Dynamic Systems and Applications 22 (2013) 503-516

A NOTE ON STOCHASTIC INCLUSIONS APPROACH FOR FUZZY STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY SEMIMARTINGALES

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ABSTRACT. In the paper we present the notion of fuzzy stochastic differential equation with respect to semimartingale integrators in connection with well established theory of stochastic inclusions. We study the existence of fuzzy solutions to such equations via stochastic inclusions approach. Such an approach extends to the stochastic case earlier studies known in the deterministic case.

Keywords and phrases. Set-valued stochastic process, semimartingale, fuzzy-valued stochastic integral, stochastic inclusion, fuzzy stochastic differential equation

2000 AMS Subject Classification. 60H05, 60H20, 37H10, 03E72.

1. INTRODUCTION

Deterministic fuzzy differential equations have been developed due to investigations of dynamic systems where an information on parameters of such systems is incomplete or vague. Investigations in this area were developed using different approaches for formulations of differential problems in a fuzzy setting (see e.g. [1], [2], [13], [14], [19], [18], [22], [23], [24], [30], [31], [32], [44], [48] and references therein). However, there are discussions that this diversity of approaches can be an advantage (see e.g. [10], [24]) and one can choose the most appropriate one to the considered situation. One of the earliest approach for fuzzy differential equations was to generalize the Hukuhara derivative of a set-valued function. This was made by Puri and Ralescu in [46] and used next by Kaleva in [22] and [23] (see also [49], [50]).

A further developments in this direction have been made among others in [8], [9], where the concept of strongly generalized differentiability was introduced. On the other hand in [18] a fuzzy differential equation was interpreted as a family of differential inclusions associated with level sets of a fuzzy right hand side of an equation.

Received November 26, 2012

Such an approach has been also used next among others in [1], [2], [10] [14], [15], [19], [30], [31], [32] and [48]. A further step it was a research concerning stochastic fuzzy differential (or integral) equations which generalize both classical stochastic differential equations and deterministic fuzzy differential equations. Such equations can be applied in modeling of phenomenons where two kinds of uncertainties, i.e., randomness and fuzziness, are incorporated. In this case, the main problem was a concept of a fuzzy stochastic integral which should cover the notion of the classical stochastic Itô integral. Studies on such integral with respect to the Wiener process and fuzzy stochastic equations driven by such a noise, were initiated in [33] and [34]. Recently, in [36] and next in [35] similar studies were extended to semimartingale integrators. The work presented in [35] extends the approach proposed in the deterministic case in [22]. The novelty of this peper is that we propose a different meaning of the notion of a fuzzy stochastic differential equation than it was considered in [33], [34] and [35]. We develop here the idea to treat a stochastic fuzzy differential equation as a system of stochastic integral inclusions. In the deterministic case our approach corresponds with ideas and comments presented in [1], [18] and [10]. This idea was used first time in stochastic case in [36] for stochastic fuzzy differential equations driven by a Wiener process. In this paper we continue our earlier studies but here we will focus on stochastic fuzzy differential equation driven by semimartingales. Moreover, we consider here strong solutions to the system of stochastic inclusions driven by such integrators. The idea used in the paper is to solve those inclusions and then apply the theorem of Negoita and Ralescu. So our studies connect the well established theory of stochastic differential (or integral) inclusions (see e.g. [4], [5], [6], [26], [27], [28], [29], [37], [38], [39], [40], [41], [42] and references therein) with a new theory of fuzzy stochastic differential equations.

The paper is organized as follows. In Section 2 we recall some notions and facts from set-valued, fuzzy-valued and stochastic analysis needed in the sequel. Next we recall the notion of fuzzy stochastic integral as well as its main properties. In the last Section 3 we establish the formulation of the fuzzy stochastic differential equation and the existence of its solution via stochastic inclusions approach.

2. PRELIMINARIES

In this section, we start with some facts from stochastic analysis needed in the sequel. We recall the notion of a fuzzy stochastic integral with respect to semimartingale integrators developed recently in [36] and next used in [35]. We present their main properties needed in the sequel. Let $T > 0$ and let $I = [0, T]$ or R_+. Let $(\Omega, \mathbf{F}, \{\mathbf{F}_t\}_{t\in I}, P)$ be a complete filtered probability space satisfying the usual hypothesis, i.e., $\{\mathbf{F}_t\}_{t\in I}$ is an increasing and right continuous family of sub-σ-fields of $\mathbf{F}$ and $\mathbf{F}_0$ contains all P-null sets. Let $\mathcal{P}$ denote the smallest σ-field on $I \times \Omega$ with respect to which every left-continuous and $\{\mathbf{F}_t\}_{t\in I}$-adapted process is measurable. An R^d-valued stochastic process x is said to be predictable if x is $\mathcal{P}$-measurable. One has $\mathcal{P} \subset \beta \otimes \mathbf{F}$, where β denotes the Borel σ-field on I. Let S^p, $(p \geqslant 1)$ denote the space of all $\{\mathbf{F}_t\}_{t\in I}$-adapted and cádlág (i.e., right continuous and with finite left-hand limits) processes $(x_t)_{t\in I}$ such that the norm $\|x\|_{S^p} := \|\sup_{t\in I}|x_t|\,\|_{L^p}$ is finite with $L^p := L^p(\Omega, R^1)$. It is well known that $(S^p, \|\ \|_{S^p})$ is a Banach space. We shall use the notation $x_{t-} := \lim_{s\nearrow t} x_t$ P-a.s. Let Z be an $\{\mathbf{F}_t\}_{t\in I}$-adapted and cádlág process with values in R^1. It is said to be a semimartingale if $Z = M + A$ where M is an $\{\mathbf{F}_t\}_{t\in I}$-adapted local martingale and A is an $\{\mathbf{F}_t\}_{t\in I}$-adapted, cádlág process with finite variation on compact intervals in I (see e.g. [45] for details). We shall assume that $Z_{0-} = Z_0 = 0$. We shall consider the class of $\mathcal{H}^2$-semimartingales, i.e., the space of $\{\mathbf{F}_t\}_{t\in I}$-adapted semimartingales with a finite $\mathcal{H}^2$-norm:

$$\|\mathrm{Z}\|_{\mathcal{H}^2} := \|[\mathrm{M},\mathrm{M}]^{1/2}_{\sup \mathrm{I}}\|_{\mathrm{L}^2} + \|\left(\int_0^{\sup \mathrm{I}} |\mathrm{dA_t}|\right)\|_{\mathrm{L}^2} < \infty,$$

where $[M, M]$ denotes the quadratic variation process for a local martingale M, while $|A|. := \int_0^{\cdot} |dA_s|$ represents the total variation of the random measure induced by the paths of the process A. By Theorem 5, p. 127 in [45] one has the following inequality

$$E\left(\sup_{t\in I}|Z_t|\right)^2 \leqslant 8\|\mathrm{Z}\|^2_{\mathcal{H}^2}. \tag{2.1}$$

