Dynamic Systems and Applications 22 (2013) 517-534

NONLINEAR BOUNDARY VALUE PROBLEMS OF FIRST ORDER IMPULSIVE INTEGRO-DIFFERENTIAL EQUATION OF VOLTERRA TYPE ON TIME SCALES

KEXUE ZHANG AND XINZHI LIU

School of Control Science and Engineering, Shandong University
Jinan 250061, China
Department of Applied Mathematics, University of Waterloo
Waterloo, Ontario N2L 3G1, Canada

ABSTRACT. This paper investigates the nonlinear boundary value problems of a class of first order impulsive integro-differential equations of Volterra type on time scales. By developing a new comparison result and using monotone iterative technique, We obtain the existence of extremal solutions and unique solution of the problem. An example is discussed to illustrate the efficiency of the obtained results.

AMS (MOS) Subject Classification. 34B15, 39A10.

1. INTRODUCTION

The theory of calculus on time scales, which has been created to unify continuous and discrete analysis, was initiated by Stefan Hilger [1]. Much of the work devoted to the time scale calculus has been summarized and organized in literature [2,3,4].

The study of boundary value problem of general differential equations has received much attention [4–9,14,15]. To obtain existence results of the boundary value problem, someone may use the method of upper and lower solutions coupled with monotone iterative technique [5]. There are large number of papers devoted to the applications of this method to differential equations with initial and boundary value problems, for details, see [6–10]. To our best knowledge, the application of this monotone method for differential equations on time scales is not much [11–13]. Xing, Han and Zheng [11] considered the following initial value problem on time scales

$$\begin{cases} x^{\Delta}(t) = f(t, x(t), \int_0^t k(t,s)x(s)\Delta s), \quad t \in [0,T]_{\mathbb{T}}, \\ x(0) = x_0, \end{cases}$$

and in [12], they considered the following periodic boundary value problem

$$\begin{cases} x^{\Delta}(t) = f(t, x(t), \int_0^t k(t,s)x(s)\Delta s), \quad t \in [0,T]_{\mathbb{T}}, \\ x(0) = x(\sigma(T)). \end{cases}$$

Received January 22, 2013

In both of these papers, the authors used monotone iterative technique to derive the existence of extremal solutions to the problems and obtained sufficient conditions under which such problems have extremal solutions. In [13], Geng, Xu and Zhu discussed the following periodic boundary value problem

$$\begin{cases} x^{\Delta}(t) = f(t, x(t)), & t \in J' = [0,T]_{\mathbb{T}} - \{t_1, t_2, \ldots, t_m\}, \\ x(t_k^+) - x(t_k^-) = I_k(x(t_k)), & k = 1,2,\ldots,m, \\ x(0) = x(\sigma(T)), \end{cases}$$

and obtained the existence of extremal solutions of the problem.

Motivated by the results mentioned above and [6,14], this paper is concerned with the existence of extremal solutions and unique solution for the nonlinear boundary value problem of a class of first order impulsive integro-differential equations of Volterra type on time scales. Section 2 contains some basic definitions and properties of calculus on time scales. In section 3, we establish a new comparison principle and discuss the existence and uniqueness of the solution for the first order impulsive integro-differential equations with linear boundary on time scales. In section 4, existence results for extremal solutions and unique solution are obtained by using method of upper and lower solutions coupled with monotone iterative technique. To illustrate the obtained results, an example is discussed in section 5.

2. SOME RESULTS ON TIME SCALES

In this section, we list the following fundamental definitions and results concerning time scales to make the paper is selfcontained. Further general details can be found in [2-4] and references therein.

Through out this paper, we denote by $\mathbb{T}$ any time scales, i.e. $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$. Let $0, T$ be points of $\mathbb{T}$, an interval $[0,T]_{\mathbb{T}}$ denoting time scale interval, that is, $[0,T]_{\mathbb{T}} = \{t \in \mathbb{T} : 0 \leq t \leq \mathbb{T}\}$. Other types of intervals are defined similarly.

Definition 2.1. For $t < \sup \mathbb{T}$ and $t > \inf \mathbb{T}$, the mappings σ, $\rho : \mathbb{T} \to \mathbb{T}$,

$$\sigma(t) = \inf\{\tau \in \mathbb{T} : \tau > t\} \in \mathbb{T}, \quad \rho(t) = \sup\{\tau \in \mathbb{T} : \tau < t\} \in \mathbb{T}$$

are called the forward jump operator and the backward jump operator , respectively.

If $\sigma(t) > t$, t is said to be right scattered, and if $\sigma(t) = t$, t is said to be right dense; if $\rho(t) < t$, t is said to be left scattered, and if $\rho(t) = t$, t is said to be left dense. If $\mathbb{T}$ has a right scattered minimum m, define $\mathbb{T}_k = \mathbb{T} - \{m\}$; otherwise set $\mathbb{T}_k = \mathbb{T}$. If $\mathbb{T}$ has a left scattered maximum M, define $\mathbb{T}^k = \mathbb{T} - \{M\}$; otherwise set $\mathbb{T}^k = \mathbb{T}$.

Definition 2.2. For $y : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$, we define the delta derivative of $y(t)$, $y^{\Delta}(t)$, to be the number (when it exists) with the property that for any $\varepsilon > 0$, there is a neighborhood U of t such that

$$|y(\sigma(t)) - y(s) - y^{\Delta}(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|, \quad \text{for all } s \in U.$$

If $F^{\Delta}(t) = f(t)$ for each $t \in \mathbb{T}^k$, then we define the delta integral by $\int_a^b f(s)\Delta s = F(b) - F(a)$, and F is called antiderivative of f.

Lemma 2.1. *Assume that $f, g : \mathbb{T} \to \mathbb{R}$ are delta differential at t, then*

$$(fg)^{\Delta} = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t))$$

A function $p : \mathbb{T} \to \mathbb{R}$ is called rd-continuous, if it is continuous at right-dense point in $\mathbb{T}$ and its left limits exist at left dense points in $\mathbb{T}$. If f is continuous at each right dense point and each left dense point, then f is said to be continuous function on $\mathbb{T}$.

Lemma 2.2. *If f is rd-continuous and $t \in \mathbb{T}^k$,then $\int_t^{\sigma(t)} = \mu(t)f(t)$, where $\mu(t) = \sigma(t) - t$ is the graininess function.*

Definition 2.3. A function $p : \mathbb{T} \to \mathbb{R}$ is regressive provided

$$1 + \mu(t)p(t) \neq 0, \quad \text{for all } t \in \mathbb{T}.$$

The set of all regressive and rd-continuous functions will be denoted by $\mathfrak{R}(\mathbb{T}, \mathbb{R})$.

For two functions $p, q \in \mathfrak{R}(\mathbb{T}, \mathbb{R})$, define a plus $\oplus$ and a minus $\ominus$ by

$$p \oplus q := p + \mu pq, \quad \ominus p := -\frac{p}{1 + \mu p}, \quad p \ominus q := p \oplus (\ominus q).$$

If $p \in \mathfrak{R}(\mathbb{T}, \mathbb{R})$, then the delta exponential function is given by

$$e_p(t, s) = \exp\left(\int_s^t g(\tau)\Delta\tau\right) \quad \text{for } s, t \in \mathbb{T},$$

with $g(\tau) = \begin{cases} p(\tau), & \text{if } \mu(\tau) = 0, \\ \frac{1}{\mu(\tau)}\mathrm{Log}(1 + \mu(\tau)p(\tau)), & \text{if } \mu(\tau) \neq 0. \end{cases}$ Here Log is the principle logarithm.

