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OSCILLATION OF SECOND-ORDER NONLINEAR NEUTRAL DYNAMIC EQUATIONS

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ABSTRACT. This work is concerned with the oscillation of a second-order neutral retarded dynamic equation on time scales. Some new oscillation criteria are presented that improve and complement those results reported in the literature.

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1. Introduction

Following the development of the theory of dynamic equations on time scales [1, 3, 4, 6], there has been much research activity concerning the oscillatory properties of neutral dynamic equations; see, e.g., [2, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16] and the references cited therein. Assuming

$$\int_{t_0}^{\infty} \frac{\Delta t}{r^{1/\gamma}(t)} = \infty, \tag{1.1}$$

Agarwal et al. [2] and Saker [10] established some oscillation criteria for the second-order neutral dynamic equation

$$(r(t)((x(t)+p(t)x(t-\tau))^{\Delta})^{\gamma})^{\Delta} + q(t)x^{\gamma}(t-\delta) = 0, \tag{1.2}$$

where γ is a quotient of odd positive integers, r, p, and q are real-valued positive rd-continuous functions defined on $\mathbb{T}$, some of which we present below for the convenience of the reader.

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Theorem 1.1 (See [2, Theorem 3.4]). *Assume* (1.1) *and let* $\tau > 0$, $\delta \geq 0$, *and* $0 \leq p(t) < 1$. *Suppose also that* $\gamma \geq 1$, $r^\Delta \geq 0$, *and there exists a positive rd-continuous* Δ*-differentiable function* α *such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[\alpha(s)q(s)(1-p(s-\delta))^\gamma - \frac{((\alpha^\Delta(s))_+)^2 r(s-\delta)}{4\gamma\left(\frac{s-\delta}{2}\right)^{\gamma-1}\alpha(s)}\right]\Delta s = \infty,$$

where $(\alpha^\Delta(t))_+ := \max\{0, \alpha^\Delta(t)\}$. *Then* (1.2) *is oscillatory.*

Theorem 1.2 (See [10, Corollary 3.1]). *Assume* (1.1) *and let* $\tau > 0$, $\delta \geq 0$, *and* $0 \leq p(t) < 1$. *Suppose further that* $\gamma \geq 1$ *and there exists a positive rd-continuous* Δ*-differentiable function* α *such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[\alpha(s)q(s)(1-p(s-\delta))^\gamma - \frac{((\alpha^\Delta(s))_+)^{\gamma+1} r(s-\delta)}{(\gamma+1)^{\gamma+1}\alpha^\gamma(s)}\right]\Delta s = \infty,$$

where $(\alpha^\Delta(t))_+ := \max\{0, \alpha^\Delta(t)\}$. *Then* (1.2) *is oscillatory.*

Following this trend, to develop the qualitative theory of neutral dynamic equations on time scales, in this paper we shall consider the second-order nonlinear neutral dynamic equation

$$(r(t)((x(t) + px(t-\tau))^\Delta)^\gamma)^\Delta + q(t)x^\gamma(\delta(t)) = 0 \tag{1.3}$$

on a time scale $\mathbb{T}$. We will assume that the time scale $\mathbb{T}$ under consideration is not bounded above, i.e., it is a time scale interval of the form $[t_0,\infty)_\mathbb{T} := [t_0,\infty) \cap \mathbb{T}$. Throughout, we always suppose that $0 \leq p < 1$ is a constant, $\gamma \geq 1$ is a ratio of odd positive integers, r and q are real-valued rd-continuous positive functions defined on $\mathbb{T}$, $\tau \geq 0$, $\{t-\tau : t \in [t_0,\infty)_\mathbb{T}\} = [t_0-\tau,\infty)_\mathbb{T}$, $\delta \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T},\mathbb{T})$, $\delta(t) \leq t$, and $\lim_{t\to\infty}\delta(t) = \infty$.

