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# UTILITY INDIFFERENCE PRICING OF PRODUCTS INTEGRATING REVERSE MORTGAGE WITH LONG-TERM CARE INSURANCE UNDER A LÉVY PROCESS FINANCIAL MARKET

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**ABSTRACT.** We study, in this article, the indifference pricing of the continuous annuity rate of insurance contract linking home reversion plan and long-term care insurance with the dynamics of home price modeled as a finite variation Lévy process. The multi-state Markov model is employed to describe the states and transitions of the combined contract. The equivalent utility principle and the exponential utility are respectively chosen as the pricing rule and utility function. As for the combined policy involving a single insured and a pair of insureds, we derive the non-linear partial-integro-differential equation system that the indifference continuous annuities satisfy. We numerically investigate the solution by an explicit finite-difference scheme, and discuss how the continuous annuity benefits vary in response to the changes of the major model parameters: the risk aversion of insurer, the force of interest, the age at the start of the combined policy, volatility of home value, the jump activity rate and the upward jump probability.

**Keywords:** Reverse mortgage; Home reversion plan; Long-term care insurance; Lévy process; Markov model; Indifference price; Hamilton-Jacobi-Bellman equation

## 1. INTRODUCTION

A reverse mortgage (abbreviated, RM) is a special type of loan, which is designed to allow the senior homeowners to release their home equity into cash to meet living expenses and/or pay medical bills or the premium for long-term care insurance. There are two ways to obtain the RM (reverse mortgage). One is mortgaging their homes to some special institutions, e.g. the American home equity conversion mortgage (HECM). The other is transferring partly or wholly their ownership of home equity to some special institutions, e.g. the home reversion plan (HRP) followed in England.

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In HECM, the borrower refunds the amounts borrowed and the interest accumulated until the mortgage's due date; while, in HRP, the borrower repays the lender by transferring his title of home equity on his death or after moving out of his home.

Since RM can make up the deficiencies of social security system for the elderly, many governments, such as American, Canadian, British, French, etc., have made available different types of RM. A considerable amount of research concentrates on the qualitative study with respect to RM, focusing on the demand, the risk, the feasibility and the effectiveness, see Chinloy and Megbolugbe (1994), Addae-Dapaah and Leong (1996), Buckley *et al.* (2003), Mitchell and Piggott (2004), Stucki (2005), Chou *et al.* (2006). Several other work pay attention to the use of the RM to finance long-term care (abbreviated to LTC), see Benjamin (1992), Stucki (2006) and Thomas (2009). However, the actuarial articles on pricing the contract linking the home reversion plan and long-term care are still exiguous. Xiao (2011) was the first to consider pricing the contract linking HRP and LTC involving a single insured, where home value is modeled as a geometric Brownian motion process. Ma, Zhang, and Kannan (2010) also follow the utility method to price the above-mentioned insurance policies involving a pair of insureds, and present the explicit representations of the HJB equation system via Feynman-Kac formula. The recent empirical researches show that the return distribution of risky asset prices possesses two principal features: leptokurtic feature and volatility smiles. These features motivate the researchers to employ jump-diffusion model to model the dynamics of risky asset, see Barndorff-Nielsen (1997), Andersen and Andeasen (2000). The interested reader is referred to Cont and Tankov (2004) and Kyprianou *et al.* (2006) for application of Lévy process to finance. In this paper, we assume the dynamics of home value is driven by a geometric Lévy process. We are not the first one to employ the jump-diffusion process to model the financial market with insurance risk, (refer to Jaimungal and Young (2005), Delong (2009), Perera (2010)).

Since the financial market driven by jump-diffusion and incorporated with insurance risk is incomplete, we use the usual indifference pricing technique to price the continuous annuity of insurance contract. Most indifference pricing literature with respect to insurance contract focus on the fixed premium pricing, see Young and Zariphopoulou (2002), Moore and Young (2003), Young (2003), Jaimungal and Young (2005). There also exist several papers that price continuous premium and continuous annuity, see Young and Zariphopoulou (2002), Young (2003), Xiao (2010). In Xiao (2011), the author uses a three-state Markov framework model to describe the actuarial structure of a single insured and the dynamics of the risky asset price follow the geometric Brownian process. In contrast to Xiao, we employ a six-state Markov framework model to describe the actuarial structure of a couple of insureds with the dynamics of risky asset price following the geometric Lévy process. For more comprehensive and detailed discussions of multiple state models for insurance we refer to Haberman and Pitacco (1999). In this article, we utilize the equivalent utility principle with Markov framework to derive the HJB system that indifferent continuous annuity rates satisfy and represent their numerical example.

The remainder of the paper is organized as follows: In section 2, we review the results of the optimal investment without the insurance risk. In Section 3, as for the insurance contract linking HRP to LTC involving a pair of insureds, we derive the integro-partial differential equation system that the indifferent annuities satisfy under the exponential utility function. In Section 4, as for the contract linking HRP and LTC for a single insured, we derive the integro-partial differential equation system that the indifferent annuities satisfies. Section 5 presents numerical examples, where we will discuss how the continuous annuity vary with respect to the major model parameters. The final Section 6 concludes and points to our future work.

## 2. THE OPTIMAL INVESTMENT WITHOUT THE INSURANCE RISK

This section will review the fundamental model of optimal portfolio investment for expected utility of terminal wealth in the absence of the insurance risk. As usual, we assume that the insurer can invest in both a riskless bond and a risky asset. The instantaneous yield of the riskfree asset is modeled as

$$dM_t = rM_t dt, \quad 0 \le t \le T.$$

with the constant force of interest $r > 0$, and the constant $T > 0$ refers to the term of the trading horizon. We also assume the price of risky asset at time $t$ follows a geometric Lévy process, i.e.,

$$dH_t = H_{t-}(\mu dt + \sigma dB_t + dJ_t), \quad 0 \le t \le T,$$

where $H_t$ is defined on a probability space $(\Omega, \mathbb{F}, \mathbb{P})$ with the natural filtration $\mathbb{F} = \{\mathcal{F}_t, 0 \le t \le T\}$ generated by $H_t$. Here $\{B_t, 0 \le t \le T\}$ is a $\mathbb{P}$-standard Brownian motion, and $\{J_t, 0 \le t \le T\}$ is a pure $\mathbb{P}$-Lévy jump process with the random jump measure $N(dy, dt)$, i.e.,

$$J_t = \int_0^t \int_{-1}^{+\infty} y \quad N(dy, ds).$$

The predictable compensator of $J_t$ is specified by $t \int_{-1}^{+\infty} y \ \nu(dy)$, where $\nu(dy)$ is the Lévy measure defined on $(-1, \infty)$, with $\nu(\{0\}) = 0$, and having finite variation, namely, $\int_{y>-1} |y|\nu(dy) < \infty$, (interested readers are referred to Sato (1999), Oksendal and Sulem (2005) for more details of Lévy process and its applications).

