Dynamic Systems and Applications 22 (2013) 543-556

SOLVABILITY OF A SEMILINEAR ANISOTROPIC HYPERBOLIC PROBLEM

DUMITRU MOTREANU

Département de Mathématiques, Université de Perpignan Via Domitia,
Perpignan 66860, France motreanu@univ-perp.fr

ABSTRACT. In this paper we prove the solvability of a unilateral dynamic problem driven by a wave equation with nonconstant coefficients in the principal part and containing a nonlinear reaction term and constraints of obstacle type on the boundary.

AMS (MOS) Subject Classification. 35L86, 49J40.

1. INTRODUCTION

In this paper we examine the solvability of a unilateral dynamic problem driven by a semilinear wave equation with nonconstant coefficients in the principal part and exhibiting constraints of obstacle type on the boundary. The precise formulation of our hyperbolic problem is as follows. Let Ω be a bounded domain in $\mathbb{R}^N$ with a C^2-boundary $\Gamma = \partial\Omega$ such that for disjoint parts Γ_1 and Γ_2 of Γ one has $\Gamma = \overline{\Gamma}_1 \cup \Gamma_2 = \Gamma_1 \cup \overline{\Gamma}_2$ and $\Gamma_1 \cap \Gamma_2$ is of class C^2. For some $T > 0$, consider the following initial-boundary value problem (P): find $u(x,t)$ such that

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \sum\limits_{i,j=1}^{N} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial u}{\partial x_j}) = f(x,t,u) & \text{in } Q = \Omega \times (0,T) \\ u(x,0) = u_0(x), \quad \frac{\partial u}{\partial t}(x,0) = u_1(x) & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_1 \times (0,T) \\ u(x,t) \geq \Phi(x) & \text{on } \Gamma_2 \times (0,T) \\ \sum\limits_{i,j=1}^{N} a_{ij}(x)\frac{\partial u}{\partial x_j}\nu_i \geq 0, \quad (u-\Phi)\sum\limits_{i,j=1}^{N} a_{ij}(x)\frac{\partial u}{\partial x_j}\nu_i = 0 & \text{on } \Gamma_2 \times (0,T), \end{cases}$$

where $\nu = (\nu_1, \ldots, \nu_N)$ is the unit exterior normal vector on $\partial\Omega$, $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$, $\Phi \in C^1(\overline{\Omega})$ with $\Phi \leq 0$ on Γ_2.

Problem (P) is an extension of the problem studied in Kim [4] by allowing that the coefficients a_{ij} depend on $x \in \Omega$ and the right-hand side f depends on the solution u. The stated problem models, among other things, a non-homogeneous anisotropic membrane with a dynamic contact on the boundary and with a solution-dependent

Received June 29, 2012 1056-2176 $15.00 ©Dynamic Publishers, Inc.

load. For earlier unilateral contact problems we refer to Lebeau and Schatzman [5] and Maruo [6] as well as to the monograph [2].

With the data above, we formulate our hypotheses:

(H_1) the function $f(x,t,s)$ is measurable with respect to $(x,t) \in Q$, $f(\cdot,\cdot,0) \in L^2(Q)$, and is Lipschitz continuous in $s \in \mathbb{R}$ uniformly in $(x,t) \in Q$, that is there exists a constant $L > 0$ such that

$$|f(x,t,s_1) - f(x,t,s_2)| \leq L|s_1 - s_2| \text{ for a.a. } (x,t) \in Q, \text{ all } s_1, s_2 \in \mathbb{R};$$

(H_2) $a_{ij} \in C^1(\overline{\Omega})$, $a_{ij} = a_{ji}$ for all $i,j = 1,\dots,N$, and

$$c|\xi|^2 \leq \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \text{ for all } (x,t) \in \overline{\Omega},\ \xi = (\xi_1,\dots,\xi_N) \in \mathbb{R}^N,$$

with a constant $c > 0$.

In order to build our functional setting, we set

$$G = \{w \in H^1(\Omega) : w = 0 \text{ on } \Gamma_1,\ w \geq \Phi \text{ on } \Gamma_2\}$$

and

$$H^1_{\Gamma_1}(\Omega) = \{w \in H^1(\Omega) : w = 0 \text{ on } \Gamma_1\},$$

which have to be understood in the sense of traces. A solution of problem (P) means any $u \in L^\infty(0,T;H^1_{\Gamma_1}(\Omega))$ provided $u' = \frac{\partial u}{\partial t} \in L^\infty(0,T;L^2(\Omega)) \cap C([0,T];H^{-\frac{1}{2}}(\Omega))$ that satisfies

$$\begin{aligned} &\langle u'(T), w(T) - u(T)\rangle - \langle u_1, w(0) - u_0\rangle \\ &\quad - \int_0^T \langle u', w' - u'\rangle dt + \int_0^T \int_\Omega \sum_{i,j=1}^{N} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial (w-u)}{\partial x_j} dx\, dt \\ &\quad \geq \int_0^T \int_\Omega f(x,t,u)(w-u) dx dt \end{aligned} \tag{1.1}$$

for all $w \in L^\infty(0,T;H^1_{\Gamma_1}(\Omega))$ with $w' \in L^\infty(0,T;L^2(\Omega))$ and $w(t) \in G$ for a.a. $t \in (0,T)$, and

$$u(x,0) = u_0(x), \quad u'(x,0) = u_1(x) \text{ a.e. in } \Omega, \tag{1.2}$$

$$u(t) \in G \text{ for a.a. } t \in (0,T). \tag{1.3}$$

The motivation for this concept of solution is given in [4] (see also [3, Chapter 7]).