Proceeding similarly as in [36] we shall introduce some measure μ_Z on the predictable σ-field $\mathcal{P}$ associated with a semimartingale Z. Since $Z \in \mathcal{H}^2$, we have $E[M, M]_t < \infty$ for all $t \in I$, and then M is a square integrable martingale such that $EM_t^2 = E[M, M]_t$ for all $t \in I$. By the same reason the process A has a square integrable total variation on I. By μ_M denote the Doléans-Dade measure for the martingale M, i.e., μ_M is an unique measure on a predictable σ-field$\mathcal{P}$ such that

$$\begin{aligned} \mu_M\left((s,t] \times A\right) &= E\left(I_A(M_t - M_s)^2\right), \\ \mu_M\left(\{0\} \times A_0\right) &= 0 \end{aligned}$$

for all $A \in \mathbf{F}_s, 0 \leqslant s < t$ and $A_0 \in \mathbf{F}_0$ (see e.g. [11]). Then for all $f \in L^2(I \times \Omega, \mathcal{P}, \mu_M; R^d)$ the stochastic integral $\int f_s dM_s$ exists and one has

$$\begin{aligned} E\left(\|\int_0^t f_s dM_s\|^2_{R^d}\right) &= \int_{[0,t]\times\Omega} \|f\|^2_{R^d} d\mu_M \\ &= E\left(\int_0^t \|f_s\|^2_{R^d} d[M,M]_s\right), \end{aligned} \tag{2.2}$$

for $t \in I$. Let us define now a random measure on I

$$\gamma(\omega, dt) := |A(\omega)|_{\sup I}|dA_t(\omega)|$$

and a measure associated with the process A by the formula:

$$\nu_A(C) := \int_\Omega \int_0^{\sup I} I_C(\omega,t)\gamma(\omega,dt)P(d\omega)$$

for every $C \in \mathcal{P}$. Then we have

$$\nu_A(I\times\Omega) = E\left(\int_0^{\sup I} |dA_s|\right)^2.$$

Hence ν_A is a finite measure on $\mathcal{P}$. Finally, we define a finite measure μ_Z associated with $Z \in\mathcal{H}^2$ by $\mu_Z := \mu_M + \nu_A$. Let us denote $L^2_{\mathcal{P}}(\mu_Z) := L^2\left(I\times\Omega, \mathcal{P}, \mu_Z; R^d\right)$. Then by Proposition 1 in [36] for every $f \in L^2_{\mathcal{P}}(\mu_Z)$ and $t \in I$ there exists a stochastic integral $\int_0^t f_s dZ_s$ and

$$E\left(\|\int_0^t f_s dZ_s\|^2_{R^d}\right) \leqslant 16\|fI_{[0,t]}\|^2_{L^2_{\mathcal{P}}(\mu_Z)}. \tag{2.3}$$

Note that since we have assumed $Z_{0-} = Z_0 = 0$ it follows that $\int_0^t f_s dZ_s = \int_{0+}^t f_s dZ_s$ where $\int_{0+}^t := \int_{(0,t]}$.

2.1. **Set-valued trajectory stochastic integral.** Let $\mathfrak{X}$ be a Banach space. By a $\mathcal{K}^b(\mathfrak{X})$ we denote the family of all nonempty closed and bounded subsets of $\mathfrak{X}$ while by $\mathcal{K}^b_c(\mathfrak{X})$ we mean those of elements from $\mathcal{K}^b(\mathfrak{X})$ that are also convex subsets of $\mathfrak{X}$. The Hausdorff metric $H_{\mathfrak{X}}$ in $\mathcal{K}^b(\mathfrak{X})$ is defined by:

$$H_{\mathfrak{X}}(A,B) := \max\{\sup_{a\in A} dist_{\mathfrak{X}}(a,B), \sup_{b\in B} dist_{\mathfrak{X}}(b,A)\}$$

where $dist_{\mathfrak{X}}(a,B) := \inf_{b\in B}\|a-b\|_{\mathfrak{X}}$ and $\|\cdot\|_{\mathfrak{X}}$ is a norm in $\mathfrak{X}$. Moreover $(\mathcal{K}^b(\mathfrak{X}), H_{\mathfrak{X}})$ is a complete metric space and $\mathcal{K}^b_c(\mathfrak{X})$ is its closed subspace. Let us assume that $A,B,C,D \in \mathcal{K}^b_c(\mathfrak{X})$. Then it holds (see [17]):

$$H_{\mathfrak{X}}(A+B,C+D) \leqslant H_{\mathfrak{X}}(A,C) + H_{\mathfrak{X}}(B,D)$$