Let $W_s$ denote the wealth of the insurer at time $s$. Assume that the insurer possesses an initial endowment of wealth $w$ and can choose investment proportion dynamically between two assets. Specifically, the insurer chooses the amounts $\pi_s$ and $W_s-\pi_s$, $t \le s \le T$, to invest in the risky real estate markets and riskless bond at time $s$, respectively. The admissible trading strategy $(\pi_s, W_s-\pi_s)$ must be $\mathcal{F}_s$-adapted and self-financing. With the self-financing assumption, the wealth process of the insurer is modeled by

$$dW_s = [rW_s + (\mu - r)\pi_{s-}]ds + \sigma\pi_{s-}dB_s + \pi_{s-}dJ_s, \quad W_t = w.$$

Define the value function of the insurer, who did not yet sign the insurance policy, as follows:

$$U^{(0)}(w,t) = \sup_{\{\pi_s \in \mathcal{A}\}} E[u(W_T) \mid W_t = w], \tag{1}$$

where the function $u$ is an increasing concave utility function of wealth representing the insurer's risk preference, and $\mathcal{A}$ is the set of the admissible trading strategies that are $\mathcal{F}_t$-adapted, self-financing, and square integrable $\int_t^T \pi_s^2 ds < \infty$. In the following, we assume that the utility function is exponential, $u(w) = -\frac{1}{\alpha}e^{-\alpha w}$, where the parameter $\alpha > 0$ is the absolute risk aversion. Then a straightforward calculation shows that

$$U^{(0)}(w,t) = -\frac{1}{\alpha}\exp\left(-m_0(T-t) - \alpha w e^{r(T-t)}\right), \tag{2}$$

where, with the notation $\bar{\pi} = \pi\alpha e^{r(T-t)}$, $m_0$ is specified as

$$\begin{aligned} m_0 &= \max_{\bar{\pi}}\left[(\mu-r)\bar{\pi} - \frac{1}{2}\sigma^2\bar{\pi}^2 - \int_{-1}^{+\infty}\left(e^{-\bar{\pi}y} - 1\right)\nu(dy)\right] \\ &= (\mu-r)\bar{\pi}_0 - \frac{1}{2}\sigma^2\bar{\pi}_0^2 - \int_{-1}^{+\infty}\left(e^{-\bar{\pi}_0 y} - 1\right)\nu(dy), \end{aligned} \tag{3}$$

and $\bar{\pi}_0$ satisfies the following equation

$$\bar{\pi}_0 - \frac{1}{\sigma^2}\int_{-1}^{+\infty} y e^{-\bar{\pi}_0 y}\nu(dy) = \frac{\mu - r}{\sigma^2}.$$

The detailed arguments can be found in Jaimungal and Young (2005).

For brevity and latter use, we introduce the following three operators ${}_{\mathbf{0}}\mathcal{L}_b^{\pi}$,${}_{\mathbf{1}}\mathcal{L}_b^{\pi}$ and $\mathcal{J}_b^{\bar{\pi}}$.

**Definition 2.1.** With the notations $\alpha(t) = \alpha e^{r(T-t)}$ and $\bar{\pi} = \pi\alpha e^{r(T-t)}$, the integro-differential operators ${}_{\mathbf{0}}\mathcal{L}_b^{\pi}$, ${}_{\mathbf{1}}\mathcal{L}_b^{\pi}$ and $\mathcal{J}_b^{\bar{\pi}}$ are defined by

(4)

$$\begin{aligned} {}_{\mathbf{0}}\mathcal{L}_b^{\pi} f(w,H,t) = &\frac{\partial f}{\partial t}(w,H,t) + (rw + (\mu-r)\pi - b)\frac{\partial f}{\partial w}(w,H,t) + \mu H\frac{\partial f}{\partial H}(w,H,t) \\ &+ \frac{1}{2}\sigma^2\left(\pi^2\frac{\partial^2 f}{\partial w^2}(w,H,t) + 2\pi H\frac{\partial^2 f}{\partial w\partial H}(w,H,t) + H^2\frac{\partial^2 f}{\partial H^2}(w,H,t)\right) \end{aligned}$$

$$+\int_{-1}^{+\infty}[f(w+\pi y, H+Hy, t)-f(w,H,t)]\nu(dy),$$

(5)

$$ {}_{\mathbf{1}}\mathcal{L}_b^{\pi} f(w,t)=\frac{\partial f}{\partial t}(w,t)+(rw+(\mu-r)\pi-b)\frac{\partial f}{\partial w}(w,t)+\frac{1}{2}\sigma^2\pi^2\frac{\partial^2 f}{\partial w^2}(w,t) + \int_{-1}^{+\infty}[f(w+\pi y,t)-f(w,t)]\nu(dy),$$

(6)

$$\mathcal{J}_b^{\bar{\pi}} f(H,t)=\frac{\partial f}{\partial t}(H,t)+\mu H\frac{\partial f}{\partial H}(H,t)+\frac{1}{2}\sigma^2 H^2\left(\frac{\partial^2 f}{\partial H^2}(H,t)+\left(\frac{\partial f}{\partial H}(H,t)\right)^2\right) + (m_0+b\alpha(t))-\bar{\pi}(\mu-r)+\frac{1}{2}\sigma^2\bar{\pi}^2-\bar{\pi}\sigma^2 H\frac{\partial f}{\partial H}(H,t) + \int_{-1}^{+\infty}[e^{f(H+Hy,t)-f(H,t)-\bar{\pi}y}-1]\nu(dy).$$

## 3. INSURANCE LINKING HRP TO LTC: PAIR OF INSUREDS

In this section, we shall show that the indifferent annuity benefits of insurance contracts solve a non-linear integro-partial differential equation system; we do this by using the financial market described in Section 2 and employing the Markov model to express the actuarial structure of insurance contracts linking HRP to LTC for a pair of insureds.

### 3.1. **Markov model for the insureds.** We make the following assumptions:

(A1) The two events that one of the insureds dies and the other enters a nursing home cannot happen simultaneously;

(A2) For the convenience of practical operation, it is forbidden that one of the insureds lives at their home and the other lives at a nursing home at the same time.