It is known that $y(t) = e_p(t, t_0)$ is the unique solution of the initial value problem $y^{\Delta} = p(t)y$, $y(t_0) = 1$.

Remark 2.1. If $1 + \mu(t)p(t) > 0$ and $t_0 \in \mathbb{T}$, then $e_p(t, t_0) > 0$; if $p(t) > 0$ for $t \geq t_0$, $t_0 \in \mathbb{T}$, then $e_p(t, t_0) \geq 1$.

Lemma 2.3. *Assume that $p, q \in \mathfrak{R}(\mathbb{T}, \mathbb{R})$, then the following hold:*

(i) $e_0(t, s) \equiv 1$ *and* $e_p(t, s) \equiv 1$*;*
(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$*;*

(iii) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t,s)$;
(iv) $e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$;
(v) $e_p(t,r)e_p(r,s) = e_p(t,s)$;
(vi) $e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s)$;
(viii) $\frac{e_p(t,s)}{e_q(t,s)} = e_{p\ominus q}(t,s)$.

3. PRELIMINARIES

In this paper, we investigate the following nonlinear boundary value problem (NBVP) on time scales:

$$\begin{cases} x^{\Delta}(t) = f(t, x(t), \int_0^t k(t,s)x(s)\Delta s), & t \in J' = J - \{t_1, t_2, \ldots, t_m, \sigma(T)\}, \\ x(t_k^+) - x(t_k^-) = I_k(x(t_k)), & k = 1, 2, \ldots, m, \\ g(x(0), x(\sigma(T))) = 0, \end{cases} \tag{3.1}$$

where $J = [0, \sigma(T)]_{\mathbb{T}}$, $f \in J \times \mathbb{R}^2 \to \mathbb{R}$ is continuous in the second and the third variables, and for any $x, y \in \mathbb{R}$, $f(\cdot, x, y)$ is continuous in J', $k(t,s) \in C(J \times J, \mathbb{R})$, $g \in C(\mathbb{R}^2, \mathbb{R})$, $I_k \in C(\mathbb{R}, \mathbb{R})$, $t_k \in J$ and $0 < t_1 < \cdots < t_m < T$, $x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t = t_k$ in the sense of time scales, and in addition, if t_k is right scattered, then $y(t_k^+) = y(t_k)$, whereas, if t_k is left scattered, then $y(t_k^-) = y(t_k)$.

We will assume for the remainder of the paper that, for each $k = 1, 2, \ldots, m$, the points of impulse t_k are right dense. Let $PC(J) = \{u : J \to \mathbb{R} : u$ is continuous for any $t \in J'$, $u(t_k^+)$ and $u(t_k^-)$ exist and $u(t_k^-) = u(t_k)$, $k = 1, 2, \ldots, m,\}$. $PC(J)$ is a Banach space with the norm $\|u\|_{PC} = \sup\{|u(t)| : t \in J\}$.

Let $\Omega = PC(J) \bigcap C^1(J')$. A function $x \in \Omega$ is called a solution of NBVP (3.1), if it satisfies (3.1).

In order to establish the comparison result, we give a new inequality as follows.

Lemma 3.1. *Assume that*

(B_0) *the sequence $\{t_k\}$ satisfies $0 \le t_0 < t_1 < t_2 < \cdots < t_k < \cdots$ with $\lim_{k\to\infty} t_k = +\infty$;*
(B_1) *$m \in PC([0,\infty)_{\mathbb{T}})$ is continuous for any $t \in [0,\infty)_{\mathbb{T}}/\{t_k\}$ and left continuous at t_k for $k = 1, 2, \ldots$;*
(B_2) *for $k = 1, 2, \ldots,$ and $t \ge t_0$, there are*

$$\begin{aligned} &m^{\Delta}(t) \le p(t)m(t) + q(t), \quad t \ne t_k, \\ &m(t_k^+) \le d_k m(t_k), \end{aligned}$$

where $p, q \in C([0,\infty)_{\mathbb{T}}, \mathbb{R})$, $-\mu(t)p(t) < 1$, for $t \in [0,\infty)_{\mathbb{T}}$, and $d_k \ge 0$ are real constants.

Then

$$m(t) \le m(t_0) \prod_{t_0<t_k<t} d_k e_p(t,t_0) + \int_{t_0}^{t} \prod_{s<t_k<t} d_k e_p(t,\sigma(s)) q(s)\Delta s.$$

Proof. Noticing that $-\mu(t)p(t) < 1$ and Remark 2.1, we know that $e_{\ominus p}(t,t_0) > 0$. Let $t \in [t_0, +\infty)_{\mathbb{T}}$, then we have

$$\begin{aligned}
(e_{\ominus p}(t,t_0)m(t))^{\Delta} &= e_{\ominus p}^{\Delta}(t,t_0)m(t) + e_{\ominus p}(\sigma(t),t_0)m^{\Delta}(t)\\
&= (\ominus p(t))e_{\ominus p}(t,t_0)m(t) + (1+\mu(t)(\ominus p(t)))e_{\ominus p}(t,t_0)m^{\Delta}(t)\\
&= \frac{-p(t)}{1+\mu(t)p(t)} e_{\ominus p}(t,t_0)m(t) + \frac{1}{1+\mu(t)p(t)} e_{\ominus p}(t,t_0)m^{\Delta}(t)\\
&= \frac{e_{\ominus p}(t,t_0)}{1+\mu(t)p(t)}(-p(t)m(t) + m^{\Delta}(t)).
\end{aligned}$$

It then follows that

$$(e_{\ominus p}(t,t_0)m(t))^{\Delta} \le e_{\ominus p}(\sigma(t),t_0)q(t), \quad t \in [t_0,+\infty)_{\mathbb{T}}. \tag{3.2}$$

Integrating (3.2) from t_0 to t, where $t \in [t_0, t_1]_{\mathbb{T}}$, we have

$$e_{\ominus p}(t,t_0)m(t) - m(t_0) \le \int_{t_0}^{t} e_{\ominus p}(\sigma(s),t_0)q(s)\Delta s,$$

then

$$\begin{aligned}
m(t) &\le m(t_0)e_p(t,t_0) + \int_{t_0}^{t} e_p(t,\sigma(s))q(s)\Delta s,\\
m(t_1^+) &\le d_1 m(t_1).
\end{aligned}$$

Similarly, for $t \in (t_1, t_2]_{\mathbb{T}}$, we have

$$\begin{aligned}
m(t) &\le m(t_1^+)e_p(t,t_1) + \int_{t_1}^{t} e_p(t,\sigma(s))q(s)\Delta s\\
&\le [m(t_0)d_1 e_p(t_1,t_0) + \int_{t_0}^{t_1} d_1 e_p(t_1,\sigma(s))q(s)\Delta s]e_p(t,t_1) + \int_{t_1}^{t} e_p(t,\sigma(s))q(s)\Delta s\\
&= m(t_0)d_1 e_p(t,t_0) + \int_{t_0}^{t} \prod_{s<t_k<t} d_k e_p(t,\sigma(s))q(s)\Delta s.
\end{aligned}$$

Repeatedly, we achieve for $t \in [t_0, +\infty)_{\mathbb{T}}$:

$$m(t) \le m(t_0) \prod_{t_0<t_k<t} d_k e_p(t,t_0) + \int_{t_0}^{t} \prod_{s<t_k<t} d_k e_p(t,\sigma(s)) q(s)\Delta s.$$

This ends the proof. □

Remark 3.1. If $p(t) \equiv M$(real constant), and $q(t) \equiv 0$, Lemma 3.1 reduces to Lemma 3.2 in [13].

