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POSITIVE SOLUTIONS FOR A $p(t)$-LAPLACIAN THREE POINT BOUNDARY VALUE PROBLEM

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ABSTRACT. We present in this paper new results concerning positive solutions for the $p(t)$-Laplacian multipoint boundary value problem

$$\begin{cases} -(\phi(t,u'(t)))' = f(t,u(t)), & t \in (0,1), \\ u(0) = \alpha u(\eta), & u'(1) = 0, \end{cases}$$

where $\mathbb{R}^+ = [0,+\infty)$, $\alpha, \eta \in (0,1)$, $\phi(t,x) = |x|^{p(t)-2}\, x$, $p \in C\left([0,1],(1,+\infty)\right)$, and $f \in C([0,1]\times \mathbb{R}^+, \mathbb{R}^+)$.

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1. INTRODUCTION

This paper deals with existence of a positive solutions to the $p(t)$-Laplacian three point boundary value problem (bvp for short)

$$\begin{cases} -(\phi(t,u'(t)))' = f(t,u(t)), & t \in (0,1), \\ u(0) = \alpha u(\eta), & u'(1) = 0, \end{cases} \tag{1.1}$$

where $\mathbb{R}^+ = [0,+\infty)$, $\alpha, \eta \in (0,1)$, $\phi(t,x) = |x|^{p(t)-2}\, x$, $p \in C\left([0,1],(1,+\infty)\right)$, and $f \in C\left([0,1] \times \mathbb{R}^+, \mathbb{R}^+\right)$.

By a positive solution to bvp (1.1) we mean a function $u \in C^1([0,1],\mathbb{R}^+)$ with $\phi(t,u'(t)) \in C^1([0,1],\mathbb{R})$ and $u\left(t_0\right) > 0$ for some $t_0 \in [0,1[$ satisfying both the differential equation and the boundary conditions in bvp (1.1).

The operator $-(\phi(t,u'(t)))'$ is known as the unidimensional $p(t)$-Laplacian and becomes the p-Laplacian when $p(t) \equiv p \in (1,+\infty)$. Because of the physical interests (see for example [2], [3], [9] and references cited therein), differential equations involving the $p(t)$-Laplacian have received a great attention in recent years and many

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interesting results have been obtained (see for example [4], [5], [6], [7], [10], [11], [12], [13], and [14]). See that the $p(t)$-Laplacian is inhomogeneous. This makes the study of differential equations involving $p(t)$-Laplacian more complicated. The difficulties encountered when studying various aspects of differential equations involing the $p(t)$-Laplacian are described in [7], [10], [12] and [13].

It is well known that in the case where p is constant and the positivity of the nonlinearity f is guaranteed, a positive solution of the bvp (1.1) is concave (see [1]). This fact makes the use of cone compression and expansion principal possible. However, in the case of general p, a positive solution to bvp (1.1) is not necessarily concave.

To overcome the difficulty caused by the loss of concavity, we have associated with the bvp (1.1), by means of an operator A depending on the function p, an auxiliary problem such that Au is a positive solution to the bvp (1.1) for all positive solution u to the auxiliary problem.

Thus, we present here an existence result for the sublinear case without many difficulties (see Theorem 3.1 and its proof). In the opposite case, an existence result for the superlinear case necessitated the use of a homotopical argument.

2. PRELIMINARIES

First, let us recall some elements related to fixed point index theory. Let X be a real Banch space. For $R > 0$, $B(0,R)$ denotes the ball of radius R centered at 0 in X. A nonempty closed convex subset K of X is said to be an ordered cone if $K \cap (-K) = \{0\}$ and $(tK) \subset K$ for all $t \geq 0$. Let $T : B(0,R) \cap K \to K$ be a compact mapping. The following two lemmas can be found in [8]. They provide fixed point index computations.