Let $z(t) := x(t) + px(t-\tau)$. By a solution of (1.3) we mean a nontrivial real-valued function x which has the properties $z \in \mathrm{C}^1_{\mathrm{rd}}[t_x,\infty)_\mathbb{T}$ and $r(z^\Delta)^\gamma \in \mathrm{C}^1_{\mathrm{rd}}[t_x,\infty)_\mathbb{T}$, $t_x \in [t_0,\infty)_\mathbb{T}$ and satisfying (1.3) for all $t \in [t_x,\infty)_\mathbb{T}$. Our attention is restricted to those solutions of (1.3) which exist on some half line $[t_x,\infty)_\mathbb{T}$ and satisfy $\sup\{|x(t)| : t \in [t_1,\infty)_\mathbb{T}\} > 0$ for any $t_1 \in [t_x,\infty)_\mathbb{T}$. As customary, a solution of (1.3) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is called nonoscillatory. Equation (1.3) is called oscillatory if all its solutions are oscillatory.

The aim of this work is to derive some new oscillation criteria for equation (1.3). This paper is organized as follows: In Section 2, we present some basic definitions concerning the calculus on time scales. In Section 3, we will give the main results.

2. Some preliminaries on time scales

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. Since we are interested in oscillatory behavior, we suppose that the time scale under consideration is not bounded above and is a time scale interval of the form $[t_0, \infty)_{\mathbb{T}}$. On any time scale we define the forward and backward jump operators by

$$\sigma(t) := \inf\{s \in \mathbb{T} | s > t\} \quad \text{and} \quad \rho(t) := \sup\{s \in \mathbb{T} | s < t\},$$

where $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$, $\emptyset$ denotes the empty set.

A point $t \in \mathbb{T}$ is said to be left-dense if $\rho(t) = t$ and $t > \inf \mathbb{T}$, right-dense if $\sigma(t) = t$ and $t < \sup \mathbb{T}$, left-scattered if $\rho(t) < t$, and right-scattered if $\sigma(t) > t$. The graininess μ of the time scale is defined by $\mu(t) := \sigma(t) - t$.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if it is continuous at each right-dense point and if there exists a finite left limit in all left-dense points. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathrm{C}_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$.

Fix $t \in \mathbb{T}$ and let $f : \mathbb{T} \to \mathbb{R}$. Define $f^{\Delta}(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$|[f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]| \leq \varepsilon |\sigma(t) - s| \quad \text{for all} \quad s \in U.$$

In this case, $f^{\Delta}(t)$ is called the (delta) derivative of f at t. f is said to be differentiable if its derivative exists. The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivative is rd-continuous function is denoted by $\mathrm{C}^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$. If f is differentiable at t, then f is continuous at t. If f is continuous at t and t is right-scattered, then f is differentiable at t with

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

If t is right-dense, then f is differentiable at t iff the limit

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number. In this case

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

If f is differentiable at t, then

$$f^{\sigma}(t) = f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t).$$

Let f be a real-valued function defined on an interval $[a, b]_{\mathbb{T}}$. We say that f is increasing, decreasing, nondecreasing, and nonincreasing on $[a, b]_{\mathbb{T}}$ if $t_1,\ t_2 \in [a, b]_{\mathbb{T}}$ and $t_2 > t_1$ imply $f(t_2) > f(t_1)$, $f(t_2) < f(t_1)$, $f(t_2) \geq f(t_1)$, and $f(t_2) \leq f(t_1)$, respectively. Let f be a differentiable function on $[a, b]_{\mathbb{T}}$. Then f is increasing, decreasing, nondecreasing, and nonincreasing on $[a,b]_{\mathbb{T}}$ if $f^{\Delta}(t)>0$, $f^{\Delta}(t)<0$, $f^{\Delta}(t)\geq 0$, and $f^{\Delta}(t)\leq 0$ for all $t\in[a,b)_{\mathbb{T}}$, respectively.

