5) \qquad x(t) = x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, x(s)) ds.$$

The initial-value problem of the Riemann-Liouville fractional differential equation is given by

$$(2.6) \qquad D^q x = f(t,x), \;\; x^0 = x(t)(t-t_o)^{1-q}|_{t=t_0}$$

and the corresponding Volterra fractional integral equation is given by

$$(2.7) \qquad x(t) = x^0(t) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, x(s)) ds$$

where $x^0(t) = \dfrac{x^0(t-t_0)^{q-1}}{\Gamma(q)}$.

At this point, note that any two functions $v, w \in C_p[[t_0, T], \mathbb{R}]$ satisfying the relations (2.1) and (2.2) are said to be lower and upper solutions, respectively, of the Riemann-Liouville fractional differential equation (2.6).

We now state an existence result in the Riemann-Liouville fractional differential equation set up.

Theorem 2.7. *Let $v, w \in C_p[[t_0, T], \mathbb{R}]$ be lower and upper solutions of initial-value problem* (2.6)*, such that $v(t) \leq w(t)$ on $[t_0, T]$. Then, there exists a solution $x(t)$ of the initial-value problem* (2.6) *satisfying $v(t) \leq x(t) \leq w(t)$ on $[t_0, T]$ where $f \in C[\Omega, \mathbb{R}]$ and*

$$\Omega = \{(t, x) : v(t) \leq x \leq w(t), \quad t \in [t_0, T]\}.$$

The proof is similar to the proof of Theorem 3.2.1 in [6], except that Lemma 2.3 is used in place of Lemma 2.3.1 in [6].

Next, we introduce the initial-value problem for Caputo fractional differential equation given by

$$ {}^cD^q x = f(t, x) + g(t, x), \tag{2.8}$$

$$ x(t_0) = x_0. \tag{2.9}$$

Then, the corresponding Volterra fractional integral equation is

$$x(t) = x_0 + \int_{t_0}^{t} (t - s)^{q-1}[f(s, x(s)) + g(s, x(s))]ds.$$

We now state a generalized quasilinearization result from [6].

Theorem 2.8. *Assume that*

(i) $f, g \in C[J \times \mathbb{R}, \mathbb{R}]$, $\alpha_0, \beta_0 \in C^q[J, \mathbb{R}]$,

$$\begin{aligned} {}^cD^q\alpha_0 &\leq f(t, \alpha_0) + g(t, \alpha_0),\ \alpha_0(t_0) \leq x_0, \\ {}^cD^q\beta_0 &\geq f(t, \beta_0) + g(t, \beta_0),\ \beta_0(t_0) \geq x_0, \end{aligned}$$

$\alpha_0(t) \leq \beta_0(t)$ on J, and $\alpha_0(t_0) \leq x_0 \leq \beta_0(t_0)$, where $J = [t_0, T]$.

(ii) *Suppose $f_x(t, x)$ exists, $f_x(t, x)$ is increasing in x for each t,*

$$f(t, x) \geq f(t, y) + f_x(t, y)(x - y), \quad x \geq y,$$

and

$$|f_x(t, x) - f_x(t, y)| \leq L_1|x - y|, \quad L_1 > 0.$$

Further suppose that $g_x(t, x)$ exists, $g_x(t, x)$ is decreasing in x for each t,

$$g(t, x) \geq g(t, y) + g_x(t, x)(x - y), \quad x \geq y,$$

and

$$|g_x(t, x) - g_x(t, y)| \leq L_2|x - y|, \quad L_2 > 0.$$

Then, there exist monotone sequences $\{\alpha_n\}$, $\{\beta_n\}$, such that $\alpha_n \to \rho$, $\beta_n \to r$ uniformly and monotonically and $\rho = r = x$ is the unique solution of IVP (2.8) *and* (2.9) *on J, and the convergence is quadratic.*

Finally, we observe that the relation between the Caputo and Riemann-Liouville fractional differential equations established in [1] enables us to apply results developed for the Riemann-Liouville fractional differential equations to the Caputo fractional differential equations.

3. MAIN RESULTS

In this section, we present a constructive method to obtain a solution of the terminal-value problem of Caputo fractional differential equation using a sequence of solutions of initial-value problems of the Caputo fractional differential equations.

Consider the terminal-value problem of the Caputo fractional differential equation given by

$$ {}^cD^q x = F(t,x), \tag{3.1} $$

$$ x(T) = A, \tag{3.2} $$

where $F \in C[[t_0,T] \times \mathbb{R}, \mathbb{R}]$.