Our main result is the following existence theorem.

Theorem 1.1. *Assume that hypotheses (H_1) and (H_2) are satisfied. Then there exists a solution of problem (P).*

Theorem 1.1 extends the main result in Kim [4] in two directions: (a) the coefficients a_{ij} are allowed to depend on $x \in \Omega$, which makes the problem anisotropic; (b) the right-hand side of the equation can depend on the solution u. It is also worth mentioning that contrary to the case in Kim [4] we do not assume that Γ_1 be of positive surface measure. This two-fold extension required to overcome serious mathematical difficulties related to the anisotropic character of the problem and the presence of an additional nonlinearity. It also broadened the area of applicability covering new situations with lack of homogeneity.

We will prove Theorem 1.1 by means of the penalty method and approximation. To this end we first regularize the term $f(x,\cdot,u)$ by introducing the regularization $f_k : \Omega \times (0,T) \times \mathbb{R} \to \mathbb{R}$ of f for any integer $k \geq 1$ as

$$f_k(x,t,s) = k\int_{-\infty}^{+\infty} \eta(k(t-\tau))f(x,\tau,s)d\tau = k\int_{-\frac{1}{k}}^{\frac{1}{k}} \eta(k(t-\tau))f(x,\tau,s)d\tau,$$

where η stands for the standard mollifier (see, e.g., [3, p. 629]). Then, for every integer $k \geq 1$, we formulate the approximate problem (P_k): find $u_k \in L^\infty(0,T;H^1_{\Gamma_1}(\Omega))$ with $u_k' \in L^\infty(0,T;H^1_{\Gamma_1}(\Omega))$ and $u_k'' = \frac{\partial^2 u_k}{\partial t^2} \in L^\infty(0,T;L^2(\Omega))$ which satisfies

$$\langle u_k''(t), v\rangle + a(u_k(t),v) = \langle f_k(\cdot,t,u_k(t)),v\rangle + \int_{\Gamma_2} (k(u_k(t)-\Phi)^- - \frac{1}{k}u_k'(t))v d\sigma \tag{1.4}$$

for all $v \in H^1_{\Gamma_1}(\Omega)$ and for a.a. $t \in (0,T)$, and

$$u_k(x,0) = u_0(x), \quad u_k'(x,0) = u_1(x) \text{ a.e. in } \Omega, \tag{1.5}$$

where we used the notation

$$a(u,v) = \sum_{i,j=1}^{N} \int_\Omega a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}dx \ \text{ for all } u,v \in H^1_{\Gamma_1}(\Omega) \tag{1.6}$$

and the notation r^- with $r \in \mathbb{R}$ which stands for the negative part of r, that is $r^- = \max\{-r, 0\}$. It is shown in Theorem 2.1 that a solution u_k of (P_k) exists. In addition, we prove in Theorem 3.1 that the found solution u_k of (P_k) satisfies some basic a priori estimates (see (3.3), (3.4)). Finally, a solution of problem (P) is obtained by passing to the limit as $k \to \infty$ in (1.4), (1.5), which is possible in view of the a priory estimates and a compensated compactness technique that we adapt from [1] and [4].

2. CONSTRUCTION OF APPROXIMATE SOLUTIONS

This section is devoted to the following result.

Theorem 2.1. *Assume that hypotheses (H_1) and (H_2) are satisfied. Then for every integer $k \geq 1$, the approximate problem (P_k) has a solution.*

Proof. We proceed through a Galerkin's approximation procedure (see, e.g., [7, p. 922]). Let $\{v_i\}_{i\geq 1}$ be a Galerkin basis of the space $H^1_{\Gamma_1}(\Omega)$ introduced in Section 1, with $v_i \in C^\infty(\overline{\Omega}) \cap H^1_{\Gamma_1}(\Omega)$ for all $i \geq 1$. Given $n \geq 1$, we seek $w_n \in C^2([0,T]; H^1_{\Gamma_1}(\Omega))$ in the form

$$w_n(x,t) = \sum_{i=1}^{m} a_{ni}(t) v_i(x) \quad \text{for all } (x,t) \in Q \tag{2.1}$$

such that

$$\begin{aligned} \langle w_n''(t), v_j\rangle + a(w_n(t), v_j) &= \langle f_k(\cdot, t, w_n(t)), v_j\rangle \\ &+ \int_{\Gamma_2} (k(w_n(t)-\Phi)^- - \frac{1}{k} w_n'(t)) v_j dt \quad \text{for all } t \in [0,T], \quad j = 1, \dots, n, \end{aligned} \tag{2.2}$$

$$w_n(0) = \text{proj } u_0, \quad w_n'(0) = \text{proj } u_1, \tag{2.3}$$

where proj u_0 and proj u_1 denote the orthogonal projections on span$\{v_1, \dots, v_n\}$ of u_0 and u_1 in $H^1_{\Gamma_1}(\Omega)$ and $L^2(\Omega)$, respectively, and the symmetric bilinear map $a(\cdot,\cdot)$ is introduced in (1.6). A standard existence and uniqueness theorem for initial value problems of ordinary differential equations yield unique functions $a_{ni} \in C^2([0,T])$, $i = 1, \dots, n$, such that w_n in (2.1) satisfies (2.2), (2.3). Here we essentially utilize the Lipschitz condition in assumption (H_1). Multiplying equation (2.2) by $a_{nj}'(t)$ and summing up over $1 \leq j \leq n$ lead to