Inspired by the idea in [14], we establish a new comparison result which plays an important role in this paper.

Lemma 3.2. *Suppose that $u \in \Omega$ satisfies*

$$\begin{cases} u^{\Delta}(t) \le -Mu(t) - N\int_0^t k(t,s)u(s)\Delta s, & t \in J', \\ u(t_k^+) - u(t_k^-) \le -L_k u(t_k), & k = 1,2,\ldots,m, \\ u(0) \le ru(\sigma(T)), \end{cases}$$

where $M > 0$, $N \ge 0$, $\mu(t)M < 1$, $0 \le L_k < 1$, $k = 1,2,\ldots,m$, and $0 < re_{\ominus(-M)}(\sigma(T),0) \le 1$. If

$$\int_0^{\sigma(T)} q(s)\Delta s \le \prod_{k=1}^{m}(1-L_k), \tag{3.3}$$

where

$$q(t) = \frac{-N}{1-\mu(t)M}\int_0^t k(t,s)\prod_{s<t_k<\sigma(T)}(1-L_k)e_{\ominus(-M)}(t,s)\Delta s, \tag{3.4}$$

then, $u(t) \le 0$ on J.

Proof. For convenience, we let $c_k = 1 - L_k$, $k = 1,2,\ldots,m$. Set

$$v(t) = \left(\prod_{t<t_k<\sigma(T)} c_k^{-1}\right) e_{\ominus(-M)}(t,0)u(t),$$

then we have

$$\begin{cases} v^{\Delta}(t) \le \prod_{t<t_k<\sigma(T)} c_k^{-1}\Big[\frac{-N}{1-\mu(t)M}\int_0^t k(t,s)\prod_{s<t_k<\sigma(T)} c_k e_{\ominus(-M)}(t,s)v(s)\Delta s\Big], & t \in J', \\ v(t_k^+) \le c_k v(t_k), \ k = 1,2,\ldots,m, \\ v(0) \le re_{\ominus(-M)}(t,0)v(\sigma(T))\prod_{k=1}^m c_k^{-1}. \end{cases} \tag{3.5}$$

Obviously, $v(t) \le 0$ implies $u(t) \le 0$.

To show $v(t) \le 0$, we suppose, on the contrary, that $v(t) > 0$ for some $t \in J$. It is enough to consider the following cases:

(i) there exists a $\bar{t} \in J$, such that $v(\bar{t}) > 0$, and $v(t) \ge 0$ for all $t \in J$;
(ii) there exist $t_1, t^* \in J$,such that $v(t_1) < 0$, $v(t^*) > 0$.

Case (i): By (3.5), we have $v^{\Delta}(t) \le 0$ for $t \ne t_k$ and $v(t_k^+) \le v(t_k)$, $k = 1,2,\ldots,m$, hence $v(t)$ is nonincreasing in J i.e. $v(\sigma(T)) \le v(0)$. If $re_{\ominus(-M)}(\sigma(T),0) < 1$, then $v(0) < v(\sigma(T))$, which is a contradiction. If $re_{\ominus(-M)}(\sigma(T),0) = 1$, then $v(0) \le v(\sigma(T))$ which implies $v(t) \equiv C > 0$. But from (3.5) we get $v^{\Delta}(t) < 0$ for $t \in J'$. Hence, $v(\sigma(T)) < v(0)$. It's again a contradiction.

Case (ii): Let $\inf_{t\in J} v(t) = -\lambda$, then $\lambda > 0$. For some $i \in \{1,2,\ldots,m\}$, there exists $t_* \in (t_i, t_{i+1}]$ such that $v(t_*) = -\lambda$ or $v(t_*^+) = -\lambda$. We only consider $v(t_*) = -\lambda$, as for the case $v(t_*^+) = -\lambda$, the proof is similar.

From (3.5), we have

$$v^{\Delta}(t) \le \lambda \left(\prod_{t<t_k<\sigma(T)} c_k^{-1} \right) q(t), \ t \in J',$$

where $q(t)$ is defined in (3.4).

Consider the inequalities

$$\begin{cases} v^{\Delta}(t) \le \lambda \left(\prod_{t<t_k<\sigma(T)} c_k^{-1} \right) q(t), & t \in J', \\ v(t_k^+) \le c_k v(t_k), & k = i+1, i+2, \dots, m, \end{cases}$$

and Lemma 3.1 implies

$$v(t) \le v(t_*) \left(\prod_{t_*<t_k<t} c_k \right) + \lambda \int_{t_*}^{t} \left(\prod_{s<t_k<t} c_k \right) \left(\prod_{s<t_k<\sigma(T)} c_k^{-1} \right) q(s)\Delta s,$$

that is

$$v(t) \le -\lambda \left(\prod_{t_*<t_k<t} c_k \right) + \lambda \int_{t_*}^{t} \left(\prod_{t\le t_k<\sigma(T)} c_k^{-1} \right) q(s)\Delta s. \tag{3.6}$$

Let $t = \sigma(T)$ in (3.6), then

$$v(\sigma(T)) \le -\lambda \left(\prod_{t_*<t_k<\sigma(T)} c_k \right) + \lambda \int_{t_*}^{\sigma(T)} q(s)\Delta s. \tag{3.7}$$

First, we assume that $t^* > t_*$. By Lemma 3.1, we have

$$0 < v(t^*) \le v(t_*) \left(\prod_{t_*<t_k<t^*} c_k \right) + \lambda \int_{t_*}^{t^*} \left(\prod_{s<t_k<t^*} c_k \right) \left(\prod_{s<t_k<\sigma(T)} c_k^{-1} \right) q(s)\Delta s,$$

which implies that

$$\int_{t_*}^{t^*} q(s)\Delta s > \prod_{t_*\le t_k<\sigma(T)} c_k.$$

Then

$$\int_0^{\sigma(T)} q(s)\Delta s \ge \int_{t_*}^{t^*} q(s)\Delta s > \prod_{t_*\le t_k<\sigma(T)} c_k \ge \prod_{k=1}^{m} (1-L_k),$$

which is a contradiction.

Next, we assume that $t^* < t_*$. By Lemma 3.1, we have

$$0 < v(t^*) \le v(0) \left(\prod_{0<t_k<t^*} c_k \right) + \lambda \int_0^{t^*} \left(\prod_{s<t_k<t^*} c_k \right) \left(\prod_{s<t_k<\sigma(T)} c_k^{-1} \right) q(s)\Delta s,$$

which implies

$$-\lambda\int_0^{t^*}\left(\prod_{t^*\le t_k<\sigma(T)} c_k^{-1}\right) q(s)\Delta s < v(0)\left(\prod_{0<t_k<t^*} c_k\right). \tag{3.8}$$

By (3.5) and (3.7), we get from (3.8) that

$$\begin{aligned} -\lambda\int_0^{t^*}(\prod_{t^*\le t_k<\sigma(T)} c_k^{-1})q(s)\Delta s &< v(\sigma(T))re_{\ominus(-M)}(\sigma(T),0)\left(\prod_{t^*\le t_k<\sigma(T)} c_k^{-1}\right) \\ &< -\lambda re_{\ominus(-M)}(\sigma(T),0)\left(\prod_{t^*\le t_k<\sigma(T)} c_k^{-1}\right)\left[\prod_{t^*<t_k<\sigma(T)} c_k - \int_{t^*}^{\sigma(T)} q(s)ds\right]. \end{aligned}$$

Hence

$$\int_0^{\sigma(T)} q(s)\Delta s > \prod_{t^*<t_k<\sigma(T)} c_k \ge \prod_{k=1}^{m} c_k,$$

which is a contradiction to (3.3). This completes the proof. $\square$

Consider the boundary value problem(BVP)

$$\begin{cases} x^{\Delta}(t) + Mx(t) + N\int_0^t k(t,s)x(s)\Delta s = h(t), \ t\in J', \\ x(t_k^+) - x(t_k^-) = -L_k x(t_k) + I_k(\eta(t_k)) + L_k\eta(t_k), \ k=1,2,\dots,m, \\ g(\eta(0),\eta(\sigma(T))) + M_1(x(0)-\eta(0)) - M_2(x(\sigma(T))-\eta(\sigma(T))) = 0, \end{cases} \tag{3.9}$$

where $M>0$, $N\ge 0$, $0\le L_k<1$, $k=1,2,\dots,m$, M_1 and M_2 are real constants, and $\eta, h\in PC(J)$.

