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SUPERLINEAR CONVERGENCE FOR CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS WITH APPLICATIONS

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ABSTRACT. We have developed generalized quasilinear method for Caputo fractional differential equations of order q when $0 < q < 1$, using coupled lower and upper solutions. The sequences we obtain are solutions of two linear system of fractional differential equations. In addition, we develop a method which combines generalized quasilinearization and generalized monotone method. In this method, the sequences are solutions of scalar equations. The method yields superlinear convergence. As an application, this mixed method yields better results than generalized monotone method in the numerical computation of coupled lower and upper solutions to any desired interval.

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1. INTRODUCTION

Nonlinear problems (nonlinear dynamic systems) occur naturally as mathematical models in many branches of science, engineering, finance, economics, etc. So far, in literature, most models are differential equations with integer derivative. However, the qualitative and quantitative study of fractional differential and integral equations has gained importance recently due to its applications. See [1, 3, 5, 6, 7, 8] for details. Generalized quasilinearisation method can be applied to differential equations when the forcing function is the sum of convex and concave functions. See [4] for details for generalized quasilinearisation method. Generalized quasilinearisation method for Caputo fractional differential equations using natural lower and upper solutions are developed in [5]. In this work, we extend the generalized quasilinearisation method for Caputo fractional differential equations using coupled lower and upper solutions. However this method requires the computation of solutions of two linear system of fractional differential equations. In order to simplify the computation we develop a method which requires the computation of two scalar fractional differential equations. We note that this method yields generalized monotone method as a special

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case. In this new method we assume the nonlinear function is the sum of a convex function and a decreasing function. However, the rate of convergence of the sequences are superlinear and not quadratic as in generalized quasilinearization method. This method has an advantage over the generalized monotone method, which yields linear convergence. Numerical computations using the new method would be more efficient than the generalized monotone method (see [9] for numerical results via generalized monotone method).

2. PRELIMINARY AND AUXILIARY RESULTS

In this section, we recall some known results, and develop a few results which are needed for our main results. Initially, we recall some definitions.

Definition 2.1. Caputo fractional derivative of order q is given by equation

$$^cD^q u(t) = \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} u'(s) ds,$$

where $0 < q < 1$.

Also, consider nonlinear Caputo fractional differential equation with initial condition of the form

$$^cD^q u(t) = f(t, u(t)), \quad u(0) = u_0, \tag{2.1}$$

where $f \in C[J \times \mathbb{R}, \mathbb{R}]$ and $J = [0, T]$.

The integral representation of (2.1) is given by equation

$$u(t) = u_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s)) ds, \tag{2.2}$$

where $\Gamma(q)$ is the Gamma function.

The equivalence of (2.1) and (2.2) is established in [3].

In order to compute the solution of linear fractional differential equation with constant coefficients we need Mittag Leffler function.

Definition 2.2. Mittag Leffler function is given by

$$E_{\alpha,\beta}(\lambda(t-t_0)^\alpha) = \sum_{k=0}^{\infty} \frac{(\lambda(t-t_0)^\alpha)^k}{\Gamma(\alpha k + \beta)},$$

where $\alpha, \beta > 0$. Also, for $t_0 = 0$, $\alpha = q$ and $\beta = 1$, we get

$$E_{q,1}(\lambda t^q) = \sum_{k=0}^{\infty} \frac{(\lambda t^q)^k}{\Gamma(qk+1)},$$

where $q > 0$.

Also, consider linear Caputo fractional differential equation,

$$^cD^q u(t) = \lambda u(t) + f(t), \quad u(0) = u_0, \text{ on } J \tag{2.3}$$

where $J = [0, T]$, λ is a constant and $f(t) \in C[J, \mathbb{R}]$.

The solution of (2.3) exists and is unique. The explicit solution of (2.3) is given by

$$u(t) = u_0 E_{q,1}(\lambda t^q) + \int_0^t (t-s)^{q-1} E_{q,q}(\lambda t^q) f(s) ds. \tag{2.4}$$

See [5] for details.

In particular, if $\lambda = 0$, the solution $u(t)$ is given by

$$u(t) = u_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s) ds, \tag{2.5}$$

where $\Gamma(q)$ is the Gamma function.

Note that if ${}^cD^q u(t) \leq \lambda u(t) + f(t)$, $u(0) = u_0$, on J in (2.3), then the conclusions in (2.4) and (2.5) will hold good with $\leq$ in place of equality. This inequalities will be useful in computing the rate of convergence in our main results.

Also we recall known results related to scalar Caputo nonlinear fractional differential equations of the following form:

$$ {}^cD^q u(t) = f(t, u) + g(t, u), \quad u(0) = u_0 \text{ on } J = [0, T], \tag{2.6}$$

where $0 < q < 1$. Here $f, g \in C(J \times \mathbb{R}, \mathbb{R})$, $f(t, u)$ is non-decreasing in u on J and $g(t, u)$ is non-increasing in u on J.

The next result is related to the Reimann-Liouville fractional derivative. For that purpose we define C_p continuous function.

Definition 2.3. Let $p = 1 - q$. A function $\phi(t) \in C[(0, T], \mathbb{R}]$ is a C_p function if $t^p \phi(t) \in C([0, T], \mathbb{R})$. The set of C_p functions is denoted $C_p(J, \mathbb{R})$. Further, given a function $\phi(t) \in C_p(J, \mathbb{R})$ we call the function $t^p \phi(t)$ the continuous extension of $\phi(t)$.