For the situation depicted in Figure 1(a), we employ a six-state Markov model in continuous-time case to illustrate the states and transitions of the insurance contract linking HRP to LTC applied jointly by a couple. Under the assumption (A1), transition $5\to 1$ cannot occur in Figure $1(a)$. Let $Z_t$ denote the continuous-time Markov chain taking values in the state space $S=\{0,1,2,3,4,5\}$, where $Z_t$ denote the state of the policy at time $t\in[0,T)$. The transition probabilities are denoted by

$$P_{ij}(s,t)=P(Z_t=j|Z_s=i),\quad s\le t \text{ and } i,j\in S,$$

with the corresponding intensities of transition defined by

$$\lambda_{ij}(t)=\lim_{h\to 0}\frac{P_{ij}(t,t+h)}{h},\quad i\neq j.$$

The corresponding policy states are as follows:

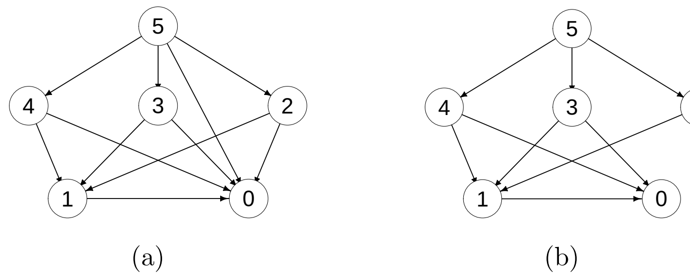



FIGURE 1. Markov models for the insurance contracts relevant to the home reversion plan

(1) State 5 represents that $(x)$ and $(y)$ both live at home;
(2) State 4 represents that $(x)$ dies and $(y)$ lives at home;
(3) State 3 represents that $(y)$ dies and $(x)$ lives at home;
(4) State 2 represents that $(x)$ and $(y)$ both live at the nursing home;
(5) State 1 represents that the only survivor lives at the nursing home;
(6) State 0 represents that the only survivor die.

Here, $(x)$ and $(y)$ denote the $x$-year old husband and $y$-year old wife, respectively.

**Remark 3.1.** Under the assumption that a pair of insureds cannot die simultaneously, we use the situation in Figure 1$(b)$ to describe the actuarial structure of the combined policy; without that assumption, we follow the situation in Figure 1$(a)$. In contrast to Figure 1$(b)$, the transitions $2 \to 0$ and $5 \to 0$ occur in Figure 1$(a)$.

3.2. **The wealth Process of the Insurer.** Let $\tau_i$ denote the time of moving into state $i$ $(i = 0, 1, 2, 3, 4)$. The $\tau_i$'s are the stopping times defined by $\tau_i = \inf\{t;\ Z_t = i\}$, $i = 0, 1, 2, 3, 4$. Our insurance policy linking HRP to LTC for a pair of insureds is designed as follows:

(I) As far as the benefits to the insureds are concerned, we assume that they will be paid with a continuous annuity at an instantaneous constant rate $b_i$ while the insureds are in state $i$, $i = 1, 2, 3, 4, 5$.
(II) As for the return to the insurer, instead of the payment on account or payment on terms, we assume that the insurer will be refunded at the time of $\min\{\tau_0, \tau_1, \tau_2\}$, with the cash of the sale of their home equity.

Then the wealth process of the insurer is governed by

$$\begin{cases} W_t = w, & \\ W_{\tau_0^+} = W_{\tau_0^-} + H_{\tau_0}, & \tau_2 = \infty,\ \tau_1 = \infty, \\ W_{\tau_1^+} = W_{\tau_1^-} + H_{\tau_1}, & \tau_2 = \infty,\ t < \tau_1 < \tau_0 < T, \\ W_{\tau_2^+} = W_{\tau_2^-} + H_{\tau_2}, & \tau_4 = \infty,\ \tau_3 = \infty,\ t < \tau_2 < \tau_0 < T, \\ dW_s = \mu_5 ds + \sigma\pi_{s-} dB_s + \pi_{s-} dJ_s, & t < s < \min(\tau_0, \tau_2, \tau_3, \tau_4), \\ dW_s = \mu_i ds + \sigma\pi_{s-} dB_s + \pi_{s-} dJ_s, & \tau_{I(i)} = \infty,\ \tau_{K(i)} = \infty,\ t < \tau_i < s < \min(\tau_0, \tau_1), \\ dW_s = \mu_1 ds + \sigma\pi_{s-} dB_s + \pi_{s-} dJ_s, & t < \tau_1 < s < \tau_0 < T, \\ dW_s = \mu_0 ds + \sigma\pi_{s-} dB_s + \pi_{s-} dJ_s, & t < \tau_0 < s < T. \end{cases}$$

Here, with $b_0 \equiv 0$,

$$\begin{aligned} \mu_i &= rW_{s-} + (\mu - r)\pi_{s-} - b_i,\ i = 0, 1, \ldots, 5, \\ I(i) &= (i-2)\mathbb{I}_{\{i=4\}} + (i-1)\mathbb{I}_{\{i=3\}} + (i+1)\mathbb{I}_{\{i=2\}},\ i = 2, 3, 4, \\ K(i) &= (i-1)\mathbb{I}_{\{i=4\}} + (i+1)\mathbb{I}_{\{i=3\}} + (i+2)\mathbb{I}_{\{i=2\}},\ i = 2, 3, 4. \end{aligned}$$

3.3. **HJB Equations for Indifference annuity Benefits.** Suppose the insurer has the opportunity to insure two lives $(x)$ and $(y)$ at time $t = 0$. We measure time from the above ages $x$ and $y$ for the husband and wife, respectively. Accordingly, we denote these two persons by $(x+t)$ and $(y+t)$, respectively, at time $t \in [0, T]$. If $(x+t)$ and $(y+t)$ enter formally into the combined plan at time $t$, the continuous annuity benefits for our product linking HRP to LTC relate to the price of home equity at time $t$. However, once the annuity benefits are set at time $t$, they are fixed for the life of the policy.

For our insurance product linking HRP to LTC, as described in Section 3.2, a value function of the insurer at state $i$, $i = 3, 4, 5$, is defined by

$$U^{(i)}(w, H, t) = \sup_{\{\pi_s \in \mathcal{A}\}} E[u(W_T) | W_t = w, H_t = H, Z_t = i].$$

When the insured moves into state $i$, $i = 1, 2$, at time $t$, the insured will be paid at a instantaneous constant rate $b_i$. Thus, the insurer will still bear the risk for the annuity payout, but the insurer will be repaid with the cash of selling the house at that time. In this instance, the maximum expected utility of terminal wealth for the insurer derived by investing optimally, which is no longer dependent on the home value, is defined by

$$U^{(i)}(w, t) = \sup_{\{\pi_s \in \mathcal{A}\}} E[u(W_T) | W_t = w, Z_t = i].$$

The following Lemma 3.1 presents the non-linear integro-partial differential equation system that the value functions $U^{(i)}(w, H, t)$, $i = 1, 2, \ldots, 5$, satisfy. See the Equation (4) and (5), respectively, for the definition of ${}_{\mathbf{0}}\mathcal{L}_b^{\pi} f(w, H, t)$ and ${}_{\mathbf{1}}\mathcal{L}_b^{\pi} f(w, t)$ which are used below.