We will make use of the following product and quotient rules for the derivative of the product fg and the quotient f/g (where $g(t)g(\sigma(t))\neq 0$) of two differentiable functions f and g

$$(fg)^{\Delta}(t)=f^{\Delta}(t)g(t)+f(\sigma(t))g^{\Delta}(t)=f(t)g^{\Delta}(t)+f^{\Delta}(t)g(\sigma(t)),$$

$$\left(\frac{f}{g}\right)^{\Delta}(t)=\frac{f^{\Delta}(t)g(t)-f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}.$$

For $a,\ b\in\mathbb{T}$ and a differentiable function f, the Cauchy integral of f^{Δ} is defined by

$$\int_a^b f^{\Delta}(t)\Delta t=f(b)-f(a).$$

The integration by parts formula reads

$$\int_a^b f^{\Delta}(t)g(t)\Delta t=f(b)g(b)-f(a)g(a)-\int_a^b f^{\sigma}(t)g^{\Delta}(t)\Delta t,$$

and infinite integrals are defined as

$$\int_a^{\infty} f(s)\Delta s=\lim_{t\to\infty}\int_a^t f(s)\Delta s.$$

3. **Main results**

In this section, using the Riccati transformation technique we obtain new oscillation results for equation (1.3). In what follows, all functional inequalities are assumed to hold eventually, that is, for all sufficiently large t.

Firstly, we give two lemmas which we will use in the proofs of the main theorem.

Lemma 3.1 (See [3, Theorem 1.90]). *If* $z\in \mathrm{C}^1_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$, *then*

$$(z^{\gamma})^{\Delta}(t)=\gamma z^{\Delta}(t)\int_0^1\left[hz^{\sigma}(t)+(1-h)z(t)\right]^{\gamma-1}\mathrm{d}h.$$

Lemma 3.2 (See [3]). *Assume* $\sup\mathbb{T}=\infty$ *and let* $v\in \mathrm{C}^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{T})$ *be a strictly increasing function and unbounded such that* $v([t_0,\infty)_{\mathbb{T}})=[v(t_0),\infty)_{\mathbb{T}}$. *Then*

$$(y(v(t)))^{\Delta}=y^{\Delta}(v(t))v^{\Delta}(t)\quad \textit{for}\quad t\in[t_0,\infty)_{\mathbb{T}},\tag{3.1}$$

where $y\in \mathrm{C}^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$.

Now we establish the main results.

Theorem 3.3. *Assume* (1.1) *and let* $r \in \mathrm{C}^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$, $r^\Delta \geq 0$, *and*

$$\int_{t_0}^{\infty} \delta^\gamma(t) q(t) \Delta t = \infty. \tag{3.2}$$

If there exist a positive rd-continuous Δ*-differentiable function* η *and a constant* $c \in (1,\infty)$ *such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[\left(\frac{1}{1+pc}\right)^\gamma \eta(s) q(s) \left(\frac{\delta(s)}{s}\right)^\gamma - \frac{1}{(\gamma+1)^{\gamma+1}} \frac{r(s)((\eta^\Delta(s))_+)^{\gamma+1}}{\eta^\gamma(s)} \right] \Delta s = \infty, \tag{3.3}$$

where $(\eta^\Delta(t))_+ := \max\{0, \eta^\Delta(t)\}$, *then* (1.3) *is oscillatory.*

Proof. Suppose to the contrary that x is a nonoscillatory solution of (1.3). Without loss of generality, we may assume that there is a $t_1 \in [t_0,\infty)_{\mathbb{T}}$ such that $x(t) > 0$, $x(t-\tau) > 0$, and $x(\delta(t)) > 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$. From (1.3), we see that

$$(r(z^\Delta)^\gamma)^\Delta(t) = -q(t) x^\gamma(\delta(t)) < 0. \tag{3.4}$$

It is easy to see that $z^\Delta > 0$ due to condition (1.1). From $r^\Delta \geq 0$, $z^\Delta > 0$, $(r(z^\Delta)^\gamma)^\Delta < 0$, and Lemma 3.1, we obtain $z^{\Delta\Delta} < 0$. By $z^\Delta > 0$ and $z^{\Delta\Delta} < 0$, we have $\lim_{t\to\infty} z^\Delta(t) = a \geq 0$, where a is finite. Define the function ω by

$$\omega(t) := \eta(t) \frac{r(t)(z^\Delta(t))^\gamma}{z^\gamma(t)}, \quad t \in [t_1,\infty)_{\mathbb{T}}. \tag{3.5}$$