Lemma 2.4. *Let $m(t) \in C_p[J, \mathbb{R}]$ (where $J = [0, T]$) be such that for some $t_1 \in (0, T]$, $m(t_1) = 0$ and $m(t) \leq 0$, on J, then $D^q m(t_1) \geq 0$.*

Proof. See [2, 5] for details. □

However note that we have not assumed $m(t)$ to be Holder continuous as in [5]. The above lemma is true for Caputo derivative also, using the relation ${}^cD^q x(t) = D^q(x(t) - x(0))$ between the Caputo derivative and the Reimann-Liouville derivative. This is the version we will be using to prove our comparison results. The next lemma states the Caputo derivative version.

Lemma 2.5. *Let $m(t) \in C^1[J, \mathbb{R}]$ (where $J = [0, T]$) be such that $m(t) \leq 0$ on J and for $t_1 > 0$, if $m(t_1) = 0$, then ${}^cD^q m(t_1) \geq 0$.*

We recall the following known definitions which are needed for our main results.

Definition 2.6. The functions $v_0, w_0 \in C^1([0,T],\mathbb{R})$ are called natural lower and upper solutions of (2.6) if :

$$^cD^q v_0(t) \leq f(t,v_0) + g(t,v_0), \quad v_0(0) \leq u_0,$$

and

$$^cD^q w_0(t) \geq f(t,w_0) + g(t,w_0), \quad w_0(0) \geq u_0.$$

Definition 2.7. The functions $v_0, w_0 \in C^1([0,T],\mathbb{R})$ are called coupled lower and upper solutions of type I of (2.6) if :

$$^cD^q v_0(t) \leq f(t,v_0) + g(t,w_0), \quad v_0(0) \leq u_0,$$

and

$$^cD^q w_0(t) \geq f(t,w_0) + g(t,v_0), \quad w_0(0) \geq u_0.$$

Consider the fractional differential equation

$$^cD^q x = f(t,u), u(0) = u_0 \tag{2.7}$$

where $f \in C[J \times R^n, R^n]$.

The above equation can be written component wise as

$$^cD^q u_i = f_i(t,u_i,[u]_{pi},[u]_{qi}), u_i(0) = u_{0i}, \tag{2.8}$$

such that $pi + qi = N - 1$.

Definition 2.8 (Mixed quasimonotone property)**.** A function $f(t,u)$ is said to possess a mixed quasimonotone property if for each i, $f_i(t,u_i,[u]_{pi},[u]_{qi})$ is monotone nondecreasing in $[u]_{pi}$ components and monotone nonincreasing in $[u]_{qi}$ components.

Definition 2.9. A pair of functions $v_i, w_i \in C^1[J,\mathbb{R}^n]$, for $i = 1,2,\ldots,N$, are called coupled upper and lower solutions of (2.8) if the following inequalities hold good.

$$\left.\begin{aligned} ^cD^q v_i &\leq f_i(t,v_i,[v]_{pi},[w]_{qi}), \quad v_i(0) \leq u_{0i}, \\ ^cD^q w_i &\geq f_i(t,w_i,[w]_{pi},[v]_{qi}), \quad w_i(0) \leq u_{0i}, \end{aligned}\right\} \tag{2.9}$$

for $i = 1,2,\ldots,N$.

Definition 2.10. A pair of functions $v_i, w_i \in C^1[J,\mathbb{R}^n]$, for $i = 1,2,\ldots,N$, are called Muller's type of coupled upper and lower solutions of (2.8) if the following inequalities hold good.

$$\left.\begin{aligned} ^cD^q v_i &\leq f_i(t,\sigma), \ \ \forall\, \sigma \mid v_i(t) = \sigma_i \text{ and } v(t) \leq \sigma \leq w(t), \\ ^cD^q w_i &\geq f_i(t,\sigma), \ \ \forall\, \sigma \mid w_i(t) = \sigma_i \text{ and } v(t) \leq \sigma \leq w(t), \end{aligned}\right\} \tag{2.10}$$

for $i = 1,2,\ldots,N$.

The following theorem provides the existence of a solution to (2.7) via coupled lower and upper solutions. Also we define $\Omega = \{(t,u) : v \leq u \leq w, t \in J\}$.

Theorem 2.11. *Let $v, w \in C^1[J, \mathbb{R}^n]$ be coupled lower and upper solutions of (2.7) and $f \in C[\Omega, \mathbb{R}^n]$. If $f(t,u)$ possesses a mixed quasimonotone property, then there exists a solution $u(t)$ of (2.7) such that $v(t) \le u(t) \le w(t)$ on J.*

Proof. If $f(t,u)$ possesses a mixed quasimonotone property, the existence of coupled lower and upper solutions of the form (2.9) imply the existence of coupled lower and upper solutions of the form (2.10). Hence we prove the next theorem which will suffice the proof of Theorem 2.11. □

Theorem 2.12. *Let $v(t), w(t) \in C^1[J, \mathbb{R}^n]$ and $f \in C(\Omega, \mathbb{R}^n)$, where $v(t) \le w(t)$ on J and such that $v(t)$ and $w(t)$ are coupled lower and upper solutions of the form (2.10) for (2.7). Then there exists a solution $u \in C^1[J, \mathbb{R}^n]$ of (2.7) such that $v(t) \le u(t) \le w(t)$ on J provided $v(0) \le u_0 \le w(0)$.*

Proof. Consider the function μ defined by $\mu_i(t,u) = \max\{v_i(t), \min\{u_i(t), w_i(t)\}\}$ for each i. Note that $v(t) \le \mu(t,u) \le w(t)$ on Ω and $f(t, \mu(t,u))$ defines a continuous extension of f to $J \times \mathbb{R}^n$, which is also bounded since f is bounded on Ω. Therefore ${}^cD^q u = f(t, \mu(t,u))$ has a solution u on J with $u(0) = u_0$. Let us show that $v(t) \le u(t) \le w(t)$ and hence solution of (2.7).