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|w_n'(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\frac{d}{dt} a(w_n(t), w_n(t)) + \frac{k}{2}\frac{d}{dt}\|(w_n(t)-\Phi)^-\|^2_{L^2(\Gamma_2)} \\ &\quad = \langle f_k(\cdot, t, w_n(t)), w_n'(t)\rangle - \frac{1}{k}\|w_n'(t)\|^2_{L^2(\Gamma_2)} \quad \text{for all } t \in [0,T]. \end{aligned}$$

By integration over $[0,t]$ with any $t \in [0,T]$ and Young's inequality, we derive that

$$\begin{aligned} &\frac{1}{2}\|w_n'(t)\|^2_{L^2(\Omega)} + \frac{1}{2} a(w_n(t), w_n(t)) + \frac{k}{2}\|(w_n(t)-\Phi)^-\|^2_{L^2(\Gamma_2)} \\ &\quad + \frac{1}{k}\int_0^t \|w_n'(s)\|^2_{L^2(\Gamma_2)} ds \\ &\quad\quad \leq \frac{1}{2}\int_0^t \|f(\cdot, s, w_n(s))\|^2_{L^2(\Omega)} ds + \frac{1}{2}\int_0^t \|w_n'(s)\|^2_{L^2(\Omega)} ds + C \end{aligned} \tag{2.4}$$

with a constant $C > 0$ depending on $\|u_0\|_{H^1(\Omega)}$ and $\|u_1\|_{L^2(\Omega)}$ as can be seen from (2.3) and the properties of convolution.

Hypothesis (H_1) implies that

$$|f(x,t,s)| \leq |f(x,t,0)| + L|s| \quad \text{for a.a. } (x,t) \in Q, \text{ all } s \in \mathbb{R}. \tag{2.5}$$

On the basis of (2.5) and assumption (H_2), through Gronwall's inequality we infer from (2.4) the estimates

$$\|w_n(t)\|_{H^1(\Omega)} \leq M \quad \text{for all } t \in [0,T], \tag{2.6}$$

$$\|w_n'(t)\|_{L^2(\Omega)} \leq M \quad \text{for all } t \in [0,T], \tag{2.7}$$

$$\|(w_n(t)-\Phi)^-\|_{L^2(\Gamma_2)} \leq M \quad \text{for all } t \in [0,T], \tag{2.8}$$

$$\int_0^T \|w_n'(t)\|^2_{L^2(\Gamma_2)} ds \leq M, \tag{2.9}$$

with a constant $M > 0$ that depends on k but not on n. We emphasize that the estimate (2.6) in the norm of $H^1(\Omega)$ is the consequence of (2.4) combined with (2.7), taking into account that

$$w_n(t) = w_n(0) + \int_0^t w_n'(s)ds.$$

Furthermore, from (2.2) in conjunction with (2.3), (2.4), we note that

$$\|w_n''(0)\|_{L^2(\Omega)} \leq M \quad \text{for all } t \in [0,T], \tag{2.10}$$

with a constant $M > 0$ independent on n.

From (H_1) we see that the partial derivative $(f_k)_t'(x,t,s)$ with respect to t of the function $f_k(x,t,s)$ fulfills the Lipschitz condition

$$|(f_k)_t'(x,t,s_1) - (f_k)_t'(x,t,s_2)| \leq L|s_1 - s_2| \tag{2.11}$$

for a.a. $(x,t) \in Q$, all $s_1, s_2 \in \mathbb{R}$. It is also worth noting that the partial derivative $(f_k)_s'(x,t,s)$ with respect to s of $f(x,t,s)$ exists for a.a. $s \in \mathbb{R}$, a.a. $(x,t) \in Q$, thanks to the Lipschitz continuity in (H_1).

In view of the regularity of f_k, differentiation of (2.2) with respect to t can be done, which implies for all $j = 1,\dots,n$ that

$$\begin{aligned}\langle w_n'''(t), v_j\rangle + a(w_n'(t), v_j) &= \langle (f_k)_t'(\cdot,t,w_n(t)), v_j\rangle + \langle (f_k)_s'(\cdot,t,w_n(t))w_n'(t), v_j\rangle \\ &+ k\int_{\Gamma_2} \frac{d}{dt}(w_n(t)-\Phi)^- v_j d\sigma - \frac{1}{k}\int_{\Gamma_2} w_n''(t) v_j d\sigma \quad \text{for a.a. } t \in (0,T).\end{aligned}$$

Multiplying the latter by $a_{nj}''(t)$, summing up over $1 \leq j \leq n$ and using Young's inequality enable us to find the estimate

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|w_n''(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\frac{d}{dt}a(w_n'(t), w_n'(t)) \\ &\quad\leq \frac{1}{2}(\|(f_k)_t'(\cdot,t,w_n(t))\|^2_{L^2(\Omega)} + \|(f_k)_s'(\cdot,t,w_n(t))w_n'(t)\|^2_{L^2(\Omega)}) \\ &\qquad + \|w_n''(t)\|^2_{L^2(\Omega)} + \frac{k}{2}\int_{\Gamma_2}\left(\frac{d}{dt}(w_n(t)-\Phi)^-\right)^2 d\sigma - \frac{1}{k}\|w_n''(t)\|^2_{L^2(\Gamma_2)}.\end{aligned}$$