**Lemma 3.1.** *$U^{(5)}(w,H,t)$ solves the HJB equation*

$$\max_{\pi}[{}_{\mathbf{0}}\mathcal{L}^{\pi}_{b_5}U^{(5)}(w,H,t)] + \sum_{i=3,4}\lambda_{5i}(t)[U^{(i)}(w,H,t) - U^{(5)}(w,H,t)] + \sum_{i=0,2}\lambda_{5i}(t)[U^{(i)}(w+H,t) - U^{(5)}(w,H,t)] = 0, \tag{7}$$

*where $U^{(i)}(w,H,t)$, $i=3,4$, and $U^{(i)}(w,t)$, $i=1,2$, respectively, satisfy the following HJB equations*

$$\max_{\pi}\left[{}_{\mathbf{0}}\mathcal{L}^{\pi}_{b_i}U^{(i)}(w,H,t)\right] + \sum_{j=0,1}\lambda_{ij}(t)[U^{(j)}(w+H,t) - U^{(i)}(w,H,t)] = 0, \tag{8}$$

$$\max_{\pi}\left[{}_{\mathbf{1}}\mathcal{L}^{\pi}_{b_2}U^{(2)}(w,t)\right] + \sum_{j=0,1}\lambda_{2j}(t)[U^{(j)}(w,t) - U^{(2)}(w,t)] = 0, \tag{9}$$

$$\max_{\pi}\left[{}_{\mathbf{1}}\mathcal{L}^{\pi}_{b_1}U^{(1)}(w,t)\right] + \lambda_{10}(t)[U^{(0)}(w,t) - U^{(1)}(w,t)] = 0. \tag{10}$$

*The Equations* (7)–(10) *possess the terminal conditions*

$$U^{(i)}(w,H,T) = u(w), \quad i=3,4,5, \;\; U^{(i)}(w,T) = u(w), \quad i=1,2.$$

The proof of Lemma 3.1 is deferred to Appendix A.

Under the indifferent annuity benefits $b_i, i=1,2,\ldots,5$, the optimal investment with the insurance risk is the same as the optimal investment without the insurance risk, i.e.

$$U^{(0)}(w,t) = U^{(5)}(w,H,t;b_1,b_2,b_3,b_4,b_5).$$

Thus, with the indifference annuity benefits, the insurer is indifferent between signing and not signing the insurance contract linking HRP to LTC insurance. The following theorem represents the non-linear partial-integro-differential equation system the indifference annuity benefits satisfy.

**Theorem 3.1.** *Under the assumption of exponential utility $u(w) = -\frac{1}{\alpha}e^{-\alpha w}$, the instantaneous constant rate $b_i$ $(i=1,2,3,4,5)$ of the indifference continuous annuities satisfy the equation*

$$\phi_5(H,t;b_1,b_2,b_3,b_4,b_5) = 0, \tag{11}$$

*where $\phi_5(H,t)$ solves the following HJB equation*

$$\min_{\bar{\pi}}[\mathcal{J}^{\bar{\pi}}_{b_5}\phi_5(H,t)] + \sum_{j=3,4}\lambda_{5j}(t)\left(e^{\phi_j(H,t)-\phi_5(H,t)} - 1\right) + \sum_{j=0,2}\lambda_{5j}(t)\left(e^{-\alpha(t)H-\phi_5(H,t)}(\phi_2(t)\mathbb{I}_{\{j=2\}} + \mathbb{I}_{\{j=0\}}) - 1\right) = 0. \tag{12}$$

*Here, $\phi_i(H,t)$, $i=3,4$ and $\phi_2(t)$ in Equation (12), respectively, satisfy the following HJB equations*

$$\min_{\bar\pi}[\mathcal{J}_{b_i}^{\bar\pi}\phi_i(H,t)] + \sum_{j=0,1}\lambda_{ij}(t)\left(e^{-\alpha(t)H-\phi_i(H,t)}(\phi_1(t)\mathbb{I}_{\{j=1\}}+\mathbb{I}_{\{j=0\}})-1\right)=0, \quad i=3,4, \tag{13}$$

$$\frac{d\phi_2(t)}{dt}+b_2\alpha(t)\phi_2(t)+\sum_{j=0,1}\lambda_{2j}(t)\left((\phi_1(t)\mathbb{I}_{\{j=1\}}+\mathbb{I}_{\{j=0\}})-\phi_2(t)\right)=0, \tag{14}$$

*in which $\phi_1(t)$ solve the following equation*

$$\frac{d\phi_1(t)}{dt}+\left(b_1\alpha(t)-\lambda_{10}(t)\right)\phi_1(t)+\lambda_{10}(t)=0. \tag{15}$$

*The Equations (12)–(15) are subject to the terminal conditions*

$$\phi_i(H,T)=0,\ i=3,4,5,\ \phi_i(T)=1,\ i=1,2.$$

*Proof.* To simplify the HJB equation system in Lemma 3.1, let

$$U^{(i)}(w,H,t)=U^{(0)}(w,t)e^{\phi_i(H,t)}\ (i=3,4,5),\quad U^{(i)}(w,t)=U^{(0)}(w,t)\phi_i(t)\ (i=1,2). \tag{16}$$

From Equation (2), we obtain the following equations

$$\begin{aligned}\frac{\partial U^{(0)}}{\partial w}(w,t) &= (-\alpha e^{r(T-t)})U^{(0)}(w,t),\\ \frac{\partial^2 U^{(0)}}{\partial w^2}(w,t) &= \left(-\alpha e^{r(T-t)}\right)^2 U^{(0)}(w,t).\end{aligned}$$

We substitute the above expressions and their corresponding derivatives of $U^{(5)}(w,H,t)$ into $\max_\pi\{{}_{\mathbf{0}}\mathcal{L}_{b_5}^{\pi}U^{(5)}(w,H,t)\}$ to obtain

$$\max_\pi\left[{}_{\mathbf{0}}\mathcal{L}_{b_5}^{\pi}U^{(5)}(w,H,t)\right]=U^{(0)}(w,t)e^{\phi_5(H,t)}\min_{\bar\pi}\left[\mathcal{J}_{b_5}^{\bar\pi}\phi_5(H,t)\right]. \tag{17}$$

Substituting the Equation (16) and using the notation $\alpha(t)=\alpha e^{r(T-t)}$, we get

$$\sum_{j=3,4}\lambda_{5j}(t)[U^{(j)}(w,H,t)-U^{(5)}(w,H,t)] = U^{(0)}(w,t)e^{\phi_5(H,t)}\sum_{j=3,4}\lambda_{5j}(t)\left(e^{\phi_j(H,t)-\phi_5(H,t)}-1\right), \tag{18}$$

$$\sum_{j=0,2}\lambda_{5j}(t)[U^{(j)}(w+H,t)-U^{(5)}(w,H,t)] = U^{(0)}(w,t)e^{\phi_5(H,t)}\sum_{j=0,2}\lambda_{5j}(t)\left(e^{-\alpha(t)H-\phi_5(H,t)}(\phi_2(t)\mathbb{I}_{\{j=2\}}+\mathbb{I}_{\{j=0\}})-1\right). \tag{19}$$

By plugging the Equations (17), (18) and (19) into the Equation (7), we obtain the Equation (12). Similar arguments yield the Equation (13). □

