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STOCHASTIC HAMILTONIAN EQUATION WITH UNIFORM MOTION AREA

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ABSTRACT. We consider a type of stochastic relativistic Hamiltonian system, and study the behavior of its solution when the coefficient of the potential diverges to ∞. In particular, we prove that under certain conditions, the solution converges to a stochastic process with jump given as a combination of a diffusion process and a uniform motion process. The precise description of the limit process is also given.

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1. Introduction

We consider the motion of a particle with its position Q_t^λ and relative velocity $V_t^\lambda = \frac{P_t^\lambda}{\sqrt{1+|P_t^\lambda|^2}}$ given by the following stochastic differential equation:

$$\begin{cases} dQ_t^\lambda = \frac{P_t^\lambda}{\sqrt{1+|P_t^\lambda|^2}}dt \\ dP_t^\lambda = \sigma(Q_t^\lambda)dB_t - \gamma\frac{P_t^\lambda}{\sqrt{1+|P_t^\lambda|^2}}dt - \lambda\nabla U(Q_t^\lambda)dt, \\ (Q_0^\lambda, P_0^\lambda) = (q_0, p_0). \end{cases} \tag{1.1}$$

Here P_t^λ stands for the momentum of the particle, Q_t^λ, V_t^λ and P_t^λ take values in $\mathbb{R}^d$, $\gamma > 0$ is a constant, and $\lambda \geq 1$ is a parameter. We assume that $\sigma \in C^\infty(\mathbb{R}^d, \mathbb{R}^{d\times d})$ is bounded and ${}^t\sigma\sigma$ is uniformly elliptic, where ^{t} means the tranpose of a matrix. Our system (1.1) can be considered as a decayed and randomized system with Hamiltonian $H(q,p) = \sqrt{1+p^2} + \lambda U(q)$.

We assume that $U \in C_0^\infty(\mathbb{R}^d;\mathbb{R})$ is a spherical symmetric function satisfying the following conditions. There exist constants $r_2 > r_1 > 0$ such that $U(x) = 0$ if $|x| \geq r_2$, $U(x) > 0$ if $|x| < r_1$, and $U(x) < 0$ if $|x| \in (r_1, r_2)$. Let h be the real-valued function such that $U(x) = h(|x|)$. Also, assume that there exists a constant $\varepsilon_0 > 0$ and a function $k \in C_0^\infty(\mathbb{R}^d;\mathbb{R})$ such that $\|k\|_\infty \leq 1$ and $|h'(|x|)| = h'(|x|)k(x)$ if $x \in A$, where $A := \{x \in \mathbb{R}^d \big| ||x| - r_1| \leq \varepsilon_0 \text{ or } |x| \geq r_2 - \varepsilon_0\}$. Without loss of generality, we assume that $\varepsilon_0 < r_1/2 \wedge (r_2 - r_1)/2$. Also, we assume that $U(q_0) = 0$.

We are interested in the behavior of the particle described by (1.1) when $\lambda \to \infty$. As in the relation between [2] and [3], this problem is also closely related to

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the problem of "mechanical models of Brownian motions" with absorbing resulting-interactions, which we will discuss in a forthcoming paper.

[2] considered a similar question for the non-relative model, in the case where U gives a reflecting force, precisely, in the case where there exist constants $r, \varepsilon > 0$ such that $U(q) = 0$ when $|q| > r$ and $U(q) > 0$ when $|q| \in (r - \varepsilon, r)$, and got a limit process given as a diffusion process reflecting at $|Q_t| = r$. In contrast, in our model, U gives an absorbing force as soon as the particle enters $|Q_t| < r_2$, which means that when $\lambda \to \infty$, P_t becomes infinity in an instant. (This constitutes the main difficulty in the treatment of our model.)