Now we integrate over $[0,t]$, with $0 \leq t \leq T$, obtaining that

$$\begin{aligned}&\frac{1}{2}\|w_n''(t)\|^2_{L^2(\Omega)} + \frac{1}{2}a(w_n'(t), w_m'(t)) \\ &\quad\leq \frac{1}{2}\|w_n''(0)\|^2_{L^2(\Omega)} + \frac{1}{2}a(w_n'(0), w_m'(0))\end{aligned} \tag{2.12}$$

$$+ C_1 \int_0^t (\|w_n(\tau)\|^2_{L^2(\Omega)} + \|w_n'(\tau)\|^2_{L^2(\Omega)} + \|w_n''(\tau)\|^2_{L^2(\Omega)})d\tau + C_2$$
$$+ \frac{k}{2} \int_0^t \int_{\Gamma_2} \left(\frac{d}{dt}(w_n(\tau) - \Phi)^- \right)^2 d\sigma\, d\tau,$$

with constants $C_1, C_2 > 0$. Here (H_1) and (2.11) have been used. Invoking now (H_2), (2.10), (2.3), (2.6), (2.7), (2.9), by means of Gronwall's inequality it turns out from (2.12) that there exists a constant $M = M(k) > 0$ independent of n such that

$$\|w_n'(t)\|_{H^1(\Omega)} \le M \quad \text{for all } t \in [0,T], \tag{2.13}$$

$$\|w_n''(t)\|_{L^2(\Omega)} \le M \quad \text{for all } t \in [0,T]. \tag{2.14}$$

By (2.6), (2.13), (2.14), it follows that along a relabeled subsequence there hold

$$w_n \to u_k \quad \text{weak}^* \text{ in } L^\infty(0,T; H^1_{\Gamma_1}(\Omega)), \tag{2.15}$$

$$w_n' \to u_k' \quad \text{weak}^* \text{ in } L^\infty(0,T; H^1_{\Gamma_1}(\Omega)), \tag{2.16}$$

$$w_n'' \to u_k'' \quad \text{weak}^* \text{ in } L^\infty(0,T; L^2(\Omega)) \tag{2.17}$$

as $n \to \infty$, for some $u_k \in L^\infty(0,T; H^1_{\Gamma_1}(\Omega))$ with $u_k' \in L^\infty(0,T; H^1_{\Gamma_1}(\Omega))$ and $u_k'' \in L^\infty(0,T; L^2(\Omega))$. The convergence in (2.15) and (2.16) implies that

$$(w_n - \Phi)^- \to (u_k - \Phi)^- \quad \text{in } C([0,T]; L^2(\partial\Omega)) \text{ as } n \to \infty \tag{2.18}$$

(see [4, Lemma 1.4]). Passing to the limit as $n \to \infty$ in (2.2), we infer from (2.17), (2.15), (2.16) and (2.18) that (1.4) holds true for all vectors $v \in \text{span}\{v_1, \dots, v_n, \dots\}$. Since the latter is dense in $H^1_{\Gamma_1}(\Omega)$, the assertion stated in (1.4) is established. From (2.3), (2.15), (2.16) we readily get $u_k(\cdot, 0) = u_0$, while (2.3), (2.16), (2.17) entail $u_k'(\cdot, 0) = u_1$. Hence (1.5) is proven, which completes the proof. □

3. A PRIORI ESTIMATES FOR THE APPROXIMATE SOLUTIONS

For $\delta > 0$ small we set

$$\Omega_\delta = \{x \in \Omega : \text{dist}(x, \partial\Omega) > \delta\}$$

and

$$S_\delta = \partial\Omega_\delta = \{x \in \Omega : \text{dist}(x, \partial\Omega) = \delta\}.$$

Choose $h = (h_1, \dots, h_N) \in C^1(\overline{\Omega})^N$ such that $h(x)$ is the exterior unit normal vector to S_δ at any $x \in S_\delta$ provided $\delta > 0$ is sufficiently small. This can be done due to the regularity assumption that Ω has a C^2-boundary. Moreover, we may assume that h is normalized as follows

$$\sum_{i,j=1}^N a_{ij} h_i h_j = 1, \tag{3.1}$$

which can be achieved by replacing h with $\Big(1/\Big(\sum_{i,j=1}^{N} a_{ij}h_ih_j\Big)^{1/2}\Big)h$ on the basis of assumption (H_2).

Theorem 3.1. *Assume that hypotheses (H_1) and (H_2) are satisfied. Then there exist constants $M > 0$ and $\delta_0 > 0$ such that for every integer $k \geq 1$ the approximate problem (P_k) has a solution u_k satisfying*

$$u_k \in L^\infty(0,T;H^2(\Omega_\delta)) \cap C([0,T];H^1_{\Gamma_1}(\Omega)),\ u_k' \in C([0,T];L^2(\Omega)) \tag{3.2}$$

and the a priori estimate

$$\begin{aligned}\frac{1}{2}\int_0^T\int_{S_\delta}(u_k')^2 d\sigma\, dt + \int_0^T\int_{S_\delta}\left(\sum_{i,j=1}^{N} a_{ij}\frac{\partial u_k}{\partial x_i}h_j\right)^2 dx\, dt \\ -\frac{1}{2}\int_0^T\int_{S_\delta}\left(\sum_{i,j=1}^{N} a_{ij}\frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right) d\sigma\, dt = E(\delta),\end{aligned} \tag{3.3}$$

with

$$|E(\delta)| \leq M \quad \textit{for all } 0 < \delta \leq \delta_0 \textit{ and all } k. \tag{3.4}$$