Lemma 2.1. *If $Tu \not\geq u$ for all $u \in \partial B(0,R) \cap K$, then $i(T, B(0,R) \cap K, K) = 1$.*

Lemma 2.2. *If $Tu \not\leq u$ for all $u \in \partial B(0,R) \cap K$, then $i(T, B(0,R) \cap K, K) = 0$.*

Throughout this paper, $\psi(t,\cdot)$ denotes the inverse function of $\phi(t,\cdot)$ and we have

$$\psi(t,x) = |x|^{q(t)-2}\, x \text{ where } q(t) = \frac{p(t)}{p(t)-1}.$$

The real numbers, p^-, p^+ are defined by

$$p^- = \min_{t\in[0,1]} p(t), \quad p^+ = \max_{t\in[0,1]} p(t).$$

E will denote the Banach space of all continuous functions defined on $[0,1]$ equipped with the sup-norm denoted by $\|\cdot\|$, and E^+, K and P are cones of E defined by

$$E^+ = \{u \in E : u \geq 0 \ in \ [0,1]\},$$

$$K = \left\{u \in E^{+} : u \text{ is nonincreasing on } [0,1]\right\}$$

and

$$P = \left\{u \in K,\ u(t) \geq (1-t)\left\|u\right\|\right\}.$$

The linear operator $L : E \to E$ is given for $u \in E$ by

$$Lu(t) = \int_t^1 u(s)ds$$

and $F, A_p, T : E^{+} \to E^{+}$ are the operators defined for $u \in E^{+}$ by

$$Fu(t) = f(t, A_p u(t)),$$

$$A_p u(t) = \frac{\alpha}{1-\alpha}\int_0^{\eta} \psi\left(s, u(s)\right) ds + \int_0^t \psi\left(s, u(s)\right) ds$$

and

$$T = LFA_p.$$

Lemma 2.3. *The operator A_p is continuous.*

Proof. Let $(u_n) \subset E^{+}$ be a sequence converging to $u \in E^{+}$. There exists $M > 0$ such that $(u_n) \subset \overline{B}(0, M)$. From the uniform continuity of ψ on the compact $[0,1] \times [-M, M]$ we deduce that for $\epsilon > 0$, there exists $\delta_\epsilon > 0$ such that, for all $s \in [0,1]$ and all $x, y \in [-M, M]$, $|x-y| \leq \delta_\epsilon$ implies $\left|\psi\left(s,x\right) - \psi\left(s,y\right)\right| \leq \epsilon$.

Thus for n large enough, we have for all $t \in [0,1]$

$$|A_p u_n(t) - A_p u(t)| \leq \int_0^t \left|\psi\left(s, u_n(s)\right) - \psi\left(s, u(s)\right)\right| ds \leq \epsilon.$$

This means that $\lim A_p u_n = A_p u$ and A_p is continuous □

Lemma 2.4. *We have that*

1. *T is completly continuous,*
2. *$T\left(E^{+}\right) \subset K$ and*
3. *If $u \in E^{+}$ is a fixed point of T then $v = A_p u$ is a positive solution to bvp* (1.1).

Proof.
1. Since all the operators $L,\ F$ and A_p are continuous, $T = LFA_p$ is continuous. Moreover the continuity of the functions ψ and f make the operators F and A_p bounded (map bounded sets into bounded sets) and Ascoli-Arzela Theorem makes L a compact operator. Thus, $T = LFA_p$ is completely continuous.
2. Let $u \in E^{+}$ and $v = Tu$. We have $v'\left(t\right) = -f\left(t, A_p u\left(t\right)\right) \leq 0$ for all $t \in (0,1)$. This, together with $v(1) = 0$, leads to v is a nonnegative and nonincreasing function on $[0,1]$. Namely $v \in K$.

3. If $u \in E^+$ is a fixed point of T, then u satisfies the auxiliary problem

$$\begin{cases} -u'(t) = f(t, A_p u(t)) \ t \in (0,1), \\ u(1) = 0. \end{cases}$$

Let $v = A_p u$. We have for all $t \in (0,1)$,

$$\phi(t, v'(t)) = \phi(t, \psi(t, u(t))) = u(t). \tag{2.1}$$

Then for all $t \in (0,1)$,

$$-(\phi(t, v'(t)))' = f(t, v(t)).$$

Also we have from (2.1), $v'(1) = u(1) = 0$, and from

$$\begin{aligned} v(0) &= A_p u(0) = \frac{\alpha}{1-\alpha} \int_0^\eta (u(s))^{\frac{1}{p(s)-1}} ds, \\ v(\eta) &= \frac{1}{1-\alpha} \int_0^\eta (u(s))^{\frac{1}{p(s)-1}} ds, \end{aligned}$$

we have $v(0) = \alpha v(\eta)$. Thus $v = A_p u$ is a positive solution to bvp (1.1).