Lemma 3.3. *$x\in\Omega$ is a solution of BVP (3.9) if and only if x is a solution of the impulsive integral equation*

$$\begin{aligned} x(t) &= Ce_{(-M)}(t,0)B\eta + \int_0^{\sigma(T)} G(t,s)[h(s) - N\int_0^s k(s,\tau)x(\tau)\Delta\tau]\Delta s \\ &\quad + \sum_{0<t_k<\sigma(T)} G(t,t_k)[-L_k x(t_k) + I_k(\eta(t_k)) + L_k\eta(t_k)], \ t\in J, \end{aligned} \tag{3.10}$$

where $B\eta = -g(\eta(0),\eta(\sigma(T)))+M_1\eta(0)-M_2\eta(\sigma(T))$, $C=[M_1-M_2e_{(-M)}(\sigma(T),0)]^{-1}$, $M_1\ne M_2e_{(-M)}(\sigma(T),0)$ and

$$G(t,s) = \begin{cases} CM_2e_{(-M)}(t,0)e_{(-M)}(\sigma(T),\sigma(s)) + e_{(-M)}(t,\sigma(s)), & 0\le\sigma(s)<t\le\sigma(T), \\ CM_2e_{(-M)}(t,0)e_{(-M)}(\sigma(T),\sigma(s)), & 0\le t\le\sigma(s)\le\sigma(T). \end{cases}$$

Proof. Assume $x\in\Omega$ is a solution of BVP (3.9). By Lemma 2.3, we have

$$\begin{aligned} &(e_{\ominus(-M)}(t,0)x(t))^{\Delta} \\ &\quad = e^{\Delta}_{\ominus(-M)}(t,0)x(t) + e_{\ominus(-M)}(\sigma(t),0)x^{\Delta}(t) \\ &\quad = \ominus(-M)e_{\ominus(-M)}(t,0)x(t) + [1+\mu(t)(\ominus(-M))]e_{\ominus(-M)}(t,0)x^{\Delta}(t) \end{aligned}$$

$$= \frac{M}{1-\mu(t)M} e_{\ominus(-M)}(t,0)x(t) + \frac{1}{1-\mu(t)M} e_{\ominus(-M)}(t,0)x^{\Delta}(t)$$
$$= \frac{e_{\ominus(-M)}(t,0)}{1-\mu(t)M}(Mx(t) + x^{\Delta}(t)),$$

then

$$(e_{\ominus(-M)}(t,0)x(t))^{\Delta} \le e_{(-M)}(0,\sigma(t)) \left[h(t) - N\int_0^t k(t,s)x(s)\Delta s\right]. \tag{3.11}$$

Integrating both sides of (3.11) from 0 to t ($t \in [0,t_1]$) yields

$$\begin{aligned} x(t) &\le x(0)e_{(-M)}(t,0) \\ &\quad + \int_0^t e_{(-M)}(t,\sigma(s)) \left[h(s) - N\int_0^s k(s,\tau)x(\tau)\Delta\tau\right]\Delta s, \quad t \in [0,t_1], \end{aligned} \tag{3.12}$$

and integrating both sides of (3.11) from t_1^+ to t ($t \in (t_1,t_2]$) yields

$$\begin{aligned} x(t) &\le x(t_1^+)e_{(-M)}(t,t_1) \\ &\quad + \int_{t_1}^t e_{(-M)}(t,\sigma(s)) \left[h(s) - N\int_0^s k(s,\tau)x(\tau)\Delta\tau\right]\Delta s, \quad t \in (t_1,t_2], \end{aligned} \tag{3.13}$$

where

$$x(t_1^+) = x(t_1) + \omega(t_1), \tag{3.14}$$

here we denote $\omega(t_k) = -L_k x(t_k) + I_k(\eta(t_k)) + L_k\eta(t_k)$, $k = 1,2,\ldots,m$.

By (3.13) and (3.14), we obtain

$$\begin{aligned} x(t) &\le x(0)e_{(-M)}(t,0) + \int_0^t e_{(-M)}(t,\sigma(s)) \left[h(s) - N\int_0^t k(s,\tau)x(\tau)\Delta\tau\right]\Delta s \\ &\quad + e_{(-M)}(t,t_1)\omega(t_1), \quad t \in (t_1,t_2], \end{aligned}$$

then

$$\begin{aligned} x(t) &= x(0)e_{(-M)}(t,0) + \int_0^t e_{(-M)}(t,\sigma(s)) \left[h(s) - N\int_0^t k(s,\tau)x(\tau)\Delta\tau\right]\Delta s \\ &\quad + \sum_{0<t_k<t} e_{(-M)}(t,t_1)\omega(t_1), \quad t \in [0,t_2]. \end{aligned}$$

Repeating the above procedure, we obtain for $t \in J$ that

$$\begin{aligned} x(t) &= x(0)e_{(-M)}(t,0) + \int_0^t e_{(-M)}(t,\sigma(s)) \left[h(s) - N\int_0^s k(s,\tau)x(\tau)\Delta\tau\right]\Delta s \\ &\quad + \sum_{0<t_k<t} e_{(-M)}(t,t_k)\omega(t_k), \ t \in J. \end{aligned} \tag{3.15}$$

Let $t = \sigma(T)$ in (3.15), then

$$\begin{aligned} &x(\sigma(T)) \\ &\quad = x(0)e_{(-M)}(\sigma(T),0) + \int_0^{\sigma(T)} e_{(-M)}(\sigma(T),\sigma(s)) \left[h(s) - N\int_0^s k(s,\tau)x(\tau)\Delta\tau\right]\Delta s \end{aligned}$$

$$+\sum_{0<t_k<\sigma(T)} e_{(-M)}(\sigma(T),t_k)\omega(t_k),\ t\in J,$$

with the boundary value condition of BVP (3.9), we have

$$\begin{aligned} x(0) &= CB\eta+\int_0^{\sigma(T)} CM_2e_{(-M)}(\sigma(T),\sigma(s))\left[h(s)-N\int_0^s k(s,\tau)x(\tau)\Delta\tau\right]\Delta s \\ &\quad +\sum_{k=1}^{m} CM_2e_{(-M)}(\sigma(T),t_k)\omega(t_k). \end{aligned} \tag{3.16}$$