For $\epsilon > 0$, consider $v_{\epsilon,i}(t) = v_i(t) - \epsilon(1 + \frac{t^q}{\Gamma(1+q)})$ and $w_{\epsilon,i}(t) = w_i(t) + \epsilon(1 + \frac{t^q}{\Gamma(1+q)})$. We have, $v_{\epsilon,i}(0) = v_i(0) - \epsilon < v_i(0) < u_i(0)$. Also $w_{\epsilon,i}(0) = w_i(0) + \epsilon > w_i(0) > u_i(0)$. Therefore $v_{\epsilon,i}(0) < u_i(0) < w_{\epsilon,i}(0)$.

Our aim is to prove that the solution of (2.7) is such that $v_{\epsilon,i}(t) < u_i(t) < w_{\epsilon,i}(t)$ on J. If the conclusion is not true, $\exists\, t_1 \in J$ such that $v_{\epsilon,j}(t) < u_j(t) < w_{\epsilon,j}(t)$ on $[0, t_1)$ and either $u_j(t_1) = v_{\epsilon,j}(t_1)$ or $u_j(t_1) = w_{\epsilon,j}(t_1)$. In addition $v_i(t_1) \le \mu_i(t_1, u_i(t_1)) \le w_i(t_1)$. First let us consider the case when $u_j(t_1) = v_{\epsilon,j}(t_1)$. Then by Lemma 2.4 we have ${}^cD^q(v_{\epsilon,j} - u_j)|_{t=t_1} \ge 0$.

Now ${}^cD^q v_{\epsilon,j}(t_1) < {}^cD^q v_j(t_1) \le f_j(t_1, v_j(t_1), [\mu(t_1, u(t_1))]_{pj}, [\mu(t_1, u(t_1))]_{qj}) = {}^cD^q u_j(t_1)$ which is a contradiction. Hence $v_{\epsilon,j}(t) < u_j(t)$. Similarly if $u_j(t_1) = w_{\epsilon,j}(t_1)$, we can get a contradiction. Therefore $v_{\epsilon,j}(t) < u_j(t) < w_{\epsilon,j}(t)$. Now as $\epsilon \longrightarrow 0$ we get, $v(t) \le u(t) \le w(t)$. This proves $\mu_j(t, u_j) = u_j$ by definition of $\mu_j(t, u_j)$, and hence the solution of (2.7). This concludes the proof. □

We prove equicontinuity for a sequence of bounded functions which is needed for our main results. The next result is precisely this. The proof is on the same lines as in [5].

Theorem 2.13. *Let $v_n(t)$ be a family of continuous functions on $[0,T]$, for each $n > 0$, where ${}^cD^q v_n(t) = f(t, v_n(t))$, $v_n(0) = v_0$ and $|f(t, v_n(t))| \le M$ for $0 < t < T$. Then the family $v_n(t)$ is equicontinuous on $[0,T]$.*

Proof. For $0 \le t_1 \le t_2$, consider

$$\begin{aligned}
|v_n(t_1) - v_n(t_2)| &= |\frac{1}{\Gamma(q)} \int_0^{t_1} (t_1 - s)^{q-1} f(s, v_n(s)) ds \\
&\quad - \frac{1}{\Gamma(q)} \int_0^{t_2} (t_2 - s)^{q-1} f(s, v_n(s)) ds| \\
&= |\frac{1}{\Gamma(q)} \int_0^{t_1} [(t_1 - s)^{q-1} - (t_2 - s)^{q-1}] f(s, v_n(s)) ds \\
&\quad - \frac{1}{\Gamma(q)} \int_{t_1}^{t_2} (t_2 - s)^{q-1} f(s, v_n(s)) ds| \\
&\le \frac{M}{\Gamma(q)} \int_0^{t_1} |[(t_1 - s)^{q-1} - (t_2 - s)^{q-1}]| ds + \frac{M}{\Gamma(q)} \int_{t_1}^{t_2} |(t_2 - s)^{q-1}| ds \\
&= \frac{M}{\Gamma(q)} \int_0^{t_1} [(t_1 - s)^{q-1} - (t_2 - s)^{q-1}] ds + \frac{M}{\Gamma(q)} \int_{t_1}^{t_2} |(t_2 - s)^{q-1}| ds \\
&= \frac{M}{\Gamma(q+1)} [t_1^q - t_2^q + 2(t_2 - t_1)^q] \\
&\le \frac{M}{\Gamma(q+1)} [2(t_2 - t_1)^q],
\end{aligned}$$

provided $|t_2 - t_1| < \delta = [\frac{\epsilon\Gamma(q+1)}{2M}]^{1/q}$. This completes the proof. □

Theorem 2.14. *Let $v, w \in C[J, \mathbb{R}]$ and $f(t, u), g(t, u) \in C[J \times \mathbb{R}, \mathbb{R}]$. Suppose that*

$$(2.11) \qquad {}^cD^q v \le f(t, u) + g(t, w) \text{ and } {}^cD^q w \ge f(t, w) + g(t, v),$$

where $f(t, u_1) - f(t, u_2) \le L(u_1 - u_2)$ and $g(t, u_1) - g(t, u_2) \ge -L(u_1 - u_2)$ for $u_1 \ge u_2$. Then $v(0) \le w(0)$ implies $v(t) \le w(t)$.