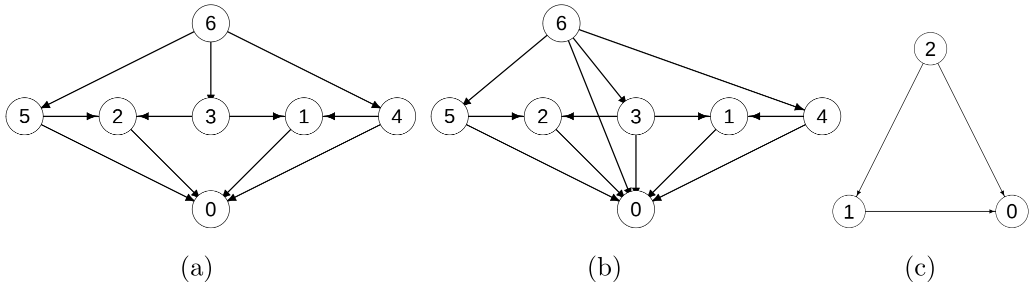


FIGURE 2. Markov models for the insurance contracts relevant to the home reversion plan

Recalling the definition of the operator ${}_{\mathbf{1}}\mathcal{L}^{\pi}_{b_1}$ from Equation (5) and the relation in Equation (3), we can employ the argument similaar to the above to obtain

$$\max_{\pi}\left[{}_{\mathbf{1}}\mathcal{L}^{\pi}_{b_2}U^{(2)}(w,t)\right] = U^{(0)}\min_{\bar{\pi}}\left\{[m_0 + b_2\alpha(t) - (\mu - r)\bar{\pi} + \frac{1}{2}\sigma^2\bar{\pi}^2 + \int_{-\infty}^{+\infty}\left(e^{-\bar{\pi}y} - 1\right)\nu(dy)\right]\phi_2(t) + \frac{d\phi_2}{dt}\Bigg\} = U^{(0)}(w,t)\left[\frac{d\phi_2(t)}{dt} + b_2\alpha(t)\phi_2(t)\right]. \tag{20}$$

Substituting $U^{(i)}(w,t) = U^{(0)}(w,t)\phi_i(t)$ $(i = 1, 2)$, we deduce

$$\sum_{j=0,1}\lambda_{2j}(t)[U^{(j)}(w,t) - U^{(2)}(w,t)] = U^{(0)}(w,t)\sum_{j=0,1}\lambda_{2j}(t)\left((\phi_1(t)\mathbb{I}_{\{j=1\}} + \mathbb{I}_{\{j=0\}}) - \phi_2(t)\right). \tag{21}$$

Substituting the Equations (20) and (21) into Equation (9), we achieve the Equation (14). We can also show, with similar arguments, that the Equation (15) holds from Equation (10).

**Remark 3.2.** The above model, depicted by Figure 1, is not considering the gender of the last survivor. When incorporating the gender factor of the last survivor, we can employ the seven state continuous-time Markov model, depicted by Figure 2($a$) and 2($b$), to illustrate the actuarial structure of the insured pair. The corresponding states are as follows:

(1) State 6: Both insureds $(x)$ and $(y)$ live at home;
(2) State 5: The insured $(x)$ dies and the insured $(y)$ lives at home;
(3) State 4: The insured $(y)$ dies and the insured $(x)$ lives at home;
(4) State 3: The insureds $(x)$ and $(y)$ both live in the nursing home;
(5) State 2: The only survivor $(y)$ lives in the nursing home;
(6) State 1: The only survivor $(x)$ lives in the nursing home;
(7) State 0: Both the insureds $(x)$ and $(y)$ die.

Under the assumption that $(x)$ and $(y)$ cannot die simultaneously, we can use Figure $2(a)$ to illustrate the state and transition of the policy. Abandoning this assumption, we can choose Figure $2(b)$ to express the actuarial structure. Comparing Figure $2(b)$ with Figure $2(a)$, the transition $6 \to 0$ and $3 \to 0$ appear in Figure $2(b)$. Under the assumption (A1), transition $6 \to 1$ and $6 \to 2$ cannot occur in Figure $2(a)$ and Figure $2(b)$.

**Remark 3.3.** Although the two models mentioned in the Remark 3.2 become more complicated than the six-sate model we studied in detail above, these changes do not affect the nature of the problems. Therefore, we can appropriately modify the arguments used above in the six-state Markov case to derive the associated HJB equations in the present case.

## 4. CONTRACTS LINKING HRP AND LTC: A SINGLE INSURED

The content of this section parallels that of Section 3, so we follow the notations similar to those used in Section 3. This should cause no confusion because there is little overlap between Section 3 and Section 4. The details of the derivations resemble those of Section 3, and hence, for the purpose of brevity, we elide the details. Whereas Xiao (2011) deals with the geometric Brownian motion, our work here is based on the Lévy process financial market.

### 4.1. The wealth Process of the Insurer.

We employ a three-state Markov model in the time-continuous case to illustrate the policy states and transitions, depicted in Figure $2(c)$. The corresponding policy states are 'healthy and living at home' (2), 'in the nursing home' (1), and 'dead' (0).We assume that recovery from state 1 is impossible.

Xiao (2011) designs the insurance product linking HRP to LTC insurance characterized by the following conditions:

(I) The continuous annuity benefit is paid at an instantaneous constant rate $b_i$ while the insured is in the state $i$, $i = 1, 2$, $b_1 < b_2$.

(II) The insurer will be repaid at the time of entering into state 0 or state 1, utilizing the cash generated from the sale of the house, whichever happens first.

Let $\tau_i$ $(i = 0, 1, 2)$ denote the time of moving into the state $i$, $(i = 0, 1, 2)$, defined by $\tau_i = \inf\{t;\ Z_t = i\}$, $(i = 0, 1, 2)$. Then the wealth process of the insurer is as follows

$$\begin{cases} W_t = w, & \\ W_{\tau_1^+} = W_{\tau_1^-} + H_{\tau_1}, & \tau_1 < \tau_0 < T, \\ W_{\tau_0^+} = W_{\tau_0^-} + H_{\tau_0}, & \tau_1 = \infty,\ \tau_0 < T, \\ dW_s = [rW_{s-} + (\mu - r)\pi_{s-} - b_2]ds + \sigma\pi_{s-}dB_s + \pi_{s-}dJ_s, & t < s < \min(\tau_0, \tau_1), \\ dW_s = [rW_{s-} + (\mu - r)\pi_{s-} - b_1]ds + \sigma\pi_{s-}dB_s + \pi_{s-}dJ_s, & \tau_1 < s < \tau_0, \\ dW_s = [rW_{s-} + (\mu - r)\pi_{s-}]ds + \sigma\pi_{s-}dB_s + \pi_{s-}dJ_s, & \tau_0 < s < T. \end{cases}$$

4.2. **HJB Equations for Indifference Annuities.** For the aforementioned insurance contract linking HRP to LTC for a single insured, the value functions are defined by

$$U^{(2)}(w, H, t; b_1, b_2) = \sup_{\{\pi_s \in \mathcal{A}\}} E[u(W_T) | W_t = w, H_t = H, Z_t = 2], \tag{22}$$

$$U^{(1)}(w, t; b_1) = \sup_{\{\pi_s \in \mathcal{A}\}} E[u(W_T) | W_t = w, Z_t = 1]. \tag{23}$$

Suitably modifying the reasoning used for Lemma 3.1, we obtain the following HJB equation for the value functions defined by Equations (22) and (23).