Proof. By applying Theorem 2.1 we know that there exists a solution u_k of the approximate problem (P_k). Inserting $v = u_k'(t)$ in (1.4), with $k \geq 1$ and $t \in (0,T)$, results in

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|u_k'(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\frac{d}{dt}a(u_k(t),u_k(t)) + \frac{k}{2}\frac{d}{dt}\|(u_k(t)-\Phi)^-\|^2_{L^2(\Gamma_2)} \\ &\quad = \langle f_k(\cdot,t,u_k(t)), u_k'(t)\rangle - \frac{1}{k}\|u_k'(t)\|^2_{L^2(\Gamma_2)},\end{aligned}$$

which by integration yields

$$\begin{aligned}&\frac{1}{2}\|u_k'(t)\|^2_{L^2(\Omega)} + \frac{1}{2}a(u_k(t),u_k(t)) \leq \frac{1}{2}\|u_1\|^2_{L^2(\Omega)} + \frac{1}{2}a(u_0,u_0) \\ &\quad + C\int_0^t(\|u_k(\tau)\|^2_{L^2(\Omega)} + \|u_k'(\tau)\|^2_{L^2(\Omega)})d\tau - \frac{k}{2}\|(u_k(t)-\Phi)^-\|^2_{L^2(\Gamma_2)} \\ &\quad - \frac{1}{k}\int_0^t \|u_k'(\tau)\|^2_{L^2(\Gamma_2)}d\tau,\end{aligned} \tag{3.5}$$

with a constant $C > 0$. In writing (3.5) we have used (1.5), (H_1) and Young's inequality. Then Gronwall's inequality and hypothesis (H_2) ensure that there exists a constant $M > 0$ independent on k such that

$$\|u_k(t)\|_{H^1(\Omega)} \leq M \quad \text{for all } t \in [0,T], \tag{3.6}$$

$$\|u_k'(t)\|_{L^2(\Omega)} \leq M \quad \text{for all } t \in [0,T], \tag{3.7}$$

$$\|(u_k(t)-\Phi)^-\|_{L^2(\Gamma_2)} \leq \frac{M}{\sqrt{k}} \quad \text{for all } t \in [0,T], \tag{3.8}$$

$$\int_0^T \|u_k'(t)\|^2_{L^2(\Gamma_2)} ds \le Mk. \tag{3.9}$$

It is seen from (1.4) that in $\mathfrak{D}'(Q)$ there holds

$$u_k''(t) - \sum_{i,j=1}^N \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u_k}{\partial x_j}\right) = f_k(x,t,u_k). \tag{3.10}$$

Since Theorem 2.1 ensures $u_k'' \in L^\infty(0,T;L^2(\Omega))$ and hypothesis (H_1) guarantees that $f(\cdot,\cdot,u_k) \in L^\infty(0,T;L^2(\Omega))$, we deduce from (3.10) the regularity information $\sum_{i,j=1}^N \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial u_k}{\partial x_j}) \in L^\infty(0,T;L^2(\Omega))$. Then the interior estimates applied to (3.10) show that the regularity property (3.2) is valid for all $\delta > 0$ sufficiently small.

We note that Fubini's theorem, integration on $t \in [0,T]$, Green's formula on Ω_δ, (3.2) and (3.1) imply that

$$\begin{aligned} \int_0^T \int_{\Omega_\delta} u_k'' \sum_{i,j=1}^N a_{ij}\frac{\partial u_k}{\partial x_i} h_j dx\, dt &= \int_{\Omega_\delta} u_k'(x,T) \sum_{i,j=1}^N a_{ij}\frac{\partial u_k}{\partial x_i}(x,T) h_j dx \\ &\quad - \int_{\Omega_\delta} u_1 \sum_{i,j=1}^N a_{ij}\frac{\partial u_0}{\partial x_i} h_j dx \\ &\quad + \frac{1}{2}\int_0^T \int_{\Omega_\delta} \left(\sum_{i,j=1}^N \frac{\partial}{\partial x_i}(a_{ij}h_j)\right)(u_k')^2 d\sigma\, dt \\ &\quad - \frac{1}{2}\int_0^T \int_{S_\delta} (u_k')^2 d\sigma\, dt. \end{aligned} \tag{3.11}$$

At this point, a direct computation based on Green's formula on Ω_δ enables us to write

$$\begin{aligned} &\int_0^T \int_{\Omega_\delta} \left(-\sum_{i,j=1}^N \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u_k}{\partial x_i}\right)\right)\left(\sum_{i,j=1}^N a_{ij}\frac{\partial u_k}{\partial x_i} h_j\right) dx\, dt \\ &\quad = -\int_0^T \int_{S_\delta} \left(\sum_{i,j=1}^N a_{ij}\frac{\partial u_k}{\partial x_i} h_j\right)^2 dx\, dt \\ &\qquad + \int_0^T \int_{\Omega_\delta} \sum_{i,n=1}^N \left(\sum_{j=1}^N a_{ij}\frac{\partial u_k}{\partial x_j}\right)\frac{\partial h_n}{\partial x_i}\left(\sum_{l=1}^N a_{ln}\frac{\partial u_k}{\partial x_l}\right) dx\, dt \\ &\qquad + \int_0^T \int_{\Omega_\delta} \sum_{i=1}^N \left(\sum_{j=1}^N a_{ij}\frac{\partial u_k}{\partial x_j}\right)\left(\sum_{l,n=1}^N h_n \frac{\partial}{\partial x_i}\left(a_{ln}\frac{\partial u_k}{\partial x_l}\right)\right) dx\, dt. \end{aligned} \tag{3.12}$$