Proof. Initially we prove when one of the inequalities in (2.11) is strict. Then we show $v(t) < w(t)$ on J. Let $v(0) < w(0)$. If the conclusion is not true $\exists\, t_1 > 0$, such that $v(t_1) = w(t_1)$ and $v(t) < w(t)$ on $[0, t_1)$. Setting $m(t) = v(t) - w(t)$ and using Lemma(2.4) we get, ${}^cD^q w(t_1) \ge^c D^q w(t_1)$. From the hypothesis we have $f(t, v(t_1)) + g(t, w(t_1)) > {}^cD^q v(t_1) \ge {}^cD^q w(t_1) \ge f(t, w(t_1) + g(t, v(t_1)) = f(t, v(t_1)) + g(t, w(t_1))$, which is a contradiction.

In order to prove the result for non-strict inequalities we define v_ϵ and w_ϵ as follows: $v_\epsilon = v - \epsilon E_{q,1}(3Lt^q)$ and $w_\epsilon = w + \epsilon E_{q,1}(3Lt^q)$, where $\epsilon > 0$.

We can see $v_\epsilon(0) \le w_\epsilon(0)$. Consider ${}^cD^q v_\epsilon \le^c D^q v - 3L\epsilon E_{q,1}(3Lt^q) = f(t, v) + g(t, w) - 3L\epsilon E_{q,1}(3Lt^q) = f(t, v) - f(t, v_\epsilon) + g(t, w) - g(t, w_\epsilon) + f(t, v_\epsilon) + g(t, w_\epsilon) - 3L\epsilon E_{q,1}(3Lt^q) \le L(v - v_\epsilon) + L(w_\epsilon - w) - 3L\epsilon E_{q,1}(3Lt^q) \le f(t, v_\epsilon) + g(t, w_\epsilon) - L\epsilon E_{q,1}(3Lt^q) < f(t, v_\epsilon) + g(t, w_\epsilon)$. Similarly we can show that ${}^cD^q w_\epsilon > f(t, w_\epsilon) + g(t, v_\epsilon)$. Using the strict inequality result, we have $v_\epsilon(t) < w_\epsilon(t)$ on J. As $\epsilon \longrightarrow 0$ we get $v(t) \le w(t)$ on J. □

3. MAIN RESULTS

In this section we develop two results relative to Caputo fractional differential equation of the form

$$^cD^q u(t) = f(t,u) + g(t,u), \quad u(0) = u_0, \tag{3.1}$$

where $f(t,u), g(t,u) \in C[J \times \mathbb{R}, \mathbb{R}]$. In the first result we consider $f(t,u)$ is convex in u and $g(t,u)$ is concave in u for each $t \in J$. We develop monotone sequences which are solutions of two linear system of Caputo fractional differential equations which converge uniformly and monotonically to the unique solution of the nonlinear Caputo fractional differential equation. Further the rate of convergence is quadratic. In the second result we assume $f(t,u)$ is convex in u and $g(t,u)$ is non increasing in u for each $t \in J$. In this case also we develop monotone sequences which converge uniformly and monotonically to coupled minimal maximal solutions. The elements of these sequences are solutions of scalar linear Caputo fractional differential equations. Further if $g_u(t,u)$ exists the sequences converge to the unique solution. Here the rate of convergence is superlinear. We note that when $g(t,u) = 0$ we have quadratic convergence and when $f(t,u) = 0$ we have linear convergence.

The first result we provide is related to the generalized quasilinearization method of (3.1) using coupled lower and upper solutions of type I.

Theorem 3.1. *Assume that*

(i) $\alpha_0, \beta_0 \in C^1[J, \mathbb{R}]$ *with* $\alpha_0 \leq \beta_0$ *on* J,

(ii) $f, g \in C[\Omega, \mathbb{R}], f_u, g_u, f_{uu}, g_{uu}$ *exist, are continuous and satisfy* $f_{uu}(t,u) \geq 0$, $g_{uu}(t,u) \leq 0$ *for* $(t,u) \in \Omega$,

(iii) $g_u(t,u) \leq 0$ *on* Ω. *Then there exit monotone sequences* $\{\alpha_n(t)\}, \{\beta_n(t)\}$ *which converge uniformly to the unique solution of* (3.1) *and the convergence is quadratic.*

Proof. The assumption that $f_{uu}(t,u) \geq 0, g_{uu}(t,u) \leq 0$ yield the following inequalities

$$f(t,u) \geq f(t,v) + f_u(t,v)(u-v), \tag{3.2}$$

$$g(t,u) \leq g(t,v) + g_u(t,v)(u-v), \tag{3.3}$$

for $u \geq v$.

From (ii) we have that f and g are Lipschitz. So choose $L_1, L_2 > 0$ such that for any $v_0(t) \leq u_2 \leq u_1 \leq w_0(t)$, $t \in J$

$$|f(t,u_1) - f(t,u_2)| \leq L_1|u_1 - u_2|,$$
$$|g(t,u_1) - g(t,u_2)| \leq L_2|u_1 - u_2|.$$

Consider the system of fractional IVP

$$(3.4)\quad {}^cD^q u \equiv F(t,\alpha_0,\beta_0;u,v) \equiv f(t,\alpha_0)+f_u(t,\alpha_0)(u-\alpha_0)+g(t,\beta_0)+g_u(t,\alpha_0)(v-\beta_0), \quad u(0)=u_0,$$

$$(3.5)\quad {}^cD^q v \equiv G(t,\alpha_0,\beta_0;v,u) \equiv f(t,\beta_0)+f_u(t,\alpha_0)(v-\beta_0)+g(t,\alpha_0)+g_u(t,\alpha_0)(u-\alpha_0), \quad v(0)=u_0.$$

We will show that α_0 and β_0 are coupled lower and upper solutions of the system (3.4) and (3.5). Equation (3.4) implies,

$$(3.6)\qquad {}^cD^q\alpha_0 \le f(t,\alpha_0)+g(t,\beta_0)\equiv F(t,\alpha_0,\beta_0;\alpha_0,\beta_0).$$