**Lemma 4.1.** *$U^{(2)}(w, H, t)$ solves the HJB equation*

$$\max_{\pi}[{}_{\mathbf{0}}\mathcal{L}^{\pi}_{b_2} U^{(2)}(w, H, t)] + \sum_{i=0,1} \lambda_{2i}(t)[U^{(i)}(w + H, t) - U^{(2)}(w, H, t)] = 0, \tag{24}$$

*where $U^{(1)}(w, t)$ satisfies the following HJB equation system*

$$\max_{\pi} \left[{}_{\mathbf{1}}\mathcal{L}^{\pi}_{b_1} U^{(1)}(w, t)\right] + \lambda_{10}(t)[U^{(1)}(w, t) - U^{(0)}(w, t)] = 0. \tag{25}$$

*The terminal conditions for Equations (24) and (25) are, respectively,*

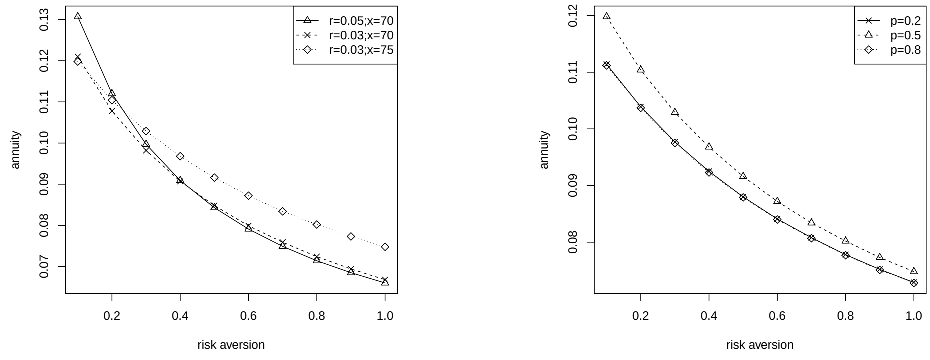
$$U^{(2)}(w, H, T) = u(w),\ U^{(1)}(w, T) = u(w).$$

We intend to compute the indifference annuity benefits $b_1, b_2$, which cause the insurer to be indifferent between signing and not signing the insurance contract; in other words, with the indifference annuity benefits, the optimal investment with the insurance risk is the same as the optimal investment in the absence of insurance risk, i.e.

$$U^{(0)}(w, t) = U^{(2)}(w, H, t; b_1, b_2).$$

**Theorem 4.1.** *Under the assumption of exponential utility $u(w) = -\frac{1}{\alpha}e^{-\alpha w}$, the instantaneous constant rate $b_i$ $(i = 1, 2)$ of the indifference continuous annuities satisfy the equation*

$$\phi_2(H, t; b_1, b_2) = 0, \tag{26}$$




(a) (b)

FIGURE 3. These diagrams depict the change of annuity with the risk aversion as the transition intensities remain fixed, the drift term $\mu = 0.08$, the volatility $\sigma = 0.2$, the jump size $\epsilon = 0.1$, and the jump intensity $v = 20$. In Figure 3(a), the upward probability is fixed as $p = 0.5$; while the force of interest and the age at the start of the combined policy are changed. In the Figure 3(b), with the varying upward probability, the force of interest and the age at the start of the combined policy, respectively, are fixed at $r = 0.03$ and $x = 75$.

*where $\phi_2(H,t)$ solves the following HJB equation*

$$\min_{\bar{\pi}}[\mathcal{J}_{b_2}^{\bar{\pi}}\phi_2(H,t)] + \sum_{j=0,1} \lambda_{2j}(t)\left(e^{-\alpha(t)H-\phi_2(H,t)}(\phi_1(t)\mathbb{I}_{\{j=1\}} + \mathbb{I}_{\{j=0\}}) - 1\right) = 0. \tag{27}$$

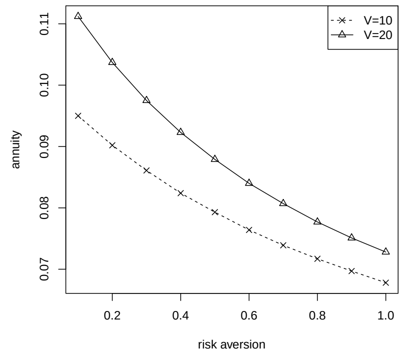
*Furthermore, $\phi_1(t)$ solves the Equation*

$$\frac{d\phi_1(t)}{dt} + (b_1\alpha(t) - \lambda_{10}(t))\,\phi_1(t) + \lambda_{10}(t) = 0. \tag{28}$$

*The Equations (27) and (28) are to satisfy the terminal conditions $\phi_2(H,T) = 0$ and $\phi_1(T) = 1$, respectively.*

## 5. NUMERICAL EXAMPLES

We devote this section to the numerical experiments of the indifference pricing of continuous annuity for a contract linking HRP to LTC that involves only a single insured. We will illustrate that the approximate constant rate of indifference annuity varies with respect to the parameter of the risk aversion of insurer, the force of interest, the age at the start of the combined policy, volatility of home value, the jump activity intensity, and the upward jump probability.



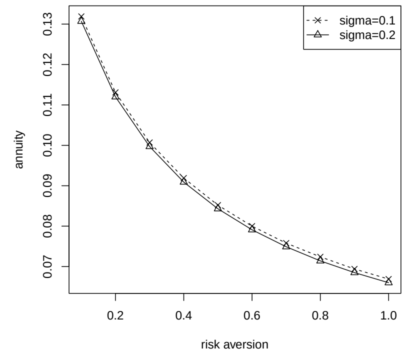




FIGURE 4. These diagrams illustrates the change of annuity with the risk aversion as the transition intensities remain unchanged, the drift term set at $\mu = 0.08$, and the jump size at $\epsilon = 0.1$. In the Figure 4(a), with varying jump intensity, we fix the volatility at $\sigma = 0.2$, the force of interest at $r = 0.03$, the upward probability as $p = 0.8$, and the age at the start of the combined policy to be $x = 75$. In the Figure 4(b), with the varying volatility, we fix the force of interest at $r = 0.05$, the jump intensity at $v = 20$, the upward probability at $p = 0.5$, and the age at the start of the combined policy at $x = 70$.