Let $F(t,x) = f(t,x) + g(t,x)$, where $f, g \in C[[t_0,T] \times \mathbb{R}, \mathbb{R}]$, $f(t,x)$ satisfies a convexity condition and $g(t,x)$ satisfies a concavity condition as given in Theorem 2.8. Then problems (3.1) and (3.2) become the terminal-value problem of the Caputo fractional differential equation given by

$$ {}^cD^q x = f(t,x) + g(t,x), \tag{3.3} $$

$$ x(T) = A. \tag{3.4} $$

Our aim is to consider the following initial-value problem of the Caputo fractional differential equation

$$ {}^cD^q x = f(t,x) + g(t,x), \tag{3.5} $$

$$ x(t_0) = x_0, \tag{3.6} $$

and find the initial value x_0 and the solution $x(t,t_0,x_0)$ such that $x(T,t_0,x_0) = A$. For this purpose we assume that lower and upper solutions, $\alpha_0(t)$ and $\beta_0(t)$, exist with $\alpha_0(t_0) = \alpha_0$ and $\beta_0(t_0) = \beta_0$, and start with $x_0 = (\alpha_0 + \beta_0)/2$. Then, using the GQL method we obtain the unique solution of (3.5) and (3.6). Next, we successively use a combination of the bisection method and the GQL method to get a better estimate of the initial value in order to obtain the unique solution $x(t,t_o,x_0)$ of (3.5) and (3.6), such that $x(T,t_0,x_0) = A$.

We apply this approach to prove the following theorem.

Theorem 3.1. *Assume that*

i) $\alpha, \beta \in C^q[J, \mathbb{R}]$ *such that*

$$^cD^q\alpha \leq f(t,\alpha) + g(t,\alpha),$$
$$^cD^q\beta \geq f(t,\beta) + g(t,\beta),$$

$\alpha(t) \leq \beta(t)$, $t \in J$ *and* $\alpha(T, t_0, \alpha(t_0)) \leq A \leq \beta(T, t_0, \beta(t_0))$, *where* $J = [t_0, T]$;

ii) $f_x(t,x)$ *exists, is increasing in* x *for each* t,
$f(t,x) \geq f(t,y) + f_x(t,y)(x-y)$, $x \geq y$, *and*
$|f_x(t,x) - f_x(t,y)| \leq L_1|x-y|, L_1 > 0$;

iii) $g_x(t,x)$ *exists, is decreasing in* x *for each* t,
$g(t,x) \geq g(t,y) + g_x(t,x)(x-y)$, $x \geq y$, *and*
$|g_x(t,x) - g_x(t,y)| \leq L_2|x-y|, L_2 > 0$.

Then, there exists a unique real number x_0 *and a unique solution* $x(t, t_0, x_0)$ *of the initial-value problem* (3.5) *and* (3.6) *such that* $x(T, t_0, x_0) = A$, *that is, there exists a unique solution of the terminal-value problem* (3.3) *and* (3.4), *and hence of* (3.1) *and* (3.2).

Proof. Since $\alpha(t) \leq \beta(t)$ on J, set

$$x_1 = \frac{\alpha(t_0) + \beta(t_0)}{2}.$$

Then we have $\alpha(t_0) \leq x_1 \leq \beta(t_0)$. Now, by assumption (i), we can apply Theorem 2.7, and claim that there exists a solution for the initial-value problem (3.5) and (3.6) with $x(t_0) = \frac{\alpha(t_0)+\beta(t_0)}{2} = x_1$. Next, observing that conditions (ii) and (iii) satisfy the hypothesis of the GQL theorem, namely Theorem 2.8, we obtain the unique solution $x_1(t, t_0, x_1)$ of the initial-value problem (3.5) and (3.6) on J.

Then, we consider the following cases.

(a) if $x_1(T, t_0, x_1) = A$, we have obtained the desired solution for the terminal-value problem (3.3) and (3.4), and hence the solution for the terminal-value problem of the (3.1) and (3.2);

(b) if $x_1(T, t_0, x_1) > A$, then set $\alpha_1(t, t_0, \alpha_1(t_0)) = \alpha(t, t_0, \alpha(t_0))$ and $\beta_1(t, t_0, \beta_1(t_0)) = x_1(t, t_0, x_1)$;

(c) if $x_1(T, t_0, x_1) < A$, then set $\alpha_1(t, t_0, \alpha_1(t_0)) = x_1(t, t_0, x_1)$ and $\beta_1(t, t_0, \beta_1(t_0)) = \beta(t, t_0, \beta(t_0))$.

Then, it is clear that $\alpha_1(t) \leq \beta_1(t)$ on J. Further, $\alpha_1(t)$ and $\beta_1(t)$ satisfy the inequalities

$$^cD^q\alpha_1(t) \leq f(t,\alpha_1) + g(t,\alpha_1), \tag{3.7}$$

$$\text{and } ^cD^q\beta_1(t) \geq f(t,\beta_1) + g(t,\beta_1). \tag{3.8}$$

Now set

$$x_2 = \frac{\alpha_1(t_0) + \beta_1(t_0)}{2}. \tag{3.9}$$

Clearly, $\alpha_1(t_0) \leq x_2 \leq \beta_1(t_0)$. Then relations (3.7) and (3.8), along with Theorem 2.7 and Theorem 2.8, yield the existence of the unique solution $x_2(t, t_0, x_2)$ of the initial-value problem (3.5) and (3.6) on J. Next, consider the value $x_2(T, t_0, x_2)$ and proceed as before.