and

$$H_{\mathfrak{X}}(A+B,C+B) = H_{\mathfrak{X}}(A,C)$$

where $A+B$ denotes the Minkowski sum of A and B. For $A \in \mathcal{K}^b(\mathfrak{X})$ we set $|||A||| := H_{\mathfrak{X}}(A,\{0\}) = \sup_{a\in A}\|a\|_{\mathfrak{X}}$. Let $(U,\mathcal{A},\mu)$ be a σ-finite measure space. Let $\mathcal{M}$ be a set of $\mathcal{A}$-measurable functions $f : U \to \mathfrak{X}$. The set $\mathcal{M}$ is said to be $\mathcal{A}$-decomposable if for every $f_1, f_2 \in \mathcal{M}$ and $A \in \mathcal{A}$ one has $I_A f_1 + I_{A^c} f_2 \in \mathcal{M}$. Let $F = (F(t))_{t\in I}$ be a set-valued stochastic process with values in $\mathcal{K}^b(R^d)$, i.e., a family of $\mathbf{F}$-measurable set-valued mappings $F(t) : \Omega \to \mathcal{K}^b(R^d)$, each $t \in I$. We call F measurable if it is $\beta\otimes\mathbf{F}$ measurable in the sense of set-valued functions, i.e., $\{(t,\omega) : F(t,\omega)\cap U \neq \emptyset\} \in \beta\otimes\mathbf{F}$ for every open set $U \subset R^d$. Similarly, F is $\{\mathbf{F}_t\}_{t\in I}$-adapted if $F(t)$ is $\mathbf{F}_t$-measurable for each $t \in I$. We call F predictable if F is $\mathcal{P}$-measurable. Let us define the set

$$S^2_{\mathcal{P}}(F,\mu_Z) := \{f \in L^2_{\mathcal{P}}(\mu_Z) : f \in F\ \mu_Z\ a.e.\}.$$

We say that F is $L^2_{\mathcal{P}}(\mu_Z)$-integrally bounded if $|||F||| \in L^2(I\times\Omega,\mathcal{P},\mu_Z;R_+)$. In this case, by Kuratowski and Ryll-Nardzewski Selection Theorem (see e.g. [25]) it follows that $S^2_{\mathcal{P}}(F,\mu_Z)\neq\emptyset$. Hence for every $t\in I$, we define the set

$$\int_0^t F_s dZ_s := \left\{\int_0^t f_s dZ_s : f\in S^2_{\mathcal{P}}(F,\mu_Z)\right\}$$

which is called the set-valued trajectory stochastic integral of F with respect to semimartingale Z. For any $0\leqslant s<t$ we set

$$\int_{s+}^t F_s dZ_s := \int_0^t I_{(s,t]}(u)F_u dZ_u = \left\{\int_{s+}^t f_u dZ_u : f\in S^2_{\mathcal{P}}(F,\mu_Z)\right\}.$$

If a semimartingale Z has continuous paths then $\int_{s+}^t f_u dZ_u = \int_s^t f_u dZ_u$ for $f\in S^2_{\mathcal{P}}(F,\mu_Z)$ and consequently $\int_{s+}^t F_s dZ_s = \int_s^t F_s dZ_s$ for any $0\leqslant s<t$ and $s,t\in I$. Below we collect the main properties of the sets $S^2_{\mathcal{P}}(F,\mu_Z)$ and $\int_0^t F_s dZ_s$.

Proposition 1 ([36])**.** *Let $F: I\times\Omega\to\mathcal{K}^b_c(R^d)$ be a predictable and $L^2_{\mathcal{P}}(\mu_Z)$-integrally bounded set-valued mapping. Then*

a) *$S^2_{\mathcal{P}}(F,\mu_Z)$ is a closed, convex, bounded, weakly compact and decomposable subset of $L^2_{\mathcal{P}}(\mu_Z)$,*
b) *$\int_0^t F_s dZ_s$ is a bounded closed, weakly compact and convex subset of $L^2(\Omega,\mathbf{F}_t,P,R^d)$ for every $t\in I$.*

Theorem 2.1 ([36])**.** *Let Z be an $\mathcal{H}^2$-semimartingale and let $F,G: I\times\Omega\to\mathcal{K}^b(R^d)$ be predictable multivalued mappings such that F and G are $L^2_{\mathcal{P}}(\mu_Z)$-integrally bounded. Then for every $0\leqslant s<t$ and $s,t\in I$ one has*

$$H^2_{L^2}\left(\int_{s+}^t F_u dZ_u, \int_{s+}^t G_u dZ_u\right) \leqslant 2\int_{(s,t]\times\Omega} H^2_{R^d}(F,G)\,d\mu_Z$$

where H_{L^2} denotes the Hausdorff distance in $L^2(\Omega,\mathbf{F},P,R^d)$.

Theorem 2.2 ([36])**.** *For each $n\geqslant 1$, let $F^{(n)}: I\times\Omega\to\mathcal{K}^b(R^d)$ be a predictable multivalued mapping such that $F^{(1)}$ is $L^2_{\mathcal{P}}(\mu_Z)$-integrally bounded and*

$$F^{(1)}\supset F^{(2)}\supset\cdots\supset F\ \ \mu_Z\text{-a.e.}$$

and let $F:=\bigcap_{n\geqslant 1}F^{(n)}$ μ_Z-a.e. Then for every $t\in I$

$$\int_0^t F_s dZ_s = \bigcap_{n\geqslant 1}\int_0^t F^{(n)}_s dZ_s.$$

2.2. **Fuzzy random variables and fuzzy stochastic integral.** By a fuzzy set u of a Banach space $\mathfrak{X}$ we mean a mapping $u:\mathfrak{X}\to[0,1]$. The space of all fuzzy sets of $\mathfrak{X}$ will be denoted by the symbol $\mathcal{F}(\mathfrak{X})$. For $\alpha\in(0,1]$ let $[u]^\alpha := \{x\in\mathfrak{X}: u(x)\geqslant\alpha\}$ and $[u]^0 := cl_{\mathfrak{X}}\{x\in\mathfrak{X}: u(x)>0\}$ where $cl_{\mathfrak{X}}$ denotes the closure in $(\mathfrak{X},\|\cdot\|_{\mathfrak{X}})$. In the sequel we deal with the following fuzzy sets

$$\mathcal{F}^b(\mathfrak{X}) = \{u\in\mathcal{F}(\mathfrak{X}) : [u]^\alpha\in\mathcal{K}^b(\mathfrak{X}) \text{ for every } \alpha\in[0,1]\},$$

$$\mathcal{F}_c^b(\mathfrak{X}) = \{u \in \mathcal{F}(\mathfrak{X}) : [u]^\alpha \in \mathcal{K}_c^b(\mathfrak{X}) \text{ for every } \alpha \in [0,1]\}.$$

We shall use a metric $D_{\mathfrak{X}}$ in $\mathcal{F}_c^b(\mathfrak{X})$ described as follows

$$D_{\mathfrak{X}}(u,v) := \sup_{\alpha\in[0,1]} H_{\mathfrak{X}}([u]^\alpha, [v]^\alpha) \text{ for } u,v \in \mathcal{F}_c^b(\mathfrak{X}).$$

It is known (c.f. [47]) that $\left(\mathcal{F}_c^b(\mathfrak{X}), D_{\mathfrak{X}}\right)$ is a complete metric space. We will use the following version of the theorem of Negoita and Ralescu.