Then $\omega(t) > 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$ and

$$\begin{aligned} \omega^\Delta(t) &= \eta(t) \frac{(r(z^\Delta)^\gamma)^\Delta(t)}{z^\gamma(t)} \\ &\quad + (r(z^\Delta)^\gamma)^\sigma(t) \frac{z^\gamma(t)\eta^\Delta(t) - \eta(t)(z^\gamma)^\Delta(t)}{z^\gamma(t) z^\gamma(\sigma(t))}. \end{aligned} \tag{3.6}$$

It follows from (3.4), (3.5), and (3.6) that

$$\begin{aligned} \omega^\Delta(t) \leq &- \eta(t) q(t) \left(\frac{x(\delta(t))}{z(t)}\right)^\gamma + \frac{(\eta^\Delta(t))_+}{\eta^\sigma(t)} \omega^\sigma(t) \\ &- \frac{\eta(t)(r(z^\Delta)^\gamma)^\sigma(t)(z^\gamma)^\Delta(t)}{z^\gamma(t) z^\gamma(\sigma(t))}. \end{aligned} \tag{3.7}$$

By Lemma 3.1 and $z^\Delta > 0$, we have

$$(z^\gamma)^\Delta(t) \geq \gamma z^{\gamma-1}(t) z^\Delta(t). \tag{3.8}$$

From (3.2) and the proof of [11, Lemma 2.1], we get that

$$\frac{z(t)}{t} \quad \text{is strictly decreasing eventually.} \tag{3.9}$$

Also by $(r(z^\Delta)^\gamma)^\Delta < 0$, we obtain

$$z^\Delta(t) \geq \left(\frac{r^\sigma(t)}{r(t)}\right)^{1/\gamma} z^\Delta(\sigma(t)). \tag{3.10}$$

Thus, from (3.5), (3.7), (3.8), (3.9), (3.10), we have

$$\begin{aligned} \omega^\Delta(t) \leq & - \eta(t)q(t)\left(\frac{\delta(t)}{t}\right)^\gamma \left(\frac{x(\delta(t))}{x(\delta(t)) + px(\delta(t)-\tau)}\right)^\gamma \\ & + \frac{(\eta^\Delta(t))_+}{\eta^\sigma(t)}\omega^\sigma(t) - \frac{\gamma\eta(t)}{r^{1/\gamma}(t)(\eta^\sigma(t))^{(\gamma+1)/\gamma}}(\omega^\sigma(t))^{(\gamma+1)/\gamma}. \end{aligned} \tag{3.11}$$

Assume first $a > 0$. Hence by Lemma 3.2, we get $x^\Delta > 0$, and so

$$\frac{x(\delta(t))}{x(\delta(t)) + px(\delta(t)-\tau)} = \frac{1}{1 + p\frac{x(\delta(t)-\tau)}{x(\delta(t))}} \geq \frac{1}{1+p}. \tag{3.12}$$

Assume now $a = 0$. It follows from $z > 0$ and $z^\Delta > 0$ that either $\lim_{t\to\infty} x(t) = b > 0$ (b is finite) or $\lim_{t\to\infty} x(t) = \infty$. Hence we have

$$\frac{x(\delta(t)-\tau)}{x(\delta(t))} < c$$

for every constant $c \in (1, \infty)$. Thus, we get

$$\frac{x(\delta(t))}{x(\delta(t)) + px(\delta(t)-\tau)} = \frac{1}{1 + p\frac{x(\delta(t)-\tau)}{x(\delta(t))}} \geq \frac{1}{1+pc}. \tag{3.13}$$

Using (3.11), (3.12), and (3.13), we see that

$$\begin{aligned} \omega^\Delta(t) \leq & - \left(\frac{1}{1+pc}\right)^\gamma \eta(t)q(t)\left(\frac{\delta(t)}{t}\right)^\gamma \\ & + \frac{(\eta^\Delta(t))_+}{\eta^\sigma(t)}\omega^\sigma(t) - \frac{\gamma\eta(t)}{r^{1/\gamma}(t)(\eta^\sigma(t))^{(\gamma+1)/\gamma}}(\omega^\sigma(t))^{(\gamma+1)/\gamma}. \end{aligned} \tag{3.14}$$