Using (3.5), (3.2) and (3.3), we have

$$\begin{aligned}(3.7)\qquad {}^cD^q\beta_0 &\ge f(t,\beta_0)+g(t,\alpha_0)\\ &\ge f(t,\alpha_0)+f_u(t,\alpha_0)(\beta_0-\alpha_0)+g(t,\beta_0)+g_u(t,\alpha_0)(\alpha_0-\beta_0)\\ &\equiv F(t,\alpha_0,\beta_0;\beta_0,\alpha_0).\end{aligned}$$

Again from (3.4) and using (3.2), (3.3), we obtain

$$\begin{aligned}(3.8)\qquad {}^cD^q\alpha_0 &\le f(t,\alpha_0)+g(t,\beta_0)\\ &\le f(t,\beta_0))+f_u(t,\alpha_0)(\alpha_0-\beta_0)+g(t,\alpha_0)+g_u(t,\alpha_0)(\beta_0-\alpha_0)\\ &\equiv G(t,\alpha_0,\beta_0;\alpha_0,\beta_0).\end{aligned}$$

Equation (3.5) implies

$$(3.9)\qquad {}^cD^q\beta_0 \ge f(t,\beta_0)+g(t,\alpha_0)\equiv G(t,\alpha_0,\beta_0;\beta_0,\alpha_0).$$

From (3.6) and (3.7) and using Theorem 2.12 there exists α_1, such that $\alpha_0 \le \alpha_1 \le \beta_0$. From (3.8) and (3.9) and using Theorem 2.12 there exists β_1, such that $\alpha_0 \le \beta_1 \le \beta_0$. Now we will show $\alpha_1 \le \beta_1$ on J. Using (3.2) and (3.3), with $g_{uu} \le 0$, and $\beta_1 \le \beta_0$ it follows

$$\begin{aligned}(3.10)\qquad {}^cD^q\alpha_1 &= F(t,\alpha_0,\beta_0;\alpha_1,\beta_1)\\ &= f(t,\alpha_0)+f_u(t,\alpha_0)(\alpha_1-\alpha_0)+g(t,\beta_0)+g_u(t,\alpha_0)(\beta_1-\beta_0)\\ &\le f(t,\alpha_1)+g(t,\beta_1)+g_u(t,\beta_1)(\beta_0-\beta_1)+g_u(t,\alpha_0)(\beta_1-\beta_0)\\ &\le f(t,\alpha_1)+g(t,\beta_1)+[g_u(t,\beta_1)-g_u(t,\alpha_0)](\beta_0-\beta_1)\\ &\le f(t,\alpha_1)+g(t,\beta_1).\end{aligned}$$

Using (3.2) and (3.3), with $f_{uu} \ge 0$, and $\beta_1 \le \beta_0$, we obtain

$$\begin{aligned}(3.11)\qquad {}^cD^q\beta_1 &= G(t,\alpha_0,\beta_0;\beta_1,\alpha_1)\\ &= f(t,\beta_0)+f_u(t,\alpha_0)(\beta_1-\beta_0)+g(t,\alpha_0)+g_u(t,\alpha_0)(\alpha_1-\alpha_0)\end{aligned}$$

$$\begin{aligned}
&\geq f(t,\beta_1) + f_u(t,\beta_1)(\beta_0 - \beta_1) + f_u(t,\alpha_0)(\beta_1 - \beta_0) + g(t,\alpha_1)\\
&\leq f(t,\beta_1) + g(t,\alpha_1) + [f_u(t,\beta_1) - f_u(t,\alpha_0)](\beta_0 - \beta_1)\\
&\leq f(t,\beta_1) + g(t,\alpha_1).
\end{aligned}$$

From (3.10), (3.11) and Theorem 2.14 we have $\alpha_1 \leq \beta_1$. This proves that $\alpha_0 \leq \alpha_1 \leq \beta_1 \leq \beta_0$.

Consider the two system of fractional IVP

$$^cD^q u = F(t,\alpha_1,\beta_1;u,v), u(0) = u_0, \tag{3.12}$$

$$^cD^q v = G(t,\alpha_1,\beta_1;v,u), v(0) = u_0, \tag{3.13}$$

where F and G are defined as earlier.

Next we will prove that α_1 and β_1 are the coupled lower and upper solutions of (3.12) and (3.13). From equation (3.10) it follows

$$^cD^q \alpha_1 \leq f(t,\alpha_1) + g(t,\beta_1) \equiv F(t,\alpha_1,\beta_1;\alpha_1,\beta_1). \tag{3.14}$$

From (3.11), (3.12) and using (3.2) we obtain

$$\begin{aligned}
^cD^q \beta_1 &\geq f(t,\beta_1) + g(t,\alpha_1)\\
&\geq f(t,\alpha_1) + f_u(t,\alpha_1)(\beta_1 - \alpha_1) + g(t,\beta_1) + g_u(t,\alpha_1)(\alpha_1 - \beta_1)\\
&\equiv F(t,\alpha_1,\beta_1;\beta_1,\alpha_1).
\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}
^cD^q \alpha_1 &\leq f(t,\alpha_1) + g(t,\beta_1)\\
&\leq f(t,\beta_1) + f_u(t,\alpha_1)(\alpha_1 - \beta_1) + g(t,\alpha_1) + g_u(t,\alpha_1)(\beta_1 - \alpha_1)\\
&\equiv G(t,\alpha_1,\beta_1;\alpha_1,\beta_1).
\end{aligned} \tag{3.16}$$

From equation (3.11) it follows

$$^cD^q \beta_1 \geq f(t,\beta_1) + g(t,\alpha_1) \equiv F(t,\alpha_1,\beta_1;\beta_1,\alpha_1). \tag{3.17}$$

This proves α_1 and β_1 are coupled lower and upper solutions of (3.12) and (3.13). By Theorem 2.12, $\exists$ unique solutions (α_2,β_2) of (3.12) and (3.13) such that $\alpha_1 \leq \alpha_2$, $\beta_2 \leq \beta_1$ on J.