Theorem 2.3 ([43])**.** *Let Y be a set and let $\{Y_\alpha, \alpha \in [0,1]\}$ be a family of subsets of Y such that:*

a) $Y_0 = Y$,
b) $\alpha_1 \leqslant \alpha_2 \Rightarrow Y_{\alpha_1} \supset Y_{\alpha_2}$,
c) $\alpha_n \nearrow \alpha \Rightarrow Y_\alpha = \bigcap_{n=1}^\infty Y_{\alpha_n}$.

Then the function $\phi : Y \to [0,1]$ defined by $\phi(x) = \sup\{\alpha \in [0,1] : x \in Y_\alpha\}$ has the property that

$$\{x \in Y : \phi(x) \geqslant \alpha\} = Y_\alpha$$

for any $\alpha \in [0,1]$.

As before let $(\Omega, \mathbf{F}, \{\mathbf{F}_t\}_{t\in I}, P)$ be a given filtered probability space and let Z be a given $\mathcal{H}^2$-semimartingale. By a fuzzy random variable (in the sense of Puri and Ralescu) we mean a function $u\colon \Omega \to \mathcal{F}_c^b(\mathfrak{X})$ such that $[u(\cdot)]^a\colon \Omega \to \mathcal{K}_c^b(\mathfrak{X})$ is an $\mathbf{F}$-measurable set-valued mapping for every $\alpha \in [0,1]$. Let $f : I \times \Omega \to \mathcal{F}_c^b\left(R^d\right)$ be a predictable fuzzy stochastic process, i.e., fuzzy-valued mapping such that the set-valued function $[f]^\alpha : I \times \Omega \to \mathcal{K}_c^b\left(R^d\right), [f]^\alpha(t,\omega) := [f(t,\omega)]^\alpha$ is a predictable set-valued stochastic process. We call f to be $L^2_{\mathcal{P}}(\mu_Z)$-integrally bounded if $|||[f]^0||| \in L^2(I \times \Omega, \mathcal{P}, \mu_Z, R_+)$. Taking such a predictable fuzzy stochastic process f let us consider the trajectory set-valued stochastic integral $Y_\alpha(t) := \int_0^t [f]^\alpha \, dZ$ for any $t \in I$ and every $\alpha \in [0,1]$. Then by Proposition 1, Theorem 2.2 and Theorem 2.3, for every fixed $t \in I$ there exists a fuzzy set $X(f,Z)_t \in \mathcal{F}_c^b\left(L^2(\Omega, \mathbf{F}_t, P, R^d)\right)$ such that $[X(f,Z)_t]^\alpha = \int_0^t [f]^\alpha \, dZ$ for every $t \in I$ and every $\alpha \in [0,1]$. Having the family of just described fuzzy sets $\{X(f,Z)_t, t \in I\}$, one can define [36]:

Definition 2.4. By a fuzzy trajectory stochastic integral of the predictable and $L^2_{\mathcal{P}}(\mu_Z)$-integrally bounded fuzzy stochastic process f with respect to the semimartingale Z we mean the family of fuzzy sets $\{X(f,Z)_t, t \in I\}$ described above. We denote it by $X(f,Z)_t := (\mathcal{F}) \int_0^t f(s) dZ_s$ for $t \in I$.

By Theorem 2.1 one can formulate the following result.

Corollary 2.5. *Let $f_1, f_2 : I \times \Omega \to \mathcal{F}_c^b(R^d)$ be predictable and $L^2_{\mathcal{P}}(\mu_Z)$-integrally bounded fuzzy stochastic processes. Then for all $\tau, t \in I, \tau \leqslant t$ it holds*

$$D^2_{L^2}((\mathcal{F})\int_{\tau+}^{t} f_1(s)dZ_s, (\mathcal{F})\int_{\tau+}^{t} f_2(s)dZ_s) \leqslant 2\int_{(\tau,t]\times\Omega} D^2_{R^d}(f_1, f_2)d\mu_Z.$$

3. STOCHASTIC INCLUSIONS AND FUZZY STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY A SEMIMARTINGALE

In this part we apply the results of the preceding sections to the theory of stochastic inclusions and fuzzy stochastic equations. Now we assume that $I = [0, T]$ for $T > 0$. Let Z be a given $\mathcal{H}^2$-semimartingale on a filtered probability space $(\Omega, \mathbf{F}, \{\mathbf{F}_t\}_{t\in I}, P)$. Let $L^2_0 := L^2(\Omega, \mathbf{F}_0, P; R^d)$. By $\langle\cdot\rangle : L^2_0 \to \mathcal{F}_c^b(L^2_0)$ we denote an embeding of L^2_0 into $\mathcal{F}_c^b(L^2_0)$, i.e., for $a \in L^2_0$ we we have $\langle a\rangle(z) = \mathbb{I}_{\{a\}}(z)$ for $z \in L^2_0$. We consider a fuzzy-valued function $f : I \times \Omega \times R^d \to \mathcal{F}_c^b(R^d)$ and $\xi \in L^2_0$. By a fuzzy stochastic differential equation we mean the formal relation

$$\begin{aligned} dX_t &= f(t, X_{t-})dZ_t, \quad t \in I \\ X_0 &= \langle\xi\rangle \end{aligned} \tag{3.1}$$