This completes the proof of the lemma □

3. MAIN RESULTS

The statements of the main results need to introduce the following notations.

$$\Lambda_1 = \left(\tfrac{1-\alpha}{1-\alpha(1-\eta)}\right)^{p^- -1}, \qquad \Lambda_2 = \tfrac{1}{1-\eta}\left(\tfrac{1-\alpha}{\eta}\right)^{p^- -1},$$
$$\Lambda_3 = \left(\tfrac{1-\alpha}{1-\alpha(1-\eta)}\right)^{p^+ -1}, \qquad \Lambda_4 = \tfrac{1}{1-\eta}\left(\tfrac{1-\alpha}{\eta}\right)^{p^+ -1},$$
$$f_0 = \liminf_{u\to 0}\left(\min_{t\in[\eta,1]} \frac{f(t,u)}{u^{p^- -1}}\right), \quad f^\infty = \limsup_{u\to +\infty}\left(\max_{t\in[0,1]} \frac{f(t,u)}{u^{p^- -1}}\right),$$
$$f^0 = \limsup_{u\to 0}\left(\max_{t\in[0,1]} \frac{f(t,u)}{u^{p^+ -1}}\right), \quad f_\infty = \liminf_{u\to +\infty}\left(\min_{t\in[\eta,1]} \frac{f(t,u)}{u^{p^+ -1}}\right).$$

Let

$$\psi^+(x) = \begin{cases} x^{\frac{1}{p^+-1}} & \text{if } x \le 1, \\ x^{\frac{1}{p^--1}} & \text{if } x \ge 1, \end{cases} \qquad \psi^-(x) = \begin{cases} x^{\frac{1}{p^--1}} & \text{if } x \le 1, \\ x^{\frac{1}{p^+-1}} & \text{if } x \ge 1. \end{cases}$$

Then we have for all $x \ge 0$

$$\psi^-(x) \le \psi(t,x) \le \psi^+(x). \tag{3.1}$$

Thus, the nonincreasing property of the functions in the cone K leads to

$$A_p u(\eta) \ge \frac{\eta}{1-\alpha}\psi^-(u(\eta)), \tag{3.2}$$

$$\|A_p u\| = A_p u(1) \le A_p u(\eta) + \psi^+(u(\eta)), \tag{3.3}$$

and

$$\|A_p u\| = A_p u(1) \leq \frac{1-\alpha(1-\eta)}{1-\alpha}\psi^+(u(0)), \tag{3.4}$$

for all $u \in K$.

Theorem 3.1 (the sublinear case)**.** *Assume that*

$$f^\infty < \Lambda_1 \leq \Lambda_2 < f_0. \tag{3.5}$$

Then bvp (1.1) *admits at least one positive solution.*

Proof. Let $\epsilon > 0$ such that $\Lambda_1 - \epsilon > f^\infty$. There exists $C > 0$ such that for all $t \in [0,1]$ and $u \geq 0$

$$f(t,u) \leq (\Lambda_1 - \epsilon)\, u^{p^- -1} + C. \tag{3.6}$$

Let $R_\infty > \max(C\Lambda_1/\epsilon, 1)$ and $u \in \partial B(0, R_\infty) \cap K$. We have from (3.6) and (3.4)

$$\begin{aligned} Tu(0) &= \int_0^1 f(s, A_p u(s))ds \leq \int_0^1 (\Lambda_1 - \epsilon)(A_p u(s))^{p^- -1} ds + C \\ &\leq (\Lambda_1 - \epsilon)\left(\frac{1-\alpha(1-\eta)}{1-\alpha}\psi^+(u(0))\right)^{p^- -1} + C \\ &= \frac{(\Lambda_1 - \epsilon)}{\Lambda_1} u(0) + C < u(0). \end{aligned}$$

This means that for all $u \in \partial B(0, R_\infty) \cap K$, $Tu \not\geq u$. So we have from Lemma 2.1, $i(T, K \cap B(0, R_\infty), K) = 1$.