Substituting $x(0)$ in (3.16) into (3.15), we have

$$\begin{aligned} x(t) &= Ce_{(-M)}(t,0)B\eta \\ &\quad +\int_0^{\sigma(T)} CM_2e_{(-M)}(t,0)e_{(-M)}(\sigma(T),\sigma(s))\left[h(s)-N\int_0^s k(s,\tau)x(\tau)\Delta\tau\right]\Delta s \\ &\quad +\sum_{k=1}^{m} CM_2e_{(-M)}(\sigma(T),t_k)e_{(-M)}(t,0)\omega(t_k) \\ &\quad +\int_0^t e_{(-M)}(t,\sigma(s))\left[h(s)-N\int_0^s k(s,\tau)x(\tau)\Delta\tau\right]\Delta s \\ &\quad +\sum_{0<t_k<t} e_{(-M)}(t,t_k)\omega(t_k) \\ &= Ce_{(-M)}(t,0)B\eta \\ &\quad +\int_t^{\sigma(T)} CM_2e_{(-M)}(t,0)e_{(-M)}(\sigma(T),\sigma(s))\left[h(s)-N\int_0^s k(s,\tau)x(\tau)\Delta\tau\right]\Delta s \\ &\quad +\int_0^t [CM_2e_{(-M)}(t,0)e_{(-M)}(\sigma(T),\sigma(s))+e_{(-M)}(t,\sigma(s))] \\ &\qquad \times\left[h(s)-N\int_0^s k(s,\tau)x(\tau)\Delta\tau\right]\Delta s \\ &\quad +\sum_{t\le t_k<\sigma(T)} [CM_2e_{(-M)}(t,0)e_{(-M)}(\sigma(T),t_k)+e_{(-M)}(t,t_k)]\omega(t_k) \\ &\quad +\sum_{0<t_k<t} CM_2e_{(-M)}(t,0)e_{(-M)}(\sigma(T),t_k)\omega(t_k) \\ &= Ce_{(-M)}(t,0)B\eta+\int_0^{\sigma(T)} G(t,s)\left[h(s)-N\int_0^s k(s,\tau)x(\tau)\Delta\tau\right]\Delta s \\ &\quad +\sum_{k=1}^{m} G(t,t_k)\omega(t_k), \end{aligned}$$

i.e. $x(t)$ is also the solution of (3.10).

On the other hand, assume $x(t)$ is a solution of (3.10). By direct computation, we have

$$G_t^{\Delta}(t,s)=\begin{cases} -M[CM_2e_{(-M)}(t,0)e_{(-M)}(\sigma(T),\sigma(s))+e_{(-M)}(t,\sigma(s))], \\ \qquad \text{if } 0\le\sigma(s)<t\le\sigma(T); \\ -M[CM_2e_{(-M)}(t,0)e_{(-M)}(\sigma(T),\sigma(s))], \\ \qquad \text{if } 0\le t\le\sigma(s)\le\sigma(T). \end{cases}$$

Hence

$$G_t^{\Delta}(t,s) = -MG(t,s).$$

Δ-differentiating (3.10) for $t \neq t_k$, then

$$x^{\Delta}(t) = -Mx(t) + h(t) - N\int_0^t k(t,s)x(s)\Delta s.$$

It is easy to verify that

$$\begin{aligned} x(t_k^+) - x(t_k^-) &= \omega(t_k), \\ M_1x(0) - M_2x(\sigma(T)) &= -g(\eta(0),\eta(\sigma(T))) + M_1\eta(0) - M_2\eta(\sigma(T)). \end{aligned}$$

This completes the proof. □

Lemma 3.4. *If $M > 0$, $N \geq 0$, $M_1 > M_2e_{(-M)}(\sigma(T),0)$ and*

$$\sup_{t\in J}\left\{\int_0^{\sigma(T)} NG(t,s)\int_0^s k(s,\tau)\Delta\tau\Delta s\right\} + (CM_2+1)\sum_{k=1}^m L_k < 1, \tag{3.17}$$

where $C = [M_1 - M_2e_{(-M)}(\sigma(T),0)]^{-1}$, $0 \leq L_k < 1$, $k = 1,2,\ldots,m$, then BVP (3.9) has a unique solution in Ω.

Proof. For any $x \in \Omega$, define an operator $F : \Omega \to \Omega$ by

$$\begin{aligned} (Fx)(t) &= Ce_{(-M)}(t,0)B\eta + \int_0^{\sigma(T)} G(t,s)\left[h(s) - N\int_0^s k(s,\tau)x(\tau)\Delta\tau\right]\Delta s \\ &\quad + \sum_{k=1}^m G(t,t_k)[-L_kx(t_k) + I_k(\eta(t_k)) + L_k\eta(t_k)],\ t \in J, \end{aligned}$$

where $B\eta$ and G are given in Lemma 3.3.

Since

$$\max_{t,s\in J} G(t,s) = CM_2 + 1,$$

for any $x, y \in \Omega$, we have

$$\begin{aligned} &|(Fx)(t) - (Fy)(t)| \\ &\quad = |\int_0^{\sigma(T)} G(t,s)N\int_0^s k(s,\tau)[y(\tau) - x(\tau)]\Delta\tau\Delta s + \sum_{k=1}^m G(t,t_k)L_k(y(t_k) - x(t_k))| \\ &\quad \leq \xi\|x - y\|_{PC}. \end{aligned}$$

Hence

$$\|Fx - Fy\|_{PC} = \sup_{t\in J}|(Fx)(t) - (Fy)(t)| \leq \xi\|x - y\|_{PC},$$

where

$$\xi = \sup_{t\in J}\left\{\int_0^{\sigma(T)} NG(t,s)\int_0^s k(s,\tau)\Delta\tau\Delta s\right\} + (CM_2+1)\sum_{k=1}^m L_k < 1.$$

Thus F is a contractive operator and Banach fixed-point theorem assures that it has a unique fixed point in x, which is the unique solution of BVP (3.9). □

4. MAIN RESULTS

We say that $\alpha_0, \beta_0 \in \Omega$ are lower and upper solutions of NBVP (3.1) if

$$\begin{cases} \alpha_0^{\Delta}(t) \leq f(t, \alpha_0(t), \int_0^t k(t,s)\alpha_0(s)\Delta s), & t \in J', \\ \alpha_0(t_k^+) - \alpha_0(t_k^-) \leq I_k(\alpha_0(t_k)), & k = 1,2,\dots,m, \\ g(\alpha_0(0), \alpha_0(\sigma(T))) \leq 0, \end{cases}$$

and

$$\begin{cases} \beta_0^{\Delta}(t) \geq f(t, \beta_0(t), \int_0^t k(t,s)\beta_0(s)\Delta s), & t \in J', \\ \beta_0(t_k^+) - \beta_0(t_k^-) \geq I_k(\beta_0(t_k)), & k = 1,2,\dots,m, \\ g(\beta_0(0), \beta_0(\sigma(T))) \geq 0. \end{cases}$$

We formulate the following theorem, which will be used in the proof of the main results.

Theorem 4.1 (11,12]). *Assume that $\{f_n\}_{n\in\mathbb{N}}$ is a function sequence on J such that*

(i) $\{f_n\}_{n\in\mathbb{N}}$ *is uniformly bounded on J;*
(ii) $\{f_n^{\Delta}\}_{n\in\mathbb{N}}$ *is uniformly bounded on J,*

then there is a subsequence of $\{f_n\}_{n\in\mathbb{N}}$ converges uniformly on J.

Let $[\alpha_0, \beta_0] = \{x \in \Omega : \alpha_0(t) \leq x(t) \leq \beta_0(t), t \in J\}$. Now we are in the position to establish the main results of this paper.

Theorem 4.2. *Assume that the following conditions hold:*

(H_1) $\alpha_0, \beta_0 \in \Omega$ *are lower and upper solutions of NBVP* (3.1), *respectively, such that* $\alpha_0 \leq \beta_0$.