which is interpreted as a family of stochastic integral inclusions

$$\begin{aligned} x_t - x_s &\in \left[(\mathcal{F})\int_s^t f(\tau, x_{\tau-})dZ_\tau\right]^\alpha, \quad 0 \leqslant s < t \leqslant T, \\ x_0 &= \xi \end{aligned}$$

or equivalently (by Definition 2.4) as

$$\begin{aligned} x_t - x_s &\in \int_s^t [f(\tau, x_{\tau-})]^\alpha dZ_\tau, \quad 0 \leqslant s < t \leqslant T, \\ x_0 &= \xi \end{aligned} \tag{3.2}$$

for $\alpha \in [0, 1]$. We will show that under appropriate assumptions the solution sets of (3.2) generate a fuzzy set in a Banach space S^2. In [36] a similar idea was used for a stochastic fuzzy differential equation driven by the Wiener process. The approach used in [36] was based on weak (or martingale) solutions to stochastic inclusions. We shall proceed in a similar way for the stochastic fuzzy differential equation (3.1) driven by a semimartingale, but here we will focus on strong solutions to the system of stochastic inclusions. Therefore, for a fixed $\alpha \in [0, 1]$ we define first the notion of a solution to (3.2). Namely, by a strong solution to stochastic inclusion (3.2) we mean a cádlág and an $\{\mathbf{F}_t\}_{t\in I}$-adapted stochastic process $x = (x_t)_{t\in I}, x \in S^2$ such that

$$x_t = \xi + \int_0^t a_s dZ_s \text{ for } a \in S^2_{\mathcal{P}}([f \circ x_-]^\alpha, \mu_Z) \text{ and } t \in I$$

where $[f \circ x_-]^\alpha (t, \omega) = [f(t, \omega, x_{t-}(\omega)]^\alpha$. Let $\Gamma(f, \xi, \alpha)$ denote the set of all strong solutions to (3.2). Then $\Gamma(f, \xi, \alpha) \subset S^2$. Suppose $\Gamma(f, \xi, \alpha) \neq \emptyset$. Thus we have the following definition of the fuzzy solution to the equation (3.1).

Definition 3.1. By the fuzzy solution to the fuzzy stochastic differential equation (3.1) we mean a fuzzy set $X(f, \xi) \in \mathcal{F}^b(S^2)$ such that $[X(f, \xi)]^\alpha = \Gamma(f, \xi, \alpha)$ for $\alpha \in [0, 1]$.

To proceed further we assume the following conditions:

(h1) Z is a given $\{\mathbf{F}_t\}_{t \in I}$-adapted, $\mathcal{H}^2$-semimartingale with a decomposition $Z = M + A$ where A is an $\{\mathbf{F}_t\}_{t \in I}$-adapted increasing predictable process such that the measure μ_Z is absolutely continuous with respect to the product measure $\lambda \otimes P$ on the σ-field $\mathcal{P}$.

(h2) The function $f : I \times \Omega \times R^d \to \mathcal{F}^b_c(R^d)$ is $\mathcal{P} \otimes \beta(R^d)$-measurable and $L^2_{\mathcal{P}}(\mu_Z)$-integrally bounded, i.e., there exists a function $m \in L^2(I \times \Omega, \mathcal{P}, \mu_Z; R)$ such that $\left\| \left| [f(\cdot, \cdot, x)]^0 \right\| \right|_{R^d} \leqslant m$ μ_Z-a.e., for every $x \in R^d$.

(h3) There exists a constant $K > 0$ such that $D_{R^d}(f(t, \omega, x), f(t, \omega, y)) \leqslant K \|x - y\|_{R^d}$ for every $x, y \in R^d$ and every $(t, \omega) \in I \times \Omega$.

(h4) There exists a constant $C > 0$ such that $D^2_{R^d}(f(t, \omega, x), \hat{\theta}) \leqslant C(1 + \|x\|^2_{R^d})$ for every $x \in R^d$ and every $(t, \omega) \in I \times \Omega$.

Then we have the following result.

Theorem 3.2. *If the conditions* (h1)–(h4) *hold then there exists a fuzzy solution in the sense of Definition* 3.1 *to the fuzzy stochastic differential equation* (3.1).

Proof. Let $\alpha \in [0, 1]$ be arbitrary and fixed and let us consider the inclusion (3.2) with the initial value ξ. As earlier by $\Gamma(f, \xi, \alpha)$ we denote the set of all strong solutions to (3.2). First, we show that $\Gamma(f, \xi, \alpha) \neq \emptyset$. Indeed, by the assumptions (h2), (h3) and (h4) the set-valued function $[f]^\alpha : I \times \Omega \times R^d \to \mathcal{K}^b_c(R^d)$ defined by $[f]^\alpha (t, \omega, x) := [f(t, \omega, x)]^\alpha$ is $\mathcal{P} \otimes \beta(R^d)$-measurable and satisfies Lipschitz and growth conditions with respect to the Hausdorff distance H_{R^d} with constants K and C, respectively. Hence using similar methods as in the proof of Th. 9.5.3 in [7] one can show that there exists an $\mathcal{P} \otimes \beta(R^d)$-measurable function (depending on α) $p : I \times \Omega \times R^d \to R^d$ such that $p(t, \omega, x) \in [f(t, \omega, x)]^\alpha$ for $(t, \omega, x) \in I \times \Omega \times R^d$ and such that the mapping $p(t, \omega, \cdot)$ is Lipschitzean. Moreover, by the assumption (h4) the function p satisfies a growth condition: $\|p(t, \omega, x)\|^2_{R^d} \leqslant C(1 + \|x\|^2_{R^d})$ for every $x \in R^d$ and every $(t, \omega) \in I \times \Omega$. Now by Th. 1 in [20] there exists a unique strong solution $x = (x_t)_{t \in I}$ to the stochastic equation:

$$x_t = \xi + \int_0^t p(s, x_{s-}) dZ_s, t \in I.$$

Clearly, we also have that $p \circ x_- \in S^2_{\mathcal{P}}([f \circ x_-]^\alpha, \mu_Z)$. Let us note that the process $\left\{\int_0^t p(s, x_{s-}) dZ_s, t \in I\right\}$ is $\mathcal{H}^2$-semimartingale. Indeed, by the definition of the measure μ_Z we have the following estimation:

$$\begin{aligned} \left\| \int_0^\cdot p(s, x_{s-}) dZ_s \right\|^2_{\mathcal{H}^2} &\leqslant 2 \left\| \left(\int_0^T \|p(s, x_{s-})\|^2_{R^d} d[M, M]_s \right)^{1/2} \right\|^2_{L^2(\Omega, \mathbf{F}, P)} \\ &\quad + 2 \left\| \int_0^T \|p(s, x_{s-})\|_{R^d} |dA_s| \right\|^2_{L^2(\Omega, \mathbf{F}, P)} \\ &\leqslant 2 \int_{I \times \Omega} \|p \circ x_-)\|^2_{R^d} d\mu_Z. \end{aligned}$$