Since $a_{ij} = a_{ji}$ for all $i,j = 1,\dots,N$ and (3.2) is valid, by applying again Green's formula and (3.1) we obtain

$$\int_0^T \int_{\Omega_\delta} \sum_{i=1}^N \left(\sum_{j=1}^N a_{ij}\frac{\partial u_k}{\partial x_j}\right)\left(\sum_{l,n=1}^N h_n \frac{\partial}{\partial x_i}\left(a_{ln}\frac{\partial u_k}{\partial x_l}\right)\right) dx\, dt \tag{3.13}$$

$$= \frac{1}{2}\int_0^T \int_{S_\delta} \sum_{i,j=1}^N a_{ij}\frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial_{x_j}} d\sigma\, dt$$

$$- \frac{1}{2}\int_0^T \int_{\Omega_\delta} \left(\sum_{i,j=1}^N \frac{\partial}{\partial x_i}(a_{ij}h_j)\right)\left(\sum_{i,j=1}^N a_{ij}\frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right) dx\, dt$$

$$- \int_0^T \int_{\Omega_\delta} \sum_{l=1}^N \left(\sum_{n=1}^N a_{ln}h_n\right)\left(\sum_{i,j=1}^N \frac{\partial a_{ij}}{\partial x_l}\frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right) dx\, dt$$

$$+ \int_0^T \int_{\Omega_\delta} \sum_{i=1}^N \left(\sum_{j=1}^N a_{ij}\frac{\partial u_k}{\partial x_j}\right)\left(\sum_{l,n=1}^N h_n\frac{\partial a_{ln}}{\partial x_i}\frac{\partial u_k}{\partial x_l}\right) dx\, dt.$$

The fact that (3.10) holds true in $L^\infty(0,T;L^2(\Omega))$ allows us to take therein as test function $\sum_{i,j=1}^N a_{ij}\frac{\partial u_k}{\partial x_i}(x,t)h_j$. Taking into account (3.11), (3.12) and (3.13), the resulting relation can be expressed in the form of (3.3). The exact expression of $E(\delta)$ can be retrieved from (3.10), (3.11), (3.12) and (3.13). These formulas, in conjunction with (H_1), (H_2), (3.6), (3.7), render that $E(\delta)$ is bounded for $\delta > 0$ sufficiently small, uniformly in k. Consequently, there exist positive constants M and δ_0 independent of k such that (3.4) holds true. The proof is thus complete. □

4. PROOF OF THEOREM 1.1

Theorem 3.1 provides the existence of a solution u_k of the approximate problem (P_k) such that (3.3) and the a priori estimate (3.4) are satisfied.

Fix any $w \in L^\infty(0,T;H^1_{\Gamma_1}(\Omega))$ with $w' \in L^\infty(0,T;L^2(\Omega))$ and $w(t) \in G$ for a.a. $t \in (0,T)$ (see Section 1 for the definition of the set G and the concept of solution). By (1.4) we know that

(4.1)
$$\int_0^T \langle u_k''(t), w(t)-u_k(t)\rangle dt + \int_0^T a(u_k(t), w(t)-u_k(t))dt$$
$$= \int_0^T \langle f_k(\cdot,t,u_k(t)), w(t)-u_k(t)\rangle dt$$
$$+ \int_0^T \int_{\Gamma_2} (k(u_k(t)-\Phi)^- - \frac{1}{k}u_k'(t))(w(t)-u_k(t)) d\sigma\, dt.$$

From $w(t) \in G$ for a.a. $t \in (0,T)$, we note that

(4.2)
$$\int_0^T \int_{\Gamma_2} (u_k(t)-\Phi)^-(w(t)-u_k(t)) d\sigma\, dt \geq 0.$$

Integration by parts in (4.1), in conjunction with (1.5) and (4.2), yields

(4.3)
$$\langle u_k'(T), w(T)-u_k(T)\rangle - \langle u_1, w(0)-u_0\rangle - \int_0^T \langle u_k'(t), w'(t)\rangle dt$$
$$+ \|u_k'\|^2_{L^2(Q)} + \int_0^T a(u_k(t), w(t))dt - \int_0^T a(u_k(t), u_k(t))dt$$

$$\geq \int_0^T \langle f_k(\cdot,t,u_k(t)), w(t)-u_k(t)\rangle dt$$
$$-\frac{1}{k}\int_0^T\int_{\Gamma_2} u_k'(t)w(t)d\sigma\, dt + \frac{1}{2k}\|u_k(T)\|^2_{L^2(\Gamma_2)} - \frac{1}{2k}\|u_0\|^2_{L^2(\Gamma_2)}.$$