To compute the numerical solution for $b_1, b_2$ in Equation (26), we introduce a new jump process $\{\widetilde{J}_s\}_{0\leq s\leq T}$ with a new jump measure $\widetilde{N}(dy, dt)$ so that

$$\widetilde{J}_s := \int_0^s \int_{-\infty}^{+\infty} (e^y - 1)\widetilde{N}(dy, dt) = \int_0^s \int_{-1}^{+\infty} y \; N(dy, dt).$$

This means that the new jump process $\widetilde{J}_s$ undergoes a jump of size $e^y - 1$ whenever the original jump process $J_s$ undergoes a jump of size $y$. Thus, the predictable compensator $\widetilde{A}_s$ of $\widetilde{J}_s$ turns out to be

$$\widetilde{A}_s = s \int_{-\infty}^{+\infty} (e^y - 1)\widetilde{\nu}(dy) = s \int_{-1}^{+\infty} y \; \nu(dy),$$

where $\widetilde{\nu}(dy)$ can be written as $\widetilde{\nu}(dy) = f(e^y - 1)e^y dy$ with the assumption that the original Lévy measure can be expressed as $\nu(dy) = f(y)dy$. Recalling the expression for the operator $\mathcal{J}_b^{\bar{\pi}}$ defined by the Equation (6), and that

$$\int_{-1}^{+\infty} [e^{\phi_2(H+Hy,t)-\phi_2(H,t)-\bar{\pi}y} - 1]\nu(dy) = \int_{-\infty}^{+\infty} [e^{\phi_2(He^y,t)-\phi_2(H,t)-\bar{\pi}(e^y-1)} - 1]\widetilde{\nu}(dy),$$

we can specify the term $\mathcal{J}_{b_2}^{\bar{\pi}}\phi_2(H,t)$. Now the Equation (27) can be expanded as follows

(29)

$$\max_{\bar{\pi}}\left\{\left(\mu-r+\sigma^2 H\frac{\partial\phi_2}{\partial H}\right)\bar{\pi}-\frac{1}{2}\sigma^2\bar{\pi}^2-\int_{-\infty}^{+\infty}\left(e^{\phi_2(He^y,t)-\phi_2(H,t)-\bar{\pi}(e^y-1)}-1\right)\widetilde{\nu}(dy)\right\}$$
$$-\frac{\partial\phi_2}{\partial t}-\mu H\frac{\partial\phi_2}{\partial H}-\frac{1}{2}\sigma^2H^2\left(\frac{\partial^2\phi_2}{\partial H^2}+\left(\frac{\partial\phi_2}{\partial H}\right)^2\right)-(m_0+b_2\alpha(t))+(\lambda_{21}(t)+\lambda_{20}(t))$$
$$-\left(\lambda_{21}(t)\phi_1(t)+\lambda_{20}(t)\right)e^{-H\alpha(t)-\phi_2(H,t)}=0.$$

To solve for the instantaneous constant rate of indifference annuities $b_1$ and $b_2$, we employ the explicit finite difference method to discretize the Equation (29). Toward this, we first make a transform $Z=\ln H$, next let $\widetilde{\phi}_2(z,t)=\phi_2(e^z,t)$, and finally we discretize the $(z,t)$ plane into meshes of size $(\Delta z,\Delta t)$ and apply an explicit scheme for the first and second partial derivatives of $\widetilde{\phi}_2(z,t)$. The maximization term will be evaluated explicitly, and its values at the points $(m\Delta x, n\Delta t)$ will be denoted by $M_m^n$. With the notations

$$\begin{aligned}
z_m &= m\Delta z, \quad (-M\le m\le M),\\
t_n &= n\Delta t, \quad (0\le n\le N),\\
\widetilde{\phi}_m^n &= \widetilde{\phi}_2(z_m,t_n),\\
a &= \frac{(2\mu-\sigma^2)\Delta t}{4\Delta z},\\
b &= \frac{\sigma^2\Delta t}{2(\Delta z)^2},
\end{aligned}$$

the resulting finite difference equation of Equation (29) can be expressed as

$$\widetilde{\phi}_m^n-C_1e^{-\widetilde{\phi}_m^n}-(b+a)\widetilde{\phi}_{m+1}^{n+1}-(1-2b)\widetilde{\phi}_m^{n+1}$$
$$-(b-a)\widetilde{\phi}_{m-1}^{n+1}-\frac{b}{4}\left(\widetilde{\phi}_{m+1}^{n+1}-\widetilde{\phi}_{m-1}^{n+1}\right)^2-C_0=0,$$

where

$$\begin{aligned}
C_0 &= \Delta t[m_0+b_2\alpha e^{r(T-t_n)}-\lambda_{21}(t_n)-\lambda_{20}(t_n)-M_m^n],\\
C_1 &= \Delta t[\lambda_{21}(t_n)\phi_1(t_n)+\lambda_{20}(t_n)]\exp(-\alpha e^{r(T-t_n)}e^{Z_m}).
\end{aligned}$$

In the numerical experiments, the specifics of the policy and the financial market are as follows:

(I) The age at the start of the combined policy is 70 or 75 years old and the upper age limit set in the life table is 100. That is, $x=70$ and the term of the policy is $T=30$; or $x=75$ and the term of the policy is $T=25$.

(II) We assume that the initial value of the home equity is $H=1$, the drift term $\mu=0.08$, and the force of interest $r=0.03, 0.05$.

(III) The continuous annuity benefits $b_1$ and $b_2$ satisfy the relation $b_1 = 2b_2$.

(IV) The mortality laws are those of the Danish technical basis G82M for males:

$$\begin{aligned} \lambda_{21}(t) &= 0.0004 + 0.0000034674 \times 10^{0.06(x+t)}, \\ \lambda_{20}(t) &= \lambda_{10}(t) = 0.0005 + 0.0000758581 \times 10^{0.038(x+t)}. \end{aligned} \tag{30}$$

(V) Consider the toy jump diffusion model with arrival rate $\lambda > 0$. The jump size only takes two possible values: $\ln|1+\epsilon|$ with probability $0 \le p \le 1$, or $\ln|1-\epsilon|$ with probability $1-p$, with $0 < \epsilon \ll 1$. The Lévy measure $\widetilde{\nu}(dy)$ corresponding to the toy jump diffusion model is

$$\widetilde{\nu}(dy) = v(p\ \delta(y - \ln|1+\epsilon|) + (1-p)\delta(y - \ln|1-\epsilon|)).$$

The jump-diffusion model is employed to the numerical experiment in Jaimungal and Young(2005). We fix the jump size as $\epsilon = 0.1$, the jump intensity as $v = 10, 20$, and the upward jump probability at $p = 0.2, 0.5, 0.8$.

Figure 3(a) reveals that: (1) The annuity benefits become greater with the increase of the age at the start of the combined policy, when the other model parameters remain unchanged;

(2) The rate of change of the annuity benefits dwindle with the decrease of the force of interest, when the other model parameters remain unchanged; in other words, the annuity benefits decrease at a more moderate pace with the lower force of interest.