(H_2) *There exist constants $M > 0$, $N \geq 0$ such that*

$$f(t, x_1, y_1) - f(t, x_2, y_2) \geq -M(x_1 - x_2) - N(y_1 - y_2),$$

$$\alpha_0(t) \leq x_2 \leq x_1 \leq \beta_0(t), \quad y_2 \leq y_1, \quad t \in J,$$

where $\mu(t)M < 1$ for $t \in J$.

(H_3) *There exist constants $0 \leq L_k < 1$ for $k = 1, 2, \dots, m$ such that*

$$I_k(x) - I_k(y) \geq -L_k(x - y),$$

$$\alpha_0(t_k) \leq y \leq x \leq \beta_0(t_k).$$

(H_4) *There exist constants M_1, M_2 such that*

$$g(x_1, y_1) - g(x_2, y_2) \leq M_1(x_1 - x_2) - M_2(y_1 - y_2),$$

$$\alpha_0(0) \leq x_2 \leq x_1 \leq \beta_0(0), \quad \alpha_0(\sigma(T)) \leq y_2 \leq y_1 \leq \beta_0(\sigma(T)),$$

wherever $0 < M_2 e_{(-M)}(\sigma(T), 0) < M_1$.

In addition, assume that inequalities (3.3) *and* (3.17) *hold, then there exist monotone sequences* $\{\alpha_n\}, \{\beta_n\} \subset \Omega$ *with* $\alpha_0 \le \cdots \le \alpha_n \le \cdots \le \beta_n \le \cdots \le \beta_0$*, such that* $\lim_{n\to\infty} \alpha_n = x_*(t)$*,* $\lim_{n\to\infty} \beta_n = x^*(t)$ *uniformly on* J*. Moreover,* $x_*(t)$*,* $x^*(t)$ *are minimal and maximal solutions of NBVP* (3.1) *in* $[\alpha_0, \beta_0]$*, respectively.*

Proof. For any $\eta \in [\alpha_0, \beta_0]$, consider BVP (3.9) with

$$h(t) = f(t, \eta(t), \int_0^t k(t,s)\eta(s)\Delta s) + M\eta(t) + N\int_0^t k(t,s)\eta(s)\Delta s.$$

By Lemma 3.4, we know that BVP (3.9) has a unique solution $x \in \Omega$. Define an operator $A : \Omega \to \Omega$ by $x = A\eta$, then the operator A has the following properties:

(a) $\alpha_0 \le A\alpha_0$, $A\beta_0 \le \beta_0$;
(b) $A\eta_1 \le A\eta_2$, if $\alpha_0 \le \eta_1 \le \eta_2 \le \beta_0$.

To prove (a), let $\alpha_1 = A\alpha_0$, and $m(t) = \alpha_0(t) - \alpha_1(t)$. Using ($H_1$) and ($H_3$), we get

$$\begin{aligned}
m^\Delta(t) &= \alpha_0^\Delta(t) - \alpha_1^\Delta(t) \\
&\le f(t, \alpha_0(t), \int_0^t k(t,s)\alpha_0(s)\Delta s) - \Big\{ - M\alpha_1(t) - N\int_0^t k(t,s)\alpha_1(s)\Delta s \\
&\quad + f(t, \alpha_0(t), \int_0^t k(t,s)\alpha_0(s)\Delta s) + M\alpha_0(t) + N\int_0^t k(t,s)\alpha_0(s)\Delta s\Big\} \\
&\le -Mm(t) - N\int_0^t k(t,s)m(s)\Delta s,
\end{aligned}$$

$$\begin{aligned}
m(t_k^+) - m(t_k^-) &= \alpha_0(t_k^+) - \alpha_0(t_k^-) - [\alpha_1(t_k^+) - \alpha_1(t_k^-)] \\
&\le I_k(\alpha_0(t_k)) - \{-L_k\alpha_1(t_k) + I_k(\alpha_0(t_k)) + L_k\alpha_0(t_k)\} \\
&\le -L_k m(t_k),
\end{aligned}$$

$$\begin{aligned}
m(0) &= \alpha_0(0) - \alpha_1(0) \\
&= \alpha_0(0) - \Big\{ - \frac{1}{M} g(\alpha_0(0), \alpha_0(\sigma(T))) + \alpha_0(0) + \frac{M_2}{M_1}(\alpha_1(\sigma(T)) - \alpha_0(\sigma(T)))\Big\} \\
&\le \frac{M_2}{M_1} m(\sigma(T)).
\end{aligned}$$

By Lemma 3.2, we get $m(t) \le 0$ for $t \in J$ i.e. $\alpha_0 \le A\alpha_0$. Similarly, we can show that $A\beta_0 \le \beta_0$.

To prove (b), set $n(t) = x_1(t) - x_2(t)$, where $x_1 = A\eta_1$ and $x_2 = A\eta_2$.

$$\begin{aligned}
n^\Delta(t) &= x_1^\Delta(t) - x_2^\Delta(t) \\
&= M(\eta_1(t) - x_1(t)) + N\int_0^t k(t,s)[\eta_1(s) - x_1(s)]\Delta s \\
&\quad + f(t, \eta_1(t), \int_0^t k(t,s)\eta_1(s)\Delta s) \\
&\quad - M(\eta_2(t) - x_2(t)) - N\int_0^t k(t,s)[\eta_2(s) - x_2(s)]\Delta s
\end{aligned}$$

$$\begin{aligned}&-f(t,\eta_2(t),\int_0^t k(t,s)\eta_2(s)\Delta s)\\ \leq\;& -Mn(t)-N\int_0^t k(t,s)n(s)\Delta s,\end{aligned}$$

$$\begin{aligned}n(t_k^+)-n(t_k^-) &= x_1(t_k^+)-x_1(t_k^-)-[x_2(t_k^+)-x_2(t_k^-)]\\ &\leq L_k(\eta_1(t_k)-x_1(t_k))+I_k(\eta_1(t_k))-L_k(\eta_2(t_k)-x_2(t_k))-I_k(\eta_2(t_k))\\ &\leq -L_k n(t_k),\end{aligned}$$

$$\begin{aligned}n(0) &= x_1(0)-x_2(0)\\ &= -\frac{1}{M}g(\eta_1(0),\eta_1(\sigma(T)))+\eta_1(0)+\frac{M_2}{M_1}(x_1(\sigma(T))-\eta_1(\sigma(T)))\\ &\quad+\frac{1}{M}g(\eta_2(0),\eta_2(\sigma(T)))-\eta_2(0)-\frac{M_2}{M_1}(x_2(\sigma(T))-\eta_2(\sigma(T)))\\ &\leq \frac{M_2}{M_1}n(\sigma(T)).\end{aligned}$$

By Lemma 3.2, we get $n(t)\leq 0$ for $t\in J$ i.e. $A\eta_1\leq A\eta_2$, then (b) is proved.

Let $\alpha_n=A\alpha_{n-1}$, and $\beta_n=A\beta_{n-1}$ for $n=1,2,3,\ldots$. By the properties (a) and (b), we have

$$\alpha_0\leq\alpha_1\leq\cdots\leq\alpha_n\leq\cdots\leq\beta_n\leq\cdots\leq\beta_1\leq\beta_0. \tag{4.1}$$

By the definition of operator A, we have that $\{\alpha_n^\Delta\}$ and $\{\beta_n^\Delta\}$ are uniformly bounded in $[\alpha_0,\beta_0]$. By Theorem 4.1 and (4.1), we know that there exist x_*, x^* in $[\alpha_0,\beta_0]$ such that

$$\lim_{n\to\infty}\alpha_n(t)=x_*(t),\ \lim_{n\to\infty}\beta_n(t)=x^*(t)\quad\text{uniformly on } J.$$

Moreover, $x_*(t)$, $x^*(t)$ are solutions of NBVP (3.1) in $[\alpha_0,\beta_0]$.