Thanks to (3.6) and (3.7) we can consider a relabeled subsequence such that

$$u_k \to u \quad \text{weak}^* \text{ in } L^\infty(0,T;H^1_{\Gamma_1}(\Omega)), \tag{4.4}$$

$$u_k' \to u' \quad \text{weak}^* \text{ in } L^\infty(0,T;L^2(\Omega)), \tag{4.5}$$

for some $u \in L^\infty(0,T;H^1_{\Gamma_1}(\Omega))$ with $u' \in L^\infty(0,T;L^2(\Omega))$. We observe that (4.4) and (4.5) imply the convergence

$$u_k \to u \quad \text{in } C([0,T];H^{\frac{1}{2}}(\Omega)) \text{ as } k\to\infty. \tag{4.6}$$

Then from (H_1) and (4.6) we deduce that

$$\int_0^T \langle f_k(\cdot,t,u_k(t)), w(t)-u_k(t)\rangle dt \to \int_0^T \langle f(\cdot,t,u(t)), w(t)-u(t)\rangle dt. \tag{4.7}$$

Furthermore, by (3.10), (4.5), (4.6), (H_1) and (H_2) it follows that

$$u_k'' \to u'' \quad \text{weak}^* \text{ in } L^\infty(0,T;H^{-1}(\Omega)). \tag{4.8}$$

On account of (4.5) and (4.8) we infer that

$$u_k' \to u' \quad \text{in } C([0,T];H^{-\frac{1}{2}}(\Omega)) \text{ as } k\to\infty, \tag{4.9}$$

which combined with (4.6) yields

$$\langle u_k'(T), w(T)-u_k(T)\rangle \to \langle u_k'(T), w(T)-u_k(T)\rangle \quad \text{as } k\to\infty. \tag{4.10}$$

We also note that

$$\frac{1}{k}\int_0^T\int_{\Gamma_2} u_k'(t)w(t)d\sigma\, dt \to 0 \quad \text{as } k\to\infty \tag{4.11}$$

because by (3.9) we have

$$\frac{1}{k}\left|\int_0^T\int_{\Gamma_2} u_k'(t)w(t)d\sigma\, dt\right| \leq \frac{1}{k}\left(\int_0^T\int_{\Gamma_2} u_k'(t)^2 d\sigma\, dt\right)^{\frac{1}{2}}\left(\int_0^T\int_{\Gamma_2} w(t)^2 d\sigma\, dt\right)^{\frac{1}{2}}$$
$$\leq \frac{M}{\sqrt{k}}\left(\int_0^T\int_{\Gamma_2} w(t)^2 d\sigma\, dt\right)^{\frac{1}{2}}.$$

We claim that

$$\limsup_{k\to\infty}\int_0^T\int_\Omega\left((u_k')^2 - \sum_{i,j=1}^N a_{ij}\frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right)dx\,dt \tag{4.12}$$
$$\leq \int_0^T\int_\Omega\left((u')^2 - \sum_{i,j=1}^N a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\right)dx\,dt.$$

Our reasoning to show (4.12) is inspired from the corresponding part in [4] handling a compensated compactness argument and strongly relying on (3.3), (3.4). We start by applying the divergence-curl property in [1, Corollary 4.3] making in that statement the following choices: $n = N+1$, $x_{N+1} = t$,

$$v_i^k = \sum_{j=1}^{N} a_{ij} \frac{\partial u_k}{\partial x_j},\ w_i^k = \frac{\partial u_k}{\partial x_i}\ (i = 1, \ldots, N),\ v_{N+1}^k = -\frac{\partial u_k}{\partial x_{N+1}},\ w_{N+1}^k = \frac{\partial u_k}{\partial x_{N+1}},$$

with any $k \geq 1$. For

$$v^k = (v_1^k, \ldots, v_N^k, v_{N+1}^k) \ \text{ and } \ w^k = (w_1^k, \ldots, w_N^k, w_{N+1}^k),$$

we see from (4.6) and (4.9) that in $L^2(Q)^{N+1}$ there hold

$$v^k \to v \ \text{ weakly and } \ w^k \to w \text{ weakly as } k \to \infty,$$

where

$$v = \left(\sum_{j=1}^{N} a_{1j} \frac{\partial u}{\partial x_j}, \ldots, \sum_{j=1}^{N} a_{Nj} \frac{\partial u}{\partial x_j}, -\frac{\partial u}{\partial x_{N+1}} \right), \quad w = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_N}, \frac{\partial u}{\partial x_{N+1}} \right).$$

We notice from (3.10) that the sequence of functions

$$\text{div } v^k = \sum_{i=1}^{N} \frac{\partial v_i^k}{\partial x_i} + \frac{\partial v_{N+1}^k}{\partial x_{N+1}} = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}(a_{ij} \frac{\partial u_k}{\partial x_j}) - \frac{\partial^2 u_k}{\partial t^2}$$

is bounded in $L^2(Q)$. Also, as $x_{N+1} = t$, it is readily seen that

$$\text{curl } w^k = \left(\frac{\partial w_i^k}{\partial x_j} - \frac{\partial w_j^k}{\partial x_i} \right)_{i,j=1,\ldots,N+1} = 0.$$

Thus we have checked that all the requirements in [1, Corollary 4.3] are fulfilled. Applying this result ensures that

$$(u_k')^2 - \sum_{i,j=1}^{N} a_{ij} \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j} \to (u')^2 - \sum_{i,j=1}^{N} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \text{ as } k \to \infty \tag{4.13}$$

in the sense of distributions. In fact, the regularity properties that we have already established show that actually the convergence in (4.13) is weak* in $L^\infty(0, T; L^1(\Omega))$.