Now let us come back to our question: what is the limit behavior of the particle evolving according to (1.1) when $\lambda \to \infty$? First notice that although P_t, instead of V_t, is the one that seems to be more natural to be considered, it is hopeless to have P_t^λ to converge as $\lambda \to \infty$ or to track the behavior of it directly: when $\lambda \to \infty$, P_t^λ actually diverges to ∞ in the domain $U(Q_t) \neq 0$, while keeping finite when $U(Q_t) = 0$. However, although P_t^λ might diverge to ∞ as $\lambda \to \infty$, we have that V_t^λ is always bounded by 1, and whenever $|P_t^\lambda| < \infty$, we always have that $P_t^\lambda = \frac{V_t^\lambda}{\sqrt{1-|V_t^\lambda|^2}}$. Also, it is V_t instead of P_t that gives the velocity of the particle. Therefore, we use (Q_t, V_t) to describe the behavior of a particle.

As it will be proven in Lemma 6.1, the distribution of $\{(Q_t^\lambda, V_t^\lambda)\}_t$ converges as $\lambda \to \infty$. But how to describe the limit process? Notice that in the limit, when the particle crosses $|Q_t| = r_2$, since the value of $|P_t|$ jumps between ∞ and a finite value as we just mentioned, we have that $|V_t|$ also jumps between 1 and a number that is strictly less than 1. So V_t is not continuous either, hence it is not so easy to describe the limit process directly. In particular, we have to find some way to determine the value of V_t when the particle enters the domain $|Q_t| > r_2$ from $|Q_t| < r_2$.

We solve this problem by defining two new stochastic processes H_t^λ and R_t^λ for any $\lambda \geq 1$. First, let us prepare some notations. For any $a, b \in \mathbb{R}^d$ with $a \neq 0$, let $\pi_a b$ and $\pi_a^\perp b$ denote the components of b that are parallel to a and perpendicular to a, respectively, *i.e.*,

$$\pi_a b = \frac{b \cdot a}{|a|^2} a, \qquad \pi_a^\perp b = b - \frac{b \cdot a}{|a|^2} a.$$

Also, we use the natural extension $|a|\pi_a b = |a|\pi_a^\perp b = 0$ if $a = 0$. Our new quantities H_t^λ and R_t^λ are defined as follows:

$$H_t^\lambda := \sqrt{1 + |P_t^\lambda|^2} + \lambda U(Q_t^\lambda) = \frac{1}{\sqrt{1 - |V_t^\lambda|^2}} + \lambda U(Q_t^\lambda),$$

$$R_t^\lambda := \pi_{Q_t^\lambda}^\perp P_t^\lambda.$$

Notice that when $\lambda \to \infty$, although P_t^λ might diverge to ∞, R_t^λ keeps finite (see Lemma 4.2 (1)). Indeed, P_t^λ diverges to ∞ only because $\nabla U(Q_t^\lambda)$ is not 0 (hence

$\lambda\nabla U(Q_t^\lambda)$ becomes infinity) in a certain domain, however, this force is parallel to Q_t^λ, so it is natural to expect that the perpendicular component R_t^λ keeps finite. In this paper, we prove that the distribution of $\{(Q_t^\lambda, V_t^\lambda, H_t^\lambda, R_t^\lambda); t \in [0,\infty)\}$ converges as $\lambda \to \infty$, and gives the characterization of the limit process. In particular, the limits of H_t^λ and R_t^λ as $\lambda \to \infty$ are continuous with respect to t. The introduction of (H_t, R_t) is one of the main ideas of this paper. See the paragraphs after Remark 1.2 for more explanations.