To prove that x_*, x^* are extremal solutions of NBVP (3.1), let $u(t)\in[\alpha_0,\beta_0]$ be any solution of NBVP (3.1). By Lemma 3.2 and induction, we get $\alpha_n(t)\leq u(t)\leq\beta_n(t)$ with $t\in J$ and $n=1,2,3,\ldots$ which implies that $x_*(t)\leq u(t)\leq x^*(t)$, i.e. x_* and x^* are minimal and maximal solution of NBVP (3.1) in $[\alpha_0,\beta_0]$, respectively. The proof is completed. □

Theorem 4.3. *Suppose that all the assumptions of Theorem 4.2 and the following conditions hold:*

(H_5) *There exist constants $\overline{M}>0$, $\overline{N}\geq 0$ such that*

$$f(t,x_1,y_1)-f(t,x_2,y_2)\leq-\overline{M}(x_1-x_2)-\overline{N}(y_1-y_2),$$

$$\alpha_0(t)\leq x_2\leq x_1\leq\beta_0(t),\quad y_2\leq y_1,\quad t\in J,$$

where $\mu(t)\overline{M}<1$ for $t\in J$.

(H_6) *There exist constants* $0 \leq \overline{L}_k < 1$, *for* $k = 1, 2, \ldots, m$, *such that*

$$I_k(x) - I_k(y) \leq -\overline{L}_k(x - y),$$

$$\alpha_0(t_k) \leq y \leq x \leq \beta_0(t_k).$$

(H_7) *There exist constants* $\overline{M}_1$, $\overline{M}_2$ *such that*

$$g(x_1, y_1) - g(x_2, y_2) \geq \overline{M}_1(x_1 - x_2) - \overline{M}_2(y_1 - y_2),$$

$$\alpha_0(0) \leq x_2 \leq x_1 \leq \beta_0(0), \quad \alpha_0(\sigma(T)) \leq y_2 \leq y_1 \leq \beta_0(\sigma(T)),$$

wherever $0 < \overline{M}_2 e_{(-M)}(\sigma(T), 0) < \overline{M}_1$.

Then NBVP (3.1) *has a unique solution in* $[\alpha_0, \beta_0]$.

Proof. By Theorem 4.2, we know that there exist x_*, $x^* \in [\alpha_0, \beta_0]$, which are minimal and maximal solutions of NBVP (3.1) with $x_*(t) \leq x^*(t)$, $t \in J$. Let $m(t) = x^*(t) - x_*(t)$.Using (H_6), (H_7) and (H_8), we get

$$\begin{aligned}
m^{\Delta}(t) &= (x^*(t))^{\Delta} - (x_*(t))^{\Delta} \\
&= f(t, x^*(t), \int_0^t k(t,s)x^*(s)\Delta s) - f\left(t, x_*(t), \int_0^t k(t,s)x_*(s)\Delta s\right) \\
&\leq -\overline{M}m(t) - \overline{N}\int_0^t k(t,s)m(s)\Delta s, \\
m(t_k^+) - m(t_k^-) &= x^*(t_k^+) - x^*(t_k^-) - [x_*(t_k^+) - x_*(t_k^-)] \\
&= I_k(x^*(t_k)) - I_k(x_*(t_k)) \\
&\leq -\overline{L}_k m(t_k), \\
m(0) &= x^*(0) - x_*(0) \\
&\leq \frac{\overline{M}_2}{\overline{M}_1}(x^*(\sigma(T)) - x_*(\sigma(T))) \\
&\quad + \frac{1}{\overline{M}_1}(g(x^*(0), x^*(\sigma(T))) - g(x_*(0), x_*(\sigma(T)))) \\
&= \frac{\overline{M}_2}{\overline{M}_1} m(\sigma(T)).
\end{aligned}$$

By Lemma 3.2, we have that $m(t) < 0$, $t \in J$ i.e. $x^*(t) \leq x_*(t)$. Hence $x^*(t) = x_*(t)$, this completes the proof. □

5. **EXAMPLE**

Consider the following problem on time scale $\mathbb{T}$:

$$\text{(5.1)} \qquad \begin{cases} x^{\Delta}(t) = t^2 - 2x(t) - e^{-4}\int_0^t (ts)x(s)\Delta s, & t \in [0, \sigma(T)]_{\mathbb{T}},\ t \neq \frac{T}{2}, \\ x(t_1^+) - x(t_1^-) = -\frac{1}{4}x(t_1), & t_1 = \frac{T}{2}, \\ x(0) = \frac{1}{2}x(\sigma(T)) - \frac{1}{6}x^2(0), \end{cases}$$

where $0, T \in \mathbb{T}$ and $T = 2$.

1. Let $\mathbb{T} = \mathbb{R}$.

It is easy to see that

$$\alpha_0(t) \equiv 0, \quad \beta_0(t) = \begin{cases} \frac{1}{2}e^t, & t \in [0,1] \\ e^t, & t \in (1,2] \end{cases}$$

are lower and upper solutions of problem (5.1) such that $\alpha_0(t) \le \beta_0(t)$ for $t \in [0,2]$.

For all $t \in [0,2]$, $x,y,u,v \in \mathbb{R}$, $x \ge u$, we have

$$f(t,x,y) - f(t,u,v) = -2(x-u) - e^{-4}(y-v),$$

$$I_1(x) - I_1(u) = -\frac{1}{4}(x-u).$$

For $x,u \in [\alpha_0(0), \beta_0(0)]$, $y,v \in [\alpha_0(2), \beta_0(2)]$, we have

$$g(x,y) - g(u,v) = [\frac{1}{6}(x+u)+1](x+u) - \frac{1}{2}(y-v) \le \frac{1}{2}e^2(x-u) - \frac{1}{2}(y-v).$$

Let $M = 2$, $N = e^{-4}$, $L_1 = \frac{1}{4}$, $M_1 = \frac{1}{2}e^2$ and $M_2 = \frac{1}{2}$, then conditions (H_1)–(H_4) of Theorem 4.2 are satisfied.

Since

$$\begin{aligned} \int_0^2 q(s)ds &= \int_0^1 N \int_0^t (ts)e^{M(t-s)}(1-L_1)dsdt + \int_1^2 N \int_0^t (ts)e^{M(t-s)}dsdt \\ &\approx 0.154 \le 1 - L_1 = 0.75, \end{aligned}$$

$$\int_0^2 NG(t,s) \int_0^s (s\tau)d\tau ds \le \frac{N}{2}(CM_2+1)\int_0^2 s^3 ds \approx 0.04,$$

$$(CM_2+1)L_1 = L_1(\frac{M_2}{M_1 - M_2e^{-2M}} + 1) \approx 0.32,$$

$$\sup_{t\in[0,2]} \left\{ \int_0^2 NG(t,s) \int_0^s (s\tau)d\tau ds \right\} + (CM_2+1)L_1 \le 0.04 + 0.32 = 0.36 < 1.$$

Hence, inequalities (3.3) and (3.17) are satisfied. By Theorem 4.2, problem (5.1) has extremal solutions in $[\alpha_0, \beta_0]$. Moreover, set $\overline{M} = 2$, $\overline{N} = e^{-4}$, $\overline{L}_1 = \frac{1}{4}$, $\overline{M}_1 = \frac{1}{2}e^2$ and $\overline{M}_2 = \frac{1}{2}$, then all the conditions of Theorem 4.3 are satisfied, that is, problem (5.1) has a unique solution in $[\alpha_0, \beta_0]$.