Since (by assumption (h2)) $|||[f(t, \omega, y)]^\alpha|||_{R^d} \leqslant m(t, \omega)$ μ_Z-a.e., for every $y \in R^d$ and $m \in L^2(I \times \Omega, \mathcal{P}, \mu_Z; R)$ it follows that $\|p \circ x_-)\|_{R^d} \leqslant m$ μ_Z-a.e. and hence we have

$$\left\| \int_0^\cdot p(s, x_{s-}) dZ_s \right\|^2_{\mathcal{H}^2} \leqslant 2 \int_{I \times \Omega} m^2 d\mu_Z < \infty.$$

Then by inequality (2.1) and equality (2.2) in Preliminaries we have

$$\begin{aligned} \|x\|^2_{S^2} &\leqslant 2E(\|\xi\|^2_{R^d}) + 2E\left(\sup_{t \in I} \| \int_0^t p(s, x_{s-}) dZ_s \|^2_{R^d} \right) \leqslant 2E(\|\xi\|^2_{R^d}) \\ &\quad + 16 \left\| \int_0^\cdot p(s, x_{s-}) dZ_s \right\|^2_{\mathcal{H}^2} \leqslant 2E(\|\xi\|^2_{R^d}) + 32 \int_{I \times \Omega} m^2_\alpha d\mu_Z < \infty. \end{aligned}$$

Thus $x \in S^2$ and clearly the process x is also the solution to the stochastic inclusion (7^α). Thus $\Gamma(f, \xi, \alpha) \neq \emptyset$. In a similar way, by assumption (h2) one can show that $\Gamma(f, \xi, \alpha)$ is a bounded subset of S^2.

Next, we show that it is a closed subset of S^2. For this aim let us take a sequence $\left(x^{(n)}\right)_{n \geqslant 1} \subset \Gamma(f, \xi, \alpha)$ such that $x^{(n)} \to x$ in S^2 as $n \to \infty$. Then we also have $x^{(n)}_- \to x_-$ in S^2 as $n \to \infty$. Consequently, $x_0 = \xi$ P-a.e. Since $\left(x^{(n)}\right)_{n \geqslant 1} \subset \Gamma(f, \xi, \alpha)$ it follows that for every $n \geqslant 1$ there exists $a^{(n)} \in S^2_{\mathcal{P}}(\left[f \circ x^{(n)}_-\right]^\alpha, \mu_Z)$ such that $x^{(n)}_t = \xi + \int_0^t a^{(n)}_s dZ_s$ for $t \in I$. Since

$$\int_{I \times \Omega} \|a^{(n)}\|^2_{R^d} d\mu_Z \leqslant \int_{I \times \Omega} \left|\left|\left| \left[f \circ x^{(n)}_-\right]^\alpha \right|\right|\right|^2_{R^d} d\mu_Z < \infty,$$

it follows that there exists a subsequence $\left(a^{(n_k)}\right)_k$ of the sequence $\left(a^{(n)}\right)_n$ and $a \in L^2_{\mathcal{P}}(\mu_Z)$ such that $a^{(n_k)} \rightharpoonup a$ (weakly) in $L^2_{\mathcal{P}}(\mu_Z)$ as $k \to \infty$. Let us fix $t \in I$ and consider a linear and (by 2.3) Lipshitz continuous mapping

$$J_t : L^2_{\mathcal{P}}(\mu_Z) \to L^2(\Omega, \mathbf{F}, P; R^d), \text{ given by } J_t(g) := \xi + \int_0^t g_s dZ_s.$$

Then by Prop. 3.4.12 in [12], it is equivalently continuous with respect to weak topologies in the spaces $L^2_{\mathcal{P}}(\mu_Z)$ and $L^2(\Omega, \mathbf{F}, P; R^d)$. Hence for every fixed $t \in I$ we get

$$x_t^{(n_k)} = \xi + \int_0^t a_s^{(n_k)} dZ_s \rightharpoonup \xi + \int_0^t a_s dZ_s \text{ (weakly) in } L^2(\Omega, \mathbf{F}, P; R^d) \text{ as } k \to \infty.$$

On the other hand, since $x^{(n)} \to x$ in S^2, it follows that $x_t^{(n)} \rightharpoonup x_t$ weakly in $L^2(\Omega, \mathbf{F}, P; R^d)$ for every $t \in I$. Thus $x_t = \xi + \int_0^t a_s dZ_s$ for every $t \in I$. To conclude that $x \in \Gamma(f, \xi, \alpha)$ it suffices to show that $a \in S^2_{\mathcal{P}}([f \circ x_-]^\alpha, \mu_Z)$. Let us first note that since $x_-^{n_k} \to x_-$ in S^2, we ensure that $x_-^{n_k} \to x_-$ $\lambda \otimes P$-a.e. on the σ-field $\mathcal{P}$ and by the assumption (h1) we also have that $x_-^{n_k} \to x_-$ μ_Z-a.e. on the σ-field $\mathcal{P}$. Thus by (h3), (h4) and the Lebesgue Dominated Convergence Theorem we conclude that

$$\int_{I]\times\Omega} H^2_{R^d}\left(\left[f \circ x_-^{(n_k)}\right]^\alpha, [f \circ x_-]^\alpha\right) d\mu_Z \to 0, k \to \infty. \tag{3.3}$$