Let $\epsilon > 0$ small enough so that $f_0 > (\Lambda_2 + \epsilon)$. There exists $\delta > 0$ such that for all $t \in [\eta, 1]$ and $u \in [0, \delta]$

$$f(t,u) \geq (\Lambda_2 + \epsilon)\, u^{p^- -1}. \tag{3.7}$$

Let $\delta_1 = \min\left(1, \delta, \Lambda_3 \delta^{p^+ -1}\right)$ and $u \in K \cap \partial B(0, \delta_1)$. We have from (3.4), $\|A_p u\| \leq \delta$ and from (3.7),

$$\begin{aligned} Tu(\eta) &= \int_\eta^1 f(s, A_p u(s))ds \\ &> \int_\eta^1 (\Lambda_2 + \epsilon)(A_p u(s))^{p^- -1} ds \\ &\geq (\Lambda_2 + \epsilon)(1-\eta)(A_p u(\eta))^{p^- -1}. \end{aligned}$$

Then from (3.2) follows

$$Tu(\eta) \geq (\Lambda_2 + \epsilon)\Lambda_2^{-1} u(\eta) > u(\eta).$$

This means that for all $u \in K \cap \partial B(0,\delta_1)$, $Tu \nleq u$. So, we deduce from Lemma 2.2 that $i(T, K \cap B(0,\delta_1), K) = 0$.

Finally, we have from additivity and solution properties of the fixed point index

$$i(T, (B(0,R_\infty) \setminus B(0,\delta_1)) \cap K, K) = 1$$

and the operator T admits a positive fixed point u with $\delta_1 < \|u\| < R_\infty$. Thus by Lemma 2.4, $A_p u$ is a positive solution to bvp (1.1) □

In what follows, we let for $u \geq 0$, $g(u) = \gamma u^{p^+-1}$ with $\gamma > \Lambda_4$, and $T_0 = LGA_p$ where $G : E^+ \to E^+$ is defined for $u \in E^+$ by $Gu\,(t) = g\,(u\,(t))$. The following lemma provides a fixed point computation for the operator T_0.

Lemma 3.2. *For all* $R > R^0 = \max\left(\frac{1}{1-\eta}, \Lambda_4\right)$,

$$i(T_0, B(0,R) \cap K, K) = 0.$$

Proof. First, we need to prove that $T_0\,(E^+) \subset P$. Let $u \in E^+$ and $v = T_0 u$. We have that $v \in K$ and $v' = -g\,(A_p u)$ is nonincreasing on $[0,1]$. This means that v is concave and reaches its maximum value at $t = 0$. Then from the concavity of v, we deduce that for all $t \in [0,1]$

$$v(t) \geq (1-t)\,v(0) = (1-t)\,\|v\|\,. \tag{3.8}$$

Now, let $R > R_0$ and $u \in \partial B(0,R) \cap P$. We have from (3.8) and (3.2), for all $t \in [0,\eta]$,

$$u\,(t) \geq u\,(\eta) \geq (1-\eta)\,R > 1,$$

and for all $t \in [\eta, 1]$,

$$A_p u\,(t) \geq A_p u\,(\eta) \geq \frac{\eta}{1-\alpha}\,(u\,(\eta))^{\frac{1}{p^+-1}} \geq \left(\frac{R}{\Lambda_4}\right)^{\frac{1}{p^+-1}} > 1.$$

Thus, we obtain from (3.2),

$$\begin{aligned} T_0 u(\eta) &> \int_\eta^1 \Lambda_4 (A_p u(s))^{p^+-1} ds \\ &\geq \Lambda_4\,(1-\eta)\,(A_p u(\eta))^{p^+-1} \\ &\geq \Lambda_4\,(1-\eta)\left(\frac{\eta}{1-\alpha}\right)^{p^+-1} u(\eta) = u(\eta). \end{aligned}$$

This means that for all $u \in \partial B(0,R) \cap P$, $T_0 u \nleq u$. So by Lemma 2.2, $i(T_0, B(0,R) \cap P, P) = 0$.