Continuing this process, we obtain a sequence of real numbers $\{x_n\}$ and a sequence of solutions $\{x_n(t, t_0, x_n)\}$ of the corresponding initial-value problems of Caputo fractional differential equations given by

$$\begin{aligned} {}^cD^q x &= f(t,x) + g(t,x), \\ x(t_0) &= x_n = \tfrac{\alpha_{n-1}(t_0)+\beta_{n-1}(t_0)}{2}. \end{aligned}$$

Now, we analyze the sequence of numbers $\{x_n\}$. Observe that $\{x_n\}$ consists of two sequences of numbers $\{\alpha_n(t_0)\}$, which is a nondecreasing sequence, and $\{\beta_n(t_0)\}$, which is a nonincreasing sequence. Thus, both sequences are monotone and bounded, and hence converge to two points x_0 and y_0, respectively. Next, consider the sequence of functions $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$. These sequences are nonincreasing and nondecreasing, respectively, and are bounded by $\alpha(t)$ and $\beta(t)$ on J. It can easily be shown that these sequences are equicontinuous and uniformly bounded. Further, since these sequences are monotone, by Arzela-Ascoli Theorem they converge to two functions, $x(t, t_0, x_0)$ and $y(t, t_0, y_0)$, respectively.

Now, we claim that $x(t, t_0, x_0)$ and $y(t, t_0, y_0)$ are solutions of the terminal-value problem. From construction, we have a decreasing sequence of nested intervals $\{[\alpha_n(t_0), \beta_n(t_0)]\}$. By Cantor's Intersection Theorem they must intersect in a point. Thus $x_0 = y_0$. Since both f and g are Lipschitz, $x(t, t_0, x_0) = y(t, t_0, x_0)$ is the unique solution of

$$\begin{aligned} {}^cD^q x &= f(t,x) + g(t,x), \\ x(t_0) &= x_0. \end{aligned}$$

If $x(T, t_0, x_0) = A$, then $x(t, t_0, x_0)$ is the solution of the terminal-value problem (3.1) and (3.2). If $x(T, t_0, x_0) \neq A$, then we obtain a contradiction by considering the sequence of nested intervals, $\{[\alpha_n(T), \beta_n(T)]\}$. By Cantor's Intersection Theorem these nested intervals must intersect at only one point, which by construction is $x(T, t_0, x_0) = A$, and hence the proof. □

Remark 3.2. In a situation where it is desired to reach the target A incrementally through a series of sub-targets A_i at times t_i, $i = 1, 2, \ldots, n$, the terminal-value problem (3.1) and (3.2) can be reformulated as follows:

Let the time interval $[t_0, T]$ be divided into n subintervals $[t_{i-1}, t_i]$, $i = 1, 2, \ldots, n$ with $t_n = T$ Consider in each subinterval $[t_{i-1}, t_i]$ the terminal-value problem

$$ {}^cD^q x = F(t,x), \tag{3.10}$$

$$ x(t_i) = A_i, \; i = 1, \; 2, \; \ldots, \; n. \tag{3.11}$$

An application of Theorem 3.1 to terminal-value problem (3.10) and (3.11) yields a unique value x_{i-1} and a unique solution $x(t, t_{i-1}, x_{i-1})$ on $[t_{i-1}, t_i]$ such that $x(t_i, t_{i-1}, x_{i-1}) = A_i$, $i = 1, 2, \ldots, n$. Now, consider the interval $[t_i, t_{i+1}]$ and the terminal-value problem (3.10) with the condition $x(t_{i+1}) = A_{i+1}$. Again, application of Theorem 3.1 yields a unique value x_i and a unique solution $x(t, t_i, x_i)$ on $[t_i, t_{i+1}]$, such that $x(t_{i+1}, t_i, x_i) = A_{i+1}$. If $A_i = x_i$ we proceed to the next interval and obtain the solution on $[t_{i+1}, t_{i+2}]$. On the other hand, if A_i is not equal to x_i we must adjust the initial value at t_i by an amount equal to the difference between the two values in order to reach the initial value x_i required to obtain the terminal value A_{i+1} at time t_{i+1}. These adjustments could be achieved through application of impulses at the fixed times t_i, $i = 1, 2, \ldots, n$. In a way, this problem resembles a terminal-value impulsive problem where the impulses are dictated by the initial value at each stage.

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