Let consider now an orthogonal projection Π from the space $L^2_{\mathcal{P}}(\mu_Z)$ on its closed and convex subset $S^2_{\mathcal{P}}([f \circ x_-]^\alpha, \mu_Z)$ i.e., for every $g \in L^2_{\mathcal{P}}(\mu_Z)$, $\Pi(g)$ is an unique element of the set $S^2_{\mathcal{P}}([f \circ x_-]^\alpha, \mu_Z)$ such that

$$\|g - \Pi(g)\|_{L^2_{\mathcal{P}}(\mu_Z)} = dist_{L^2_{\mathcal{P}}(\mu_Z)}\left(g, S^2_{\mathcal{P}}([f \circ x_-]^\alpha, \mu_Z)\right).$$

Now, we define a sequence $\hat{a}^{(n_k)} := \Pi(a^{(n_k)})$ for $k \geqslant 1$. Hence, by Theorem 2.2 in [16] get

$$\begin{aligned}
\|\hat{a}^{(n_k)} - a^{(n_k)}\|^2_{L^2_{\mathcal{P}}(\mu_Z)} &= dist^2_{L^2_{\mathcal{P}}(\mu_Z)}\left(a^{(n_k)}, S^2_{\mathcal{P}}([f \circ x_-]^\alpha, \mu_Z)\right) \\
&= \int_{I\times\Omega} dist^2_{R^d}\left(a^{(n_k)}, [f \circ x_-]^\alpha\right) d\mu_Z \\
&\leqslant \int_{I\times\Omega} H^2_{R^d}\left(\left[f \circ x_-^{(n_k)}\right]^\alpha, [f \circ x_-]^\alpha\right) d\mu_Z.
\end{aligned}$$

By (3.3) we ensure that $\hat{a}^{(n_k)} - a^{(n_k)} \rightharpoonup 0$ in $L^2_{\mathcal{P}}(\mu_Z)$ as $k \to \infty$. But the sequence $\left(a^{(n_k)}\right)_k$ converges weakly in $L^2_{\mathcal{P}}(\mu_Z)$ to the process a, hence we also have that $\hat{a}^{(n_k)} \rightharpoonup a$ in $L^2_{\mathcal{P}}(\mu_Z)$. By Proposition 1 the set $S^2_{\mathcal{P}}([f \circ x_-]^\alpha, \mu_Z)$ is weakly compact in $L^2_{\mathcal{P}}(\mu_Z)$ and since $\hat{a}^{(n_k)} \in S^2_{\mathcal{P}}([f \circ x_-]^\alpha, \mu_Z)$, it follows that $a \in S^2_{\mathcal{P}}([f \circ x_-]^\alpha, \mu_Z)$ as well. This proves the closedness of the set $\Gamma(f, \xi, \alpha)$.

Finally, we show that there exists a fuzzy set $X(f, \xi) \in \mathcal{F}^b(S^2)$ such that $[X(f, \xi)]^\alpha = \Gamma(f, \xi, \alpha)$ for $\alpha \in [0, 1]$. It is sufficient to show that the family $\{\Gamma(f, \xi, \alpha), \alpha \in [0, 1]\}$ fulfills the assumptions of Theorem 2.3. By the previous parts of the proof, $\Gamma(f, \xi, \alpha)$ is a nonempty bounded and closed subset of S^2. Since $f : I \times \Omega \times R^d \to \mathcal{F}^b_c(R^d)$ is $\mathcal{P} \otimes \beta(R^d)$-measurable and $L^2_{\mathcal{P}}(\mu_Z)$-integrally bounded, it follows that $S^2_{\mathcal{P}}([f \circ x_-]^\alpha, \mu_Z) \supset S^2_{\mathcal{P}}([f \circ x_-]^\beta, \mu_Z)$ for $0 \leqslant \alpha \leqslant \beta \leqslant 1$ and any $x \in S^2$. Hence, $\Gamma(f, \xi, \alpha) \supset \Gamma(f, \xi, \beta)$ for $0 \leqslant \alpha \leqslant \beta \leqslant 1$. Let $\alpha_n \in [0, 1]$ and $\alpha_n \nearrow \alpha$ as $n \to \infty$. Then we have $\bigcap_{n\geqslant 1} \Gamma(f, \xi, \alpha_n) \supset \Gamma(f, \xi, \alpha)$. Conversely, let $x \in \bigcap_{n\geqslant 1} \Gamma(f, \xi, \alpha_n)$. Then

$x \in S^2$ and for every $n \geqslant 1$ there exists $a^{(n)} \in S^2_{\mathcal{P}}([f \circ x_-]^{\alpha_n}$ such that

$$x_t = \xi + \int_0^t a_s^{(n)} dZ_\tau \text{ for } t \in I \text{ and for every } n \geqslant 1.$$

By Proposition 1 for every $n \geqslant 1$ the set $S^2_{\mathcal{P}}([f \circ x_-]^{\alpha_n}, \mu_Z)$ is weakly compact in $L^2_{\mathcal{P}}(\mu_Z)$. Since

$$[f \circ x_-]^{\alpha_1} \supset [f \circ x_-]^{\alpha_2} \supset \cdots \supset [f \circ x_-]^{\alpha}, \quad \mu_Z - a.e.$$

and

$$[f \circ x_-]^{\alpha} = \bigcap_{n=1}^{\infty} [f \circ x_-]^{\alpha_n}, \quad \mu_Z - a.e.$$

we get

$$S^2_{\mathcal{P}}([f \circ x_-]^{\alpha_1}, \mu_Z) \supset S^2_{\mathcal{P}}([f \circ x_-]^{\alpha_2}, \mu_Z) \supset \cdots \supset S^2_{\mathcal{P}}([f \circ x_-]^{\alpha}, \mu_Z)$$

and

$$S^2_{\mathcal{P}}([f \circ x_-]^{\alpha}, \mu_Z) = \bigcap_{n=1}^{\infty} S^2_{\mathcal{P}}([f \circ x_-]^{\alpha_n}, \mu_Z).$$