Then from the permanence property of the fixed point index, we get $i(T_0, B(0,R) \cap K, K) = 0$ □

Theorem 3.3 (the superlinear case)**.** *Assume that*

$$f^0 < \Lambda_3 \leq \Lambda_4 < f_\infty. \tag{3.9}$$

Then bvp (1.1) *admits a positive solution.*

Proof. Let $\epsilon > 0$ such that $f^0 < (\Lambda_3 - \epsilon)$. There exists $\delta > 0$ such that, for all $t \in [0,1]$ and $u \in [0, \delta]$,

$$f(t,u) \leq (\Lambda_3 - \epsilon)\, u^{p^+-1}.$$

Let $\delta_2 = \min\left(1, \delta, \Lambda_3 \delta^{p^+-1}\right)$ and $u \in \partial B(0, \delta_2) \cap K$. We have from (3.4)

$$\|A_p u\| = A_p u\,(1) \leq \frac{1 - \alpha\,(1-\eta)}{1-\alpha} \delta_2^{p^+-1} \leq \delta.$$

The above estimates together with (3.4) lead to

$$\begin{aligned} Tu(0) &= \int_0^1 f(s, A_p u(s)) ds \leq \int_0^1 (\Lambda_3 - \epsilon)\,(A_p u(s))^{p^+-1}\, ds \\ &\leq (\Lambda_3 - \epsilon)\, \Lambda_3^{-1} u(0) < u(0). \end{aligned}$$

This means that $Tu \ngeq u$ for all $u \in \partial B(0, \delta_2) \cap K$, so, we deduce from Lemma 2.1 that $i(T, K \cap B(0, r_0), K) = 1$.

In order to conclude by the additivity and solution properties of the fixed point index as in the proof of Theorem 3.1, we have to prove the existence of an $R^\infty > \delta_2$ such that

$$i(T, K \cap B(0, R^\infty), K) = 0.$$

Consider for $\theta \in [0,1]$, the fixed point equation

$$u = (1-\theta)\, Tu + \theta T_0 u. \tag{3.10}$$

We claim that there exists $R^\infty > \delta_2$ such that equation (3.10) has no solution in $K \cap \partial B\,(0, R^\infty)$. Indeed, to the contrary, assume that for all integer $n \geq 1$ there exist $\theta_n \in [0,1]$ and $u_n \in K \cap \partial B(0,n)$ such that

$$u_n = (1-\theta_n)\, Tu_n + \theta_n T_0 u_n. \tag{3.11}$$

We have that $\lim_{n\to\infty} \|Au_n\| = \lim_{n\to\infty} A_p u_n\,(1) = +\infty$. Indeed, if there exists a subsequence (u_{n_k}) such that $\|A_p u_{n_k}\|$ is bounded by a constant $M > 0$ and

$$M_1 = \sup\left\{f(s,x) + g(x) : s \in [0,1],\ x \in [0,M]\right\},$$

then we have the contradiction

$$\|u_{n_k}\| = u_{n_k}(0) = (1-\theta_{n_k}) \int_0^1 f(s, A_p u_{n_k}(s)) ds + \theta_{n_k} \int_0^1 g(A_p u_{n_k}(s)) ds \leq M_1.$$

The remainder of the proof involves showing that $\lim_{n\to\infty} Au_n(\eta) = +\infty$. We distinguish two cases:

- If $\lim_{n\to\infty} u_n(\eta) = +\infty$, then we have from (3.2), $\lim_{n\to\infty} Au_n(\eta) = +\infty$.
- If there exists a subsequence (u_{n_k}) such that $(u_{n_k}(\eta))$ is bounded and

$$m = \sup_{k\in\mathbb{N}} \left(u_{n_k}(\eta)\right),$$

then we have from (3.3) the contradiction

$$\|Au_{n_k}\| = Au_{n_k}(1) \leq Au_{n_k}(\eta) + \psi^{+}(m).$$

Let $\epsilon > 0$ be such that $f_\infty, \gamma > (\Lambda_4 + \epsilon)$. There exists $R_2 > 0$ such that, for all $t \in [\eta, 1]$ and $u > R_2$,

$$f(t,u) > (\Lambda_4 + \epsilon)\, u^{p^+-1}.$$

Let $R^\infty > \max(R_2, R^0)$. We have from (3.11) and (3.2) for n large, the contradiction

$$\begin{aligned} u_n(\eta) &= \int_\eta^1 \left((1-\theta_n) f(s, A_p u_n(s)) + \theta_n g(A_p u_n(s))\right) ds \\ &> \int_\eta^1 (\Lambda_4 + \epsilon)\left(A_p u_n(s)\right)^{p^+-1} ds \\ &\geq (1-\eta)(\Lambda_4+\epsilon)\left(A_p u_n(\eta)\right)^{p^+-1} \\ &\geq (\Lambda_4+\epsilon)\Lambda_4^{-1} u_n(\eta) \\ &> u_n(\eta). \end{aligned}$$