Now let us formulate our results. Let $\widetilde{W}^d := C([0,\infty);\mathbb{R}^d) \times D([0,\infty);\mathbb{R}^d) \times C([0,\infty);\mathbb{R}) \times C([0,\infty);\mathbb{R}^d)$, with metric function $dist(\cdot,\cdot)$ given by

$$\begin{aligned} dist(w_1,w_2) \;:=\; & \sum_{n=1}^{\infty} 2^{-n}\Big(1 \wedge \Big[\max_{t\in[0,n]} |q_1(t)-q_2(t)| + \Big(\int_0^n |v_1(t)-v_2(t)|^n\Big)^{1/n} \\ & + \max_{t\in[0,n]} |h_1(t)-h_2(t)| + \max_{t\in[0,n]} |r_1(t)-r_2(t)|\Big]\Big) \end{aligned}$$

for any $w_i(\cdot) = (q_i(\cdot), v_i(\cdot), h_i(\cdot), r_i(\cdot)) \in \widetilde{W}^d$, $i = 1,2$. Here $D([0,\infty);\mathbb{R}^d)$ denotes the set of $\mathbb{R}^d$-valued functions defined on $[0,\infty)$ that are right-continuous with left limit which exists at every point. The Skorohod metric on it is also considered (see Section 2 for more details). Let μ_λ denote the distribution of $\{(Q_t^\lambda, V_t^\lambda, H_t^\lambda, R_t^\lambda); t \in [0,\infty)\}$. We also use the notation $B(r) := \{y \in \mathbb{R}^d \,\big|\, |y| < r\}$ for any $r > 0$.

In order to present our limit process, let us first prepare some notations. For any $q, v \in \mathbb{R}^d$, let

$$\begin{aligned} A_1^h(q,v) &= {}^t v\sigma(q), \\ A_2^h(q,v) &= -\gamma|v|^2 + \frac{1}{2}\sqrt{1-|v|^2}\sum_{i,j=1}^d \sigma_{ij}^2(q) - \frac{1}{2}\sqrt{1-|v|^2}\Big|{}^t\sigma(q)v\Big|^2, \\ A_1^r(q,v) &= \sigma(q) - \frac{1}{|q|^2} q\,{}^t q\sigma(q), \\ A_2^r(q,v,r) &= -\gamma\sqrt{1-|v|^2}r - \sqrt{1-|v|^2}|r|^2\frac{q}{|q|^2} - \frac{(q,v)}{|q|^2}r, \\ A_1^v(q,v) &= \sqrt{1-|v|^2}\Big(\sigma(q) - v\,{}^t v\sigma(q)\Big), \\ A_2^v(q,v) &= -\gamma(1-|v|^2)^{3/2}v - \frac{1}{2}(1-|v|^2)\Big(\sum_{i,j=1}^d \sigma_{ij}^2(q)\Big)v \\ &\quad + \frac{3}{2}(1-|v|^2)\Big|{}^t\sigma(q)v\Big|^2 v - (1-|v|^2)\sigma(q)\,{}^t\sigma(q)v, \end{aligned}$$

let $K_1(q,v)$ be the $(3d+1)\times d$-matrix and let $K_2(q,v,r)$ be the $(3d+1)\times 1$-matrix given by the following, respectively:

$$K_1(q,v) = \begin{pmatrix} 0 \\ 1_{\{|q|>r_2\}}A_1^v(q,v) \\ A_1^h(q,v) \\ A_1^r(q,v) \end{pmatrix}, \qquad K_2(q,v,r) = \begin{pmatrix} v \\ 1_{\{|q|>r_2\}}A_2^v(q,v) \\ A_2^h(q,v) \\ A_2^r(q,v,r) \end{pmatrix}.$$

Finally, let L be the generator

$$Lf(q,v,h,r) = \frac{1}{2}\sum_{i,j=d+1}^{3d+1}\Big(K_1(q,v)^t K_1(q,v)\Big)_{ij}\nabla_i\nabla_j f(q,v,h,r) + K_2(q,v,r)\cdot\nabla f(q,v,h,r).$$

Here $*_{ij}$ stands for the (i,j)-element of the matrix $*$, $\nabla = {}^t(\nabla_1,\cdots,\nabla_{3d+1})$, and

$$\nabla_i = \begin{cases} \nabla_{q_i}, & i=1,\cdots,d, \\ \nabla_{v_{i-d}}, & i=d+1,\cdots,2d, \\ \nabla_h, & i=2d+1, \\ \nabla_{r_{i-2d-1}}, & i=2d+2,\cdots,3d+1. \end{cases}$$

Our main result is the following.