Then there exists a sequence $\left(a^{(n_k)}\right) \subset \left(a^{(n)}\right)$ and $a \in L^2_{\mathcal{P}}(\mu_Z)$ such that $a^{(n_k)} \rightharpoonup a$ in $L^2_{\mathcal{P}}(\mu_Z)$. Hence, it follows that $a \in S^2_{\mathcal{P}}([f \circ x_-]^{\alpha}, \mu_Z)$. Using a similar argumentation as earlier, one can show that $x_t = \xi + \int_0^t a_s^{(n_k)} dZ_\tau \rightharpoonup \xi + \int_0^t a_s dZ_\tau$ as $k \to \infty$ for every $t \in I$. Thus we get $x \in \Gamma(f, \xi, \alpha)$ what completes the proof of Theorem 3.2. □

Remark 3.3. Here we present some examples of semimartingales satisfying the assumption (h1) of Theorem 3.2. If Z is the standard Wiener process on $[0,T]$, then $d\mu_Z = dt \times dP$. More generally, let Z be the Lévy process on the interval $[0,T]$, with the local Lévy-Khintchine characteristics (b, σ^2, ν) having a finite second moment (see [21] for details). Hence $Z_t = M_t + tEZ_1$, where M is a square integrable martingale. Then

$$\|\mathrm{Z}\|_{\mathcal{H}^2} \leqslant \left(\sigma^2 + \int_R x^2 \nu(dx) + (EZ_1)^2 T\right) T$$

and $d\mu_Z = d\mu_M + (EZ_1)^2 T(dt \times dP)$. If $d\mu_M \ll dt \times dP$ then we have $d\mu_Z \ll dt \times dP$. In a particular case, if Z is a homogenous Poisson process with intensity $a > 0$ $(a = EZ_1)$, then $M_t = Z_t - at$, $A_t = at$ and $d\mu_Z = (a + a^2 T)(dt \times dP)$.

In a similar way as above one can also consider the following fuzzy stochastic differential equation:

$$\begin{aligned} dX_t &= f(t, X_{t-}) dA_t + g(t, X_{t-}) dM_t, \quad t \in [0,T] \\ X_0 &= \langle \xi \rangle \end{aligned} \tag{3.4}$$

where $f, g : [0,T] \times \Omega \times R^d \to \mathcal{F}_c^b(R^d)$ and $\xi \in L_0^2$. As in the case of equation (3.1) we interprete the relation (3.4) as a system of stochastic integral inclusions:

$$x_t - x_s \in \int_s^t [f(\tau, x_{\tau-})]^{\alpha} dA_\tau + \int_s^t [g(\tau, x_{\tau-})]^{\alpha} dM_\tau, \tag{3.5}$$

$$x_0 \;=\; \xi$$

for $\alpha \in [0,1]$ and $0 \leqslant s < t \leqslant T$.

Let $\Gamma(f,g,\xi,\alpha)$ denote the set of all strong solutions to stochastic inclusion (3.5) for fixed $\alpha \in [0,1]$, that is the set $\Gamma(f,g,\xi,\alpha)$ consits of all processes $x \in S^2$ such that $x_t = \xi + \int_0^t a_s dA_s + \int_0^t b_s dM_s$ for $0 \leqslant t \leqslant T$ where $a \in S^2_{\mathcal{P}}([f \circ x_-]^\alpha, \nu_A)$ and $b \in S^2_{\mathcal{P}}([g \circ x_-]^\alpha, \mu_M)$.

Then using a similar agrumentation as in the proof of Theorem 3.2 one can show the following result.

Theorem 3.4. *Let a semimartingale $Z := M + A$ satisfies* (h1) *and the fuzzy-valued functions f and g satisfy* (h2)*, with respect to the measures ν_A and μ_M, respectively. Let also f and g fulfill the conditions* (h3) *and* (h4)*. Then there exits a fuzzy set $X(f,g,\xi) \in \mathcal{F}^b(S^2)$ (called a fuzzy solution to* (3.4)*), such that $[X(f,g,\xi)]^\alpha = \Gamma(f,g,\xi,\alpha)$ for $\alpha \in [0,1]$.*

Remark 3.5. Let us note that taking in a special case $\xi \in R^d$ and $Z_t = t$ in (3.1) or $A_t = t$ and $M_t = 0$ in (3.4) for $t \in [0,T]$ the condition (h1) is satisfied. Then both equation (3.1) and (3.4) reduce in this case to the special form of deterministic fuzzy differential equation studied in [1]:

$$dx(t) = f(t, x(t))dt, \quad x(0) = \langle \xi \rangle \tag{3.6}$$

interpreted as a family of deterministic integral inclusions:

$$x(t) - x(s) \in \int_s^t [f(\tau, x(\tau))]^\alpha \, d\tau, \quad x(0) = \xi$$

for $\alpha \in [0,1]$. Here $f : [0,T] \times R^d \to \mathcal{F}^b_c(R^d)$. Moreover, in this case the space S^2 reduces to the space of continuous functions $C(I, R^d)$ while the space $L^2_{\mathcal{P}}(\mu_Z)$ reduces to $L^2(I, \beta, \lambda; R^d)$. Then we have:

$$\begin{aligned} \Gamma(f,\xi,\alpha) \;=\; & \Big\{ x \in C(I, R^d) : x(t) - x(s) \in \\ & \int_s^t [f(\tau, x(\tau))]^\alpha \, d\tau, 0 \leqslant s \leqslant t \leqslant T, x(0) = \xi \Big\}. \end{aligned}$$

Then under conditions similar to (h2), (h3) and (h4) there exists a fuzzy set $X(f,\xi) \in \mathcal{F}^b(C([0,T], R^d)$ being a fuzzy solution to the equation (3.6) and such that $[X(f,\xi)]^\alpha = \Gamma(f,\xi,\alpha)$ for every $\alpha \in [0,1]$. In fact, in this case, the set $\Gamma(f,\xi,\alpha)$ is nonempty and compact in $C([0,T], R^d)$ for every $\alpha \in [0,1]$, that is the fuzzy solution $X(f,\xi)$ has compact levelsets (c.f. [1]).

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