Setting

$$A := \frac{\gamma\eta(t)}{r^{1/\gamma}(t)(\eta^\sigma(t))^{(\gamma+1)/\gamma}}, \quad B := \frac{(\eta^\Delta(t))_+}{\eta^\sigma(t)}, \quad u := \omega^\sigma(t),$$

and applying the inequality

$$Bu - Au^{(\gamma+1)/\gamma} \leq \frac{\gamma^\gamma}{(\gamma+1)^{\gamma+1}}\frac{B^{\gamma+1}}{A^\gamma}, \quad A > 0,$$

we have

$$\begin{aligned} \frac{(\eta^\Delta(t))_+}{\eta^\sigma(t)}\omega^\sigma(t) & - \frac{\gamma\eta(t)}{r^{1/\gamma}(t)(\eta^\sigma(t))^{(\gamma+1)/\gamma}}(\omega^\sigma(t))^{(\gamma+1)/\gamma} \\ & \leq \frac{1}{(\gamma+1)^{\gamma+1}}\frac{r(t)((\eta^\Delta(t))_+)^{\gamma+1}}{\eta^\gamma(t)}. \end{aligned}$$

Substituting the latter inequality into (3.14), we get

$$\omega^\Delta(t) \leq -\left(\frac{1}{1+pc}\right)^\gamma \eta(t)q(t)\left(\frac{\delta(t)}{t}\right)^\gamma + \frac{1}{(\gamma+1)^{\gamma+1}}\frac{r(t)((\eta^\Delta(t))_+)^{\gamma+1}}{\eta^\gamma(t)}.$$

Integrating the last inequality from t_1 to t, we see that

$$\int_{t_1}^{t}\left[\left(\frac{1}{1+pc}\right)^{\gamma}\eta(s)q(s)\left(\frac{\delta(s)}{s}\right)^{\gamma}-\frac{1}{(\gamma+1)^{\gamma+1}}\frac{r(s)((\eta^{\Delta}(s))_+)^{\gamma+1}}{\eta^{\gamma}(s)}\right]\Delta s\leq\omega(t_1),$$

which contradicts condition (3.3). The proof is complete. □

From Theorem 3.3, one can establish different sufficient conditions for oscillation of (1.3) by different choices of η. For instance, if $\eta(t)=t$, then we get the following result.

Corollary 3.4. Assume (1.1) and (3.2), and let $r\in \mathrm{C}^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$, $r^{\Delta}\geq 0$. If there exists a constant $c\in(1,\infty)$ such that

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\left(\frac{1}{1+pc}\right)^{\gamma}sq(s)\left(\frac{\delta(s)}{s}\right)^{\gamma}-\frac{1}{(\gamma+1)^{\gamma+1}}\frac{r(s)}{s^{\gamma}}\right]\Delta s=\infty,$$

then (1.3) is oscillatory.

As an application of the main results, we provide the following example.

Example 3.5. Consider the neutral dynamic equation

$$[x(t)+px(t-\tau)]^{\Delta\Delta}+\frac{\lambda}{t^2}x(t)=0,\quad t\in[t_0,\infty)_{\mathbb{T}}, \tag{3.15}$$

where $\lambda>0$ and $p\in(0,1)$ are constants. Let $\gamma=1$, $r(t)=1$, $q(t)=\lambda/t^2$, and $\delta(t)=t$. Using Corollary 3.4, we see that equation (3.15) is oscillatory if $\lambda>(1+pc)/4$ for some constant $c\in(1,1/1-p)$. Using Theorem 1.1 or Theorem 1.2, we obtain that equation (3.15) is oscillatory if $\lambda>1/4(1-p)$. One can easily see that

$$\frac{1}{4(1-p)}>\frac{1+pc}{4}$$

for every constant $c\in(1,1/1-p)$, and hence our result improves those.

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