We construct a special test function to be used with (4.13). Namely, for $0 < \delta \leq \delta_0$ with $\delta_0 > 0$ small (see Theorem 3.1), we take a continuous function with compact support $\psi_\delta : \Omega \to \mathbb{R}$ as follows

$$\psi_\delta(x) = \rho_\delta(\text{dist}(\mathrm{x}, \partial\Omega)) \ \text{ for all } x \in \Omega,$$

where $\rho_\delta : [0, +\infty) \to [0, 1]$ is continuous, nondecreasing, $\rho_\delta = 0$ near 0 and $\rho_\delta(s) = 1$ whenever $s \geq \delta$. Then the weak* convergence in $L^\infty(0, T; L^1(\Omega))$ stated in (4.13) implies that

$$\int_0^T \int_\Omega \psi_\delta \left((u_k')^2 - \sum_{i,j=1}^N a_{ij} \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j} \right) dx\, dt \tag{4.14}$$
$$\to \int_0^T \int_\Omega \psi_\delta \left((u')^2 - \sum_{i,j=1}^N a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \right) dx\, dt \quad \text{as } k \to \infty.$$

On the other hand, by virtue of (3.3), (3.4), the definition of the function ψ_δ and the coarea formula (see [3, p. 628]), we derive that

$$| \int_0^T \int_\Omega (\psi_\delta - 1) \left[\frac{1}{2} \left((u_k')^2 - \sum_{i,j=1}^N a_{ij} \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j} \right) \right. \tag{4.15}$$
$$\left. + \left(\sum_{i,j=1}^N a_{ij} \frac{\partial u_k}{\partial x_i} h_j \right)^2 \right] dx\, dt | \le M\delta$$

for all $k \ge 1$ and $\delta > 0$ sufficiently small, with a constant $M > 0$ independent of k and δ.

Let $\varepsilon > 0$. In view of the absolute continuity of the integral, for $\delta > 0$ sufficiently small we have

$$| \int_0^T \int_\Omega (1 - \psi_\delta) \left((u')^2 - \sum_{i,j=1}^N a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \right) dx\, dt| \tag{4.16}$$
$$\le \int_0^T \int_{\{x \in \Omega : \text{dist}(\text{x}, \Gamma) < \delta\}} | \left((u')^2 - \sum_{i,j=1}^N a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \right) | dx\, dt < \varepsilon$$

because $\psi_\delta = 1$ on $\overline{\Omega_\delta}$. By using (4.16), (4.14), (4.15) with $\delta > 0$ small enough, we arrive at

$$\int_0^T \int_\Omega \left((u')^2 - \sum_{i,j=1}^N a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \right) dx\, dt$$
$$> -\varepsilon + \lim_{k\to\infty} \int_0^T \int_\Omega \psi_\delta \left((u_k')^2 - \sum_{i,j=1}^N a_{ij} \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j} \right) dx\, dt$$
$$\ge -\varepsilon + \limsup_{k\to\infty} \int_0^T \int_\Omega \left((u_k')^2 - \sum_{i,j=1}^N a_{ij} \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j} \right) dx\, dt$$
$$+ \liminf_{k\to\infty} \int_0^T \int_\Omega (\psi_\delta - 1) \left[(u_k')^2 - \sum_{i,j=1}^N a_{ij} \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j} + 2 \left(\sum_{i,j=1}^N a_{ij} \frac{\partial u_k}{\partial x_i} h_j \right)^2 \right] dx\, dt$$
$$\ge -2\varepsilon + \limsup_{k\to\infty} \int_0^T \int_\Omega \left((u_k')^2 - \sum_{i,j=1}^N a_{ij} \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j} \right) dx\, dt.$$

Since $\varepsilon > 0$ is arbitrary, this establishes (4.12).

Now we pass to the limit superior in (4.3) as $k \to \infty$. Then, taking into account (4.10), (4.9), (4.12), (4.7), (4.11), we obtain (1.1) in the limit. Moreover, on the basis of (4.6), (4.9), (3.8), (1.5), we prove that $u(t) \in H^1_{\Gamma_1}(\Omega))$ and $u(t) \in G$ for a.a. $t \in (0, T)$, which completes the proof.

REFERENCES

[1] B. Dacorognia, *Weak continuity and weak lower semicontinuity of nonlinear functionals*, Lecture Notes in Mathematics 922, Springer-Verlag, Berlin-New York, 1982.

[2] C. Eck, J. Jarušek and M. Krbc, *Unilateral Contact Problems in Mechanics. Variational Methods and Existence Theorems*, Monographs & Textbooks in Pure & Appl. Math. 270, Chapman & Hall/CRC, Boca Raton – London – New York – Singapore, 2005.

[3] L. C. Evans, *Partial differential equations*, Graduate Studies in Mathematics 19, American Mathematical Society, Providence, RI, 1998.

[4] J. U. Kim, A boundary thin obstacle problem for a wave equation, Comm. Partial Differential Equations 14 (1989), 1011–1026.

[5] G. Lebeau. and M. Schatzman, A wave problem in a half-space with a unilateral constraint at the boundary, J. Differential Equations 53 (1984), 309–361.

[6] K. Maruo, Existence of solutions of some nonlinear wave equations, Osaka Math. J. 22 (1984), 21–30.

[7] E. Zeidler, *Nonlinear functional analysis and its applications*, II/B. Nonlinear monotone operators, Springer-Verlag, New York, 1990.