There is a slight difference between the annuity benefits with the upward probability $p = 0.2$ and those with the upward probability $p = 0.8$. Consequently, in the Figure 3(b), the line under the upward probability $p = 0.2$ almost coincides with that one under upward probability $p = 0.8$.

Figure 4(a) shows that, with the other model parameters being unchanged, the annuity benefits will increase as the jump intensity increases.

Figure 4(b) illustrates that, when the other parameters are kept fixed, the annuity benefits become smaller as the volatility of the home price becomes larger.

All of the diagrams in Figure 3 and Figure 4 indicate that the annuity benefits diminish as the risk aversion of the insurer increases.

These results are consistent with our intuitions.

## 6. CONCLUSION

The article studies the pricing of indifference continuous annuity of contracts linking HRP and LTC with the principle of equivalent utility. We employ (1) the Markov model to describe the state, (2) the transition of insurance contract involving a single insured or a pair of insureds, and (3) jump-diffusion process to model the movement of the house value, extending the results of Xiao (2011). In addition, we conduct numerical experiments to illustrate how the indifference continuous annuity vary with respect to the major model parameters.

This method can be extended to price the more complicated insurance contracts. In fact, the constant risk-free interest rate is too simple to describe the real movement of interest rate, so the future work will consider the more general assumption of stochastic interest rate.

## APPENDIX

### Derivation of the Hamilton-Jacobi-Bellman Equation

Assume that the insurer fixes an investment policy $\{\pi_s\}$ at $\{\pi\}$ in the time interval $[t, t+h]$, where $\{\pi\}$ is not necessarily the optimal investment policy, and, after time $t+h$, the insurer follows the optimal investment policy $\{\pi_s^*\}$ until the maturity time $T$. We consider all possible transitions at state 5 in the time interval $[t, t+h]$ and optimal investments in the time interval $[t+h, T]$. The details are as follows:

(1) If the insured $(x+t)$ and $(y+t)$ both live at home until time $t+h$, which happens with probability $P_{55}(t, t+h)$, the maximum expected utility derived by investing optimally in $[t+h, T]$ is $U^{(5)}(W_{t+h}, H_{t+h}, t+h)$.

(2) If $(x+t)$ dies in the time interval $[t, t+h]$ and $(y+t)$ still survives until time $t+h$, which happens with probability $P_{54}(t, t+h)$, the maximum expected utility derived by investing optimally in $[t+h, T]$ is $U^{(4)}(W_{t+h}, H_{t+h}, t+h)$.

(3) If $(x+t)$ survives until time $t+h$ and $(y+t)$ dies before time $(y+t)$, which happens with probability $P_{53}(t, t+h)$, the maximum expected utility derived by investing optimally in $[t+h, T]$ is $U^{(3)}(W_{t+h}, H_{t+h}, t+h)$.

(4) If $(x+t)$ and $(y+t)$ move together into a nursing home, which happens with probability $P_{52}(t, t+h)$, the maximum expected utility derived by investing optimally in $[t+h, T]$ is $U^{(2)}(W_{t+h}, t+h)$.

(5) However, if $(x+t)$ and $(y+t)$ both dies before time $t+h$, which happens with probability $P_{50}(t, t+h)$, the maximum expected utility derived by investing optimally in $[t+h, T]$ is $U^{(0)}(W_{t+h}, t+h)$.

Thus, we obtain by the definition of $U^{(5)}(w, H, t)$ that

$$
\begin{aligned}
(31) \qquad U^{(5)}(w, H, t) \geq & \sum_{i=3,4,5} P_{5i}(t, t+h) E^{w,H,t}[U^{(i)}(W_{t+h}, H_{t+h}, t+h)] \\
& + \sum_{i=0,2} P_{5i}(t, t+h) E^{w,H,t}[U^{(i)}(W_{t+h} + H_{t+h}, t+h)],
\end{aligned}
$$

where the notation $E^{w,H,t}$ represents that the expectation is conditioned with respect to $W_t = w, H_t = H$.

With sufficient differentiability assumption on $U^{(i)}$ $(i = 0, 2, 3, 4, 5)$, we can apply Itô formula to $U^{(i)}(W_t, H_t, t)$ $(i = 3, 4, 5)$ and $U^{(i)}(W_t, t)$ $(i = 0, 2)$ to obtain

$$\begin{aligned} U^{(i)}(W_{t+h}, H_{t+h}, t+h) = U^{(i)}(w, H, t) + \int_t^{t+h} {}_{\mathbf{0}}\mathcal{L}^{\pi}_{b_i} U^{(i)}(W_{s-}, H_{s-}, s) ds, \\ + \int_t^{t+h} \left[ \sigma H_{s-} U^{(i)}_H (W_{s-}, H_{s-}, s) + \sigma\pi U^{(i)}_w (W_{s-}, H_{s-}, s) \right] dB_s \\ + \int_t^{t+h} \int_{-1}^{+\infty} [U^{(i)}(W_{s-} + \pi y, H_{s-} + H_{s-} y, s) \\ - U^{(i)}(W_{s-}, H_{s-}, s)](N(dy, ds) - \nu(dy) ds) \end{aligned}$$

and

$$U^{(i)}(W_{t+h}, t+h) = U^{(i)}(w, t) + \int_t^{t+h} {}_{\mathbf{1}}\mathcal{L}^{\pi}_{b_i} U^{i}(W_{s-}, s) ds + \int_t^{t+h} \sigma\pi U^{(i)}_w (W_{s-}, s) dB_s.$$

In the above equation, $b_0 = 0$. Substituting these expressions into Equation (31), rearranging terms, dividing both sides by $h$, letting $h \to 0$, and recalling that as $h \to 0$: $P_{55}(t, t+h) \to 1$, $P_{5j}(t, t+h) \to 0, j = 0, 2, 3, 4$, and

$$\frac{P_{5j}(t, t+h)}{h} \to \lambda_{5j}, j = 0, 2, 3, 4, \quad \sum_{j=2}^{5} P_{5j}(t, t+h) + P_{50}(t, t+h) = 1,$$

we obtain

$$\begin{aligned} (32) \qquad 0 \;\geq\; & {}_{\mathbf{0}}\mathcal{L}^{\pi}_{b_5} U^{(5)}(w, H, t) + \sum_{i=3,4} \lambda_{5i}(t)[U^{(i)}(w, H, t) - U^{(5)}(w, H, t)] \\ & + \sum_{i=0,2} \lambda_{5i}(t)[U^{(i)}(w + H, t) - U^{(5)}(w, H, t)] = 0. \end{aligned}$$

Finally, along the optimal path $\pi = \pi^*$, equality holds in (31), and therefore in (32). Thus, we have derived the Equation (7). With similar arguments, we can show the Equations (8)–(10) hold.

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