Now we have,

$$\begin{aligned}
^cD^q \alpha_2 &= F(t,\alpha_1,\beta_1;\alpha_2,\beta_2),\\
^cD^q \beta_2 &= G(t,\alpha_1,\beta_1;\beta_2,\alpha_2).
\end{aligned}$$

Arguing as before, we can get

$$^cD^q \alpha_2 \leq f(t,\alpha_2) + g(t,\beta_2),$$

$$^cD^q \beta_2 \geq f(t,\beta_2) + g(t,\alpha_2),$$

which yields, $\alpha_2(t) \le \beta_2(t)$ on J. Therefore, $\alpha_0 \le \alpha_1 \le \alpha_2 \le \beta_2 \le \beta_1 \le \beta_0$ on J. The process can be continued successively to arrive at

$$\alpha_0 \le \alpha_1 \le \cdots \le \alpha_n \le \beta_n \le \cdots \le \beta_1 \le \beta_0 \quad \text{on } J. \tag{3.18}$$

Here, $\alpha_n(t)$, $\beta_n(t)$ are unique solutions of the system of fractional IVP

$${}^cD^q\alpha_{n+1} = F(t, \alpha_n, \beta_n; \alpha_{n+1}, \beta_{n+1}); \alpha_{n+1}(0) = u_0, \tag{3.19}$$

$${}^cD^q\beta_{n+1} = G(t, \alpha_n, \beta_n; \beta_{n+1}, \alpha_{n+1}); \beta_{n+1}(0) = u_0. \tag{3.20}$$

It can be shown that $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ are uniformly bounded on J. Using Theorem 2.13 it can be shown that $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ are equicontinuous on J. Hence by Ascoli-Arzela theorem, there exists subsequences that converges uniformly on J. Since the sequence is monotone the entire sequences $\{\alpha_n\}$, $\{\beta_n\}$ converge uniformly and monotonically to α, β respectively.

Using (3.19) and (3.20), α and β are coupled lower and upper solutions of (3.1) on J. Hence α and β will satisfy ${}^cD^q\alpha = f(t, \alpha) + g(t, \beta)$, ${}^cD^q\beta = f(t, \beta) + g(t, \alpha)$. This proves $\alpha \le \beta$. We will use Lipshitz condition on f and g and Theorem 2.14 to show that $\beta \le \alpha$. Hence this proves $\alpha \equiv \beta \equiv u$, where u is the unique solution of (3.1) on J.

Next we prove that the rate of convergence of these sequences is quadratic. For this purpose we set $p_n(t) = u(t) - \alpha_n(t)$ with $p_n(0) = 0$ and $q_n(t) = u(t) - \alpha_n(t)$ with $q_n(0) = 0$. Then using (3.19), (3.20), the hypothesis (ii) and the mean value theorem, we obtain the following differential inequality:

$$\begin{aligned}
{}^cD^qp_n(t) &= {}^c D^q u(t) - {}^cD^q\alpha_n(t) \\
&= f(t, u) + g(t, u) - [f(t, \alpha_{n-1}) + f_u(t, \alpha_{n-1})(\alpha_n - \alpha_{n-1}) + g(t, \beta_{n-1}) \\
&\quad + g_u(t, \alpha_{n-1})(\beta_n - \beta_{n-1})] \\
&= f_u(t, \xi)p_{n-1} - g_u(t, \sigma)q_{n-1} - f_u(t, \alpha_{n-1})(-p_n + p_{n-1}) \\
&\quad - g_u(t, \alpha_{n-1})(q_n - q_{n-1}) \\
&\le [f_u(t, \xi)p_{n-1} - f_u(t, \alpha_{n-1})p_{n-1}] + f_u(t, \alpha_{n-1})p_n + g_u(t, \alpha_{n-1})q_{n-1} \\
&\quad - g_u(t, \sigma)q_{n-1} - g_u(t, \alpha_{n-1})q_n \\
&\le f_{uu}(t, \xi_1)p_{n-1}^2 + f_u(t, \alpha_{n-1})p_n + g_{uu}(t, \sigma_1)(\alpha_{n-1} - \beta_{n-1}) - g_u(t, \alpha_{n-1})q_n \\
&\le f_{uu}(t, \xi_1)p_{n-1}^2 + g_{uu}(t, \sigma_1)(p_{n-1} + q_{n-1})q_{n-1} + f_u(t, \alpha_{n-1})p_n - g_u(t, \alpha_{n-1})q_n \\
&\le N_1p_{n-1}^2 - g_{uu}(t, \sigma_1)(\beta_{n-1} - \alpha_{n-1})q_{n-1} + M_1p_n + M_2q_n \\
&\le N_1p_{n-1}^2 - N_2(p_{n-1} + q_{n-1})q_{n-1} + M_1p_n + M_2q_n \\
&\le N_1p_{n-1}^2 + N_2(q_{n-1}^2 + p_{n-1}q_{n-1}) + M_1p_n + M_2q_n \\
&\le N_1p_{n-1}^2 + N_2q_{n-1}^2 + N_2p_{n-1}q_{n-1} + M_1p_n + M_2q_n
\end{aligned}$$

$$\leq N_1 p_{n-1}^2 + N_2 q_{n-1}^2 + N_2 \frac{p_{n-1}^2 + q_{n-1}^2}{2} + M_1 p_n + M_2 q_n.$$