2. Let $\mathbb{T} = \{0\} \bigcup [\frac{1}{3}, \frac{5}{3}] \bigcup [2, \infty)$.

Obviously, $\alpha_0 \equiv 0$ and $\beta_0 \equiv 2$ are lower and upper solutions of problem (5.1). If $t \in \{0, \frac{5}{3}\}$, $\mu(t) = \sigma(t) - t = \frac{1}{3}$; if $t \in \mathbb{T}/\{0, \frac{5}{3}\}$, $\mu(t) \equiv 0$. Let $M = 2$, $N = e^{-4}$, $L_1 = \frac{1}{4}$, $M_1 = \frac{1}{2}e^2$ and $M_2 = \frac{1}{2}$, then conditions (H_1)–(H_4) of Theorem 4.1 are satisfied.

By direct computation, we have

$$\begin{aligned} &\int_{\frac{1}{3}}^{\frac{5}{3}} \frac{N}{1-\mu(t)M} \int_0^t (ts) \prod_{s<t_k<2} (1-L_k) e_{\ominus(-M)}(t,s)\Delta s \Delta t \\ &= \int_{\frac{1}{3}}^{\frac{5}{3}} N \int_0^t (ts) \prod_{s<t_k<2} (1-L_k) e^{M(t-s)} dsdt \\ &\approx 0.06. \end{aligned}$$

By Lemma 2.2, we have

$$\int_0^{\frac{1}{3}} \frac{N}{1-\mu(t)M} \int_0^t (ts) \prod_{s<t_k<2} (1-L_k) e_{\ominus(-M)}(t,s) \Delta s \Delta t$$
$$= \int_0^{\sigma(0)} \frac{N}{1-\mu(t)M} \int_0^t (ts) \prod_{s<t_k<2} (1-L_k) e_{\ominus(-M)}(t,s) \Delta s \Delta t = 0,$$

$$\begin{aligned}
&\int_{\frac{5}{3}}^{2} \frac{N}{1-\mu(t)M} \int_0^t (ts) \prod_{s<t_k<2} (1-L_k) e_{\ominus(-M)}(t,s) \Delta s \Delta t \\
&= \int_{\frac{5}{3}}^{\sigma(\frac{5}{3})} \frac{N}{1-\mu(t)M} \int_0^t (ts) \prod_{s<t_k<2} (1-L_k) e_{\ominus(-M)}(t,s) \Delta s \Delta t \\
&= \frac{5}{3e^4} \int_0^{\frac{5}{3}} s \prod_{s<t_k<2} (1-L_k) e_{\ominus(-M)}(\frac{5}{3}, s) \Delta s \\
&= \frac{5}{3e^4} \Big[\int_0^{\frac{1}{3}} s \prod_{s<t_k<2} (1-L_k) e_{\ominus(-M)}(\frac{5}{3}, s) \Delta s + \int_{\frac{1}{3}}^{\frac{5}{3}} s \prod_{s<t_k<2} (1-L_k) e_{\ominus(-M)}(\frac{5}{3}, s) \Delta s \Big] \\
&= \frac{5}{3e^4} \int_{\frac{1}{3}}^{\frac{5}{3}} s \prod_{s<t_k<2} (1-L_k) e^{M(\frac{5}{3}-s)} ds \\
&\approx 0.40.
\end{aligned}$$

Hence

$$\int_0^2 q(s)\Delta s \approx 0.40 + 0.06 < 1 - L_1 = 0.75.$$

Since

$$\int_0^2 e^{-4} G(t,s) \int_s^\tau s\tau \Delta\tau \Delta s \le 8e^{-4}(CM_2+1),$$

we have

$$\sup_{t\in[0,2]_{\mathbb{T}}} \left\{ \int_0^2 NG(t,s) \int_0^s s\tau\Delta\tau\Delta s \right\} + (CM_2+1)L_1 \le (8e^{-4} + \frac{1}{4})(CM_2+1) \approx 0.13 < 1,$$

thus, inequalities (3.3) and (3.17) are satisfied. By Theorem 4.2, we know that there exist extremal solutions in $[\alpha_0, \beta_0]$. Moreover, let $\overline{M} = 2$, $\overline{N} = e^{-4}$, $\overline{L}_1 = \frac{1}{4}$, $\overline{M}_1 = \frac{1}{2}e^2$ and $\overline{M}_2 = \frac{1}{2}$, then conditions ($H_5$)-($H_7$) of Theorem 4.3 are satisfied, i.e. problem (5.1) has an unique solution in $[\alpha_0, \beta_0]$.

6. ACKNOWLEDGEMENT

This research was partially supported by the Natural Science and Engineering Consul of Canada.

REFERENCES

[1] S. Hilger, Analysis on measure chains - a unified approach to continuous and discrete calculus, *Results Math*, 18:18–56, 1990.

[2] M. Bohnor, A. Peterson, *Dynamic equations on time scales: an introduction with applications*, Bikhäuser, New York, 2001.

[3] M. Bohnor, A. Peterson, *Adavances in dynamic equations on time scales*, Bikhäuser, Boston, 2003.

[4] V. Lakshmikantham, S. Sivasundaram, B. Kaymakcalan, *Dynamic systems on measure chains*, Kluwer Acad Publ. Dordrecht, 1996.

[5] D. Guo, V. Lakshmikanthan, *Nonlinear probles in abstract cones*, Academic Press, NewYork, 1988.

[6] D. France, E. Liz, J. J. Nieto, Y. Rogovchenko, A contribution to the study of functional differential equations with impulses, *Math. Nachr.*, 218:49–60, 2000.

[7] J. J. Nieto, R. Rodríguez-López, Boundary value problems for a class of impulsive functional equations, *Comput. Math. Appl.*, 55:2715–2731, 2008.

[8] L. Chen, J. Sun, Nonlinear boundary value problem of first order impulsive functional differential equations, *J. Math. Anal. Appl.*, 318:726–741, 2006.

[9] L. Li, J. Shen, Periodic boundary value problems for functional differential equations with impulses, *Mathematica Scientia*, 25A:237–244, 2005.

[10] X. Z. Liu, D. J. Guo, Periodic boundary value problems for a class of second- order impulsive integro-differential equations in Banach space, *J. Math. Anal. Appl.*, 216:284–302, 1997.

[11] Y. Xing, M. Han, G. Zheng, Initial value problem for first-order integro-differential equation of volterra type on time scales, *Nonlinear Anal.*, 60:429–442, 2005.

[12] Y. Xing, M. Han, G. Zheng, Periodic boundary value problem of integro-differential equation of volterra type on time scales, *Nonlinear Anal.*, 68:127–138, 2008.

[13] F. Geng, Y. Xu, D. Zhu, Periodic boundary value problems for first-order impulsive dynamic equations on time scales, *Nonlinear Anal.*, 69:4074–4087, 2008.

[14] Z. Luo, J. J. Nieto, New results for the periodic boundary value problem for impulsive integro-differential equations, *Nonlinear Anal.*, 70:2248–2260, 2009.

[15] T. Jankowski, Advanced differential eqautions with nonlinear boundary conditions, *J. Math. Anal. Appl.*, 304:490–503, 2005.