The claim is proved, and by the homotopy property of the fixed point index, we have

$$i(T, B(0,R)\cap K, K) = i(T_0, B(0,R)\cap K, K) = 0.$$

This completes the proof □

Remark 3.4. Theorem 3.1 and Theorem 3.3 cover, respectively, the cases

$$\lim_{u\to 0} \frac{f(t,u)}{\phi(t,u)} = +\infty \text{ and } \lim_{u\to+\infty} \frac{f(t,u)}{\phi(t,u)} = 0 \text{ uniformly on } [0,1],$$

and

$$\lim_{u\to 0} \frac{f(t,u)}{\phi(t,u)} = 0 \text{ and } \lim_{u\to+\infty} \frac{f(t,u)}{\phi(t,u)} = +\infty \text{ uniformly on } [0,1].$$

Thus, if $f(t,u) = u^{q(t)-1}$, then bvp (1.1) admits a positive solution in both the cases $1 < q(t) < p(t)$ and $1 < p(t) < q(t)$.

Example 3.5. Consider the bvp (1.1) with

$$p(t) = \frac{3+t}{1+t},\ f(t,u) = \frac{Au + Bu^2}{1+u} \text{ and } \alpha = \eta = \frac{1}{2}$$

where A, B are positive real numbers.

By simple computations we get that $p^+ = 3$, $p^- = 2$, and

$$\Lambda_1 = \frac{2}{3}, \quad \Lambda_2 = 2, \quad f_0 = A, \quad f^\infty = B.$$

We deduce from Theorem 3.1 that bvp (1.1) admits a positive solution if $B < \frac{2}{3} < 2 < A$.

Example 3.6. Consider the bvp (1.1) with $f(t,u) = Au^{\sigma}\exp(u)$ where σ and A are positive real numbers. By simple computations we get that

$$f_{\infty} = +\infty \text{ and } f^0 = \begin{cases} 0 & \text{if } \sigma > p^+ - 1 \\ A & \text{if } \sigma = p^+ - 1 \\ +\infty & \text{if } \sigma < p^+ - 1. \end{cases}$$

We deduce from Theorem 3.3 that bvp (1.1) admits a positive solution if $\sigma > p^+ - 1$ or $\sigma = p^+ - 1$ and $A < \left(\frac{1-\alpha}{1-\alpha(1-\eta)}\right)^{p^+-1}$.

Example 3.7. Consider the bvp (1.1) with $f(t,u) = u^{\theta}$ where θ is a positive real number. We have

$$f_0 = \begin{cases} 0 & \text{if } \theta > p^- - 1 \\ 1 & \text{if } \theta = p^- - 1 \\ +\infty & \text{if } \theta < p^- - 1 \end{cases} \qquad f^{\infty} = \begin{cases} +\infty & \text{if } \theta > p^- - 1 \\ 1 & \text{if } \theta = p^- - 1 \\ 0 & \text{if } \theta < p^- - 1 \end{cases}$$

$$f^0 = \begin{cases} 0 & \text{if } \theta > p^+ - 1 \\ 1 & \text{if } \theta = p^+ - 1 \\ +\infty & \text{if } \theta < p^+ - 1 \end{cases} \qquad f_{\infty} = \begin{cases} +\infty & \text{if } \theta > p^+ - 1 \\ 1 & \text{if } \theta = p^+ - 1 \\ 0 & \text{if } \theta < p^+ - 1. \end{cases}$$

So we deduce from Theorem 3.1 and Theorem 3.3 that bvp (1.1) admits a positive solution if $\theta \in (0, p^- - 1) \cup (p^+ - 1, +\infty)$.

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