So we have

$$^cD^q p_n(t) \leq (N_1 + \frac{1}{2}N_2)p_{n-1}^2 + \frac{3}{2}N_2 q_{n-1}^2 + M_1 p_n + M_2 q_n, \tag{3.21}$$

where $\alpha_{n-1} \leq \xi \leq u$, $u \leq \sigma \leq \beta_{n-1}$, $\alpha_{n-1} \leq \xi_1 \leq u$, $\alpha_{n-1} \leq \sigma_1 \leq \beta_{n-1}$, and $\max|f_{uu}(t,u)| \leq N_1$, $\max|g_{uu}(t,u)| \leq N_2$, $\max|f_u(t,u)| \leq M_1$, $\max|g_u(t,u)| \leq M_2$. Similarly,

$$\begin{aligned}
^cD^q q_n(t) &= ^cD^q \beta_n(t) - ^cD^q u(t)\\
&= [f(t,\beta_{n-1}) + f_u(t,\alpha_{n-1})(\beta_n - \beta_{n-1}) + g(t,\alpha_{n-1})\\
&\quad + g_u(t,\alpha_{n-1})(\alpha_n - \alpha_{n-1})] - f(t,u) - g(t,u)\\
&= f_u(t,\xi)q_{n-1} - g_u(t,\sigma)p_{n-1} + f_u(t,\alpha_{n-1})(q_n - q_{n-1})\\
&\quad + g_u(t,\alpha_{n-1})(-p_n + p_{n-1})\\
&\leq [f_u(t,\xi) - f_u(t,\alpha_{n-1})]q_{n-1} + g_u(t,\sigma)p_n + g_u(t,\alpha_{n-1})p_{n-1}\\
&\quad + f_u(t,\alpha_{n-1})q_n - g_u(t,\alpha_{n-1})p_n\\
&\leq f_{uu}(t,\xi_1)(\beta_{n-1} - \alpha_{n-1})q_{n-1} - g_{uu}(t,\sigma_1)p_{n-1}^2 + f_u(t,\alpha_{n-1})q_n - g_u(t,\alpha_{n-1})p_n\\
&\leq N_1(p_{n-1} + q_{n-1})q_{n-1} + N_2 p_{n-1}^2 + M_1 q_n + M_2 p_n\\
&\leq N_1 p_{n-1} q_{n-1} + N_1 q_{n-1}^2 + N_2 p_{n-1}^2 + M_1 q_n + M_2 p_n\\
&\leq N_1(\frac{p_{n-1}^2 + q_{n-1}^2}{2}) + N_1 q_{n-1}^2 + N_2 p_{n-1}^2 + M_1 q_n + M_2 p_n.
\end{aligned}$$

So, we have

$$^cD^q q_n(t) \leq \frac{3}{2}N_1 q_{n-1}^2 + (\frac{1}{2}N_1 + N_2)p_{n-1}^2 + M_1 q_n + M_2 p_n, \tag{3.22}$$

where $u \leq \xi \leq \beta_{n-1}$, $\alpha_{n-1} \leq \sigma \leq u$, $\alpha_{n-1} \leq \xi_1 \leq \beta_{n-1}$, $\alpha_{n-1} \leq \sigma_1 \leq u$, and $\max|f_{uu}(t,u)| \leq N_1$, $\max|g_{uu}(t,u)| \leq N_2$, $\max|f_u(t,u)| \leq M_1$, $\max|g_u(t,u)| \leq M_2$. Adding (3.21) and (3.22), we have

$$\begin{aligned}
^cD^q(p_n + q_n) &\leq (N_1 + N_2)p_{n-1}^2 + 3N_2 q_{n-1}^2 + (M_1 + M_2)p_n + (M_1 + M_2)q_n\\
&\leq (N_1 + N_2)p_{n-1}^2 + 3N_2 q_{n-1}^2 + (M_1 + M_2)(p_n + q_n).
\end{aligned}$$

Using the corresponding inequality estimates of (2.4), we have

$$\begin{aligned}
p_n + q_n &\leq 0 + \int_0^t (t-s)^{q-1} E_{q,q}((M_1 + M_2)(t-s)^q)[(N_1 + N_2)p_{n-1}^2 + 3N_2 q_{n-1}^2]ds\\
&\leq [|(N_1 + N_2)p_{n-1}^2| + |3N_2 q_{n-1}^2|] \int_0^t (t-s)^{q-1} E_{q,q}((M_1 + M_2)(t-s)^q)ds\\
&\leq [|(N_1 + N_2)p_{n-1}^2| + |3N_2 q_{n-1}^2|] \frac{E_{q,1}((M_1 + M_2)t^q)}{(M_1 + M_2)}\\
&\leq K|(N_1 + N_2)p_{n-1}^2| + K|3N_2 q_{n-1}^2|,
\end{aligned}$$

where $K = \frac{E_{q,1}((M_1+M_2)t^q)}{(M_1+M_2)}$

$$\leq M|p_{n-1}^2| + N|q_{n-1}^2|,$$

where $M = K|(N_1 + N_2)|, N = K|3N_2|$

$$\leq L(|p_{n-1}^2| + |q_{n-1}^2|),$$

where $L = \max\{M, N\}$

$$\leq L|p_{n-1} + q_{n-1}|^2.$$

Using this we have, $\max_J|p_n + q_n| \leq L \max_J|p_{n-1} + q_{n-1}|^2$, which proves the quadratic convergence. □

The next theorem is proved under the weaker assumption on $g(t, u)$. However, the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in Theorem 3.1 are solutions of two coupled linear system of Caputo fractional differential equations. The computation of the solution of coupled linear system of Caputo fractional differential equations is not easy. In order to compute the solution of the system, we decouple the coupled system. This is achieved by dropping the terms $g_u(t, \alpha_0)(v - \beta_0)$ and $g_u(t, \alpha_0)(u - \beta_0)$ in (3.4) and (3.5) respectively. And we obtain a superlinear convergence of the solution.

Theorem 3.2. *Assume that*

(i) $\alpha_0, \beta_0 \in C^1[J, \mathbb{R}]$ *with* $\alpha_0 \leq \beta_0$ *on* J,

(ii) $f, g \in C[\Omega, \mathbb{R}], f_u, g_u, f_{uu}$ *exist, are continuous and satisfy* $f_{uu}(t, u) \geq 0$ *for* $(t, u) \in \Omega$,

(iii) $g_u(t, u) \leq 0$ *on* Ω. *Then there exit monotone sequences* $\{\alpha_n(t)\}, \{\beta_n(t)\}$ *which converge uniformly to the unique solution of* (3.1) *and the convergence is superlinear.*

Proof. The proof follows on the same lines of Theorem 3.1. Here we consider the decoupled system of fractional IVP of the form

$$(3.23) \qquad {}^cD^q u \equiv F(t, \alpha_0, \beta_0; u, v) \equiv f(t, \alpha_0) + f_u(t, \alpha_0)(u - \alpha_0) + g(t, \beta_0), \quad u(0) = u_0,$$

$$(3.24) \qquad {}^cD^q v \equiv G(t, \alpha_0, \beta_0; v, u) \equiv f(t, \beta_0) + f_u(t, \alpha_0)(v - \beta_0) + g(t, \alpha_0), \quad v(0) = u_0.$$

Similarly as in Theorem 3.1 we obtain sequences $\{\alpha_n\}, \{\beta_n\}$ as solutions of the system of fractional IVP

$$(3.25) \qquad {}^cD^q \alpha_{n+1} = f(t, \alpha_n) + f_u(t, \alpha_n)(\alpha_{n+1} - \alpha_n) + g(t, \beta_n),$$

$$ {}^cD^q\beta_{n+1} = f(t,\beta_n) + f_u(t,\alpha_n)(\beta_{n+1}-\beta_n) + g(t,\alpha_n). \tag{3.26}$$

We can prove $\alpha_0 \le \alpha_1 \le \cdots \le \alpha_n \le \beta_n \le \cdots \le \beta_1 \le \beta_0$ on iJ. Further applying Ascoli-Arzela theorem, we can prove $\alpha_{n+1} \longrightarrow \alpha$ and $\beta_{n+1} \longrightarrow \beta$, uniformly and monotonically on J. Also we can prove α, β are coupled minimal and maximal solutions of (3.1) on J such that $\alpha \le \beta$ on J. Thus α and β satisfy the system

$$\begin{aligned} {}^cD^q\alpha &= f(t,\alpha) + g(t,\beta), \quad \alpha(0) = u_0 \\ {}^cD^q\beta &= f(t,\beta) + g(t,\alpha), \quad \beta(0) = u_0. \end{aligned}$$

Since f and g satisfy one sided Lipschitz condition, using Theorem 2.14 we can prove $\alpha \ge \beta$. This proves $\alpha \equiv \beta \equiv u$, where u is the unique solution of (3.1) on J. In order to prove superlinear convergence we let $p_n(t) = u(t) - \alpha_n(t)$ and $q_n(t) = \beta_n(t) - u(t)$. It is easy to see that $p_n(0) = 0$, $q_n(0) = 0$. On the same lines as in the proof of Theorem 3.1 we can now prove,

$$ {}^cD^q p_n(t) \le N_1 p_{n-1}^2 + M_2 q_{n-1} + M_1 p_n, \tag{3.27}$$

$$ {}^cD^q q_n(t) \le \frac{3}{2}N_1 q_{n-1}^2 + \frac{1}{2}N_1 p_{n-1}^2 + M_2 p_{n-1} + M_1 q_n. \tag{3.28}$$

Adding (3.27) and (3.28), we have

$$\begin{aligned} {}^cD^q(p_n+q_n) &\le \frac{3}{2}N_1(p_{n-1}^2 + q_{n-1}^2) + M_2(p_{n-1}+q_{n-1}) + M_1(p_n+q_n) \\ &\le \frac{3}{2}N_1(p_{n-1}+q_{n-1})^2 + M_2(p_{n-1}+q_{n-1}) + M_1(p_n+q_n). \end{aligned}$$

Using the corresponding inequality estimates of (2.4), we have $p_n + q_n \le L(|(p_{n-1} + q_{n-1})|^2 + |(p_{n-1}+q_{n-1})|)$. Using this we have $\max_J|p_n+q_n| \le \max_J(|(p_{n-1}+q_{n-1})|^2 + |(p_{n-1}+q_{n-1})|)$ which proves superlinear convergence. □

4. **CONCLUSION**

In [9] we have developed numerical method to compute coupled lower and upper solutions to any desired interval using generalized monotone method and natural lower and upper solutions. However the rate of convergence using the generalized monotone method is linear. Now we can use Theorem 3.2 to compute coupled lower and upper solutions to any desired interval using natural lower and upper solutions. Tha advantage of Theorem 3.2 is that the rate of convergence of the sequences is superlinear. Further we can develop Gauss-Seidel method for the sequences in Theorem 3.2 to obtain a faster convergence compared with the superlinear convergence we have already obtained.

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