Theorem 1.1. 1. *There exists a unique probability measure μ on $\widetilde{W}^d$ that satisfies the following:*

(μ1) $\mu\Big(Q_0 = q_0, V_0 = \frac{p_0}{\sqrt{1+|p_0|^2}}, H_0 = \sqrt{1+|p_0|^2}, R_0 = \pi_{q_0}^{\perp}p_0\Big) = 1.$

(μ2) $\mu(|Q(t)| \geq r_1, |V(t)| \leq 1, t\in[0,\infty)) = 1.$

(μ3) *For any $f\in C_0^\infty(\mathbb{R}^{3d+1})$ with supp$f \subset \Big\{\Big(B(r_2)\setminus\overline{B(r_1)}\Big)\cup(\overline{B(r_2)})^C\Big\}\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d$, we have that $\Big\{f(Q_t,V_t,H_t,R_t) - \int_0^t Lf(Q_s,V_s,H_s,R_s)ds; t\geq 0\Big\}$ is a continuous martingale under μ.*

(μ4) *We have μ-almost surely the following: For any $t\in[0,\infty)$, $|Q_t|\in(r_1,r_2)$ implies that $V_t = \pm\frac{Q_t}{|Q_t|}$ and that $V_t = V_{t-}$, also, $|Q_t| = r_1$ implies that $V_t = \frac{Q_t}{|Q_t|}$.*

(μ5) *We have μ-almost surely that for $t\in[0,\infty)$ with $|Q_t| = r_2$,*
   (1) *if $Q_t\cdot V_{t-} < 0$, then $V_t = -\frac{Q_t}{|Q_t|}$;*
   (2) *if $Q_t\cdot V_{t-} > 0$ and $H_t < \sqrt{1+|R_t|^2}$, then $V_t = -\frac{Q_t}{|Q_t|}$;*
   (3) *if $Q_t\cdot V_{t-} > 0$ and $H_t > \sqrt{1+|R_t|^2}$, then $V_t = \frac{\sqrt{H_t^2-1-|R_t|^2}Q_t/|Q_t| + R_t}{H_t}$.*

2. *In addition, we assume that $h'(r_1) < 0$ and $\lim_{a\to r_2-0}\frac{h'(a)}{h(a)} = -\infty$. Then when $\lambda\to\infty$, $\mu_\lambda\to\mu$ as probability measures on $(\widetilde{W}^d, dist)$.*

Notice that under μ, we have that Q_t, H_t and R_t are continuous, and V_t is right-continuous with left limit at each t.

Remark 1.2. The elements $1_{\{|Q_t|>r_2\}}A_i^v(Q_t,V_t)$ $(i=1,2)$ of $K_1(Q_t,V_t)$ and $K_2(Q_t,V_t,R_t)$ are not 0 only if $|Q_t| > r_2$, and in this domain, we get by a simple calculation that under μ, the following holds: (1) $|V_t| < 1$, (2) the distribution of $(Q_t, \frac{V_t}{\sqrt{1-|V_t|^2}})$ is a solution of the martingale problem corresponding to $dQ_t = V_t dt, d(\frac{V_t}{\sqrt{1-|V_t|^2}}) = \sigma(Q_t)dB_t - \gamma V_t dt$, equivalently, $(Q_t, \frac{V_t}{\sqrt{1-|V_t|^2}})$ satisfies (1.1) with $\lambda = 0$, (3) (H_t, R_t) is actually completely determined by Q_t and V_t: $H_t = \frac{1}{\sqrt{1-|V_t|^2}}$ and $R_t = \frac{1}{\sqrt{1-|V_t|^2}}\pi^{\perp}_{Q_t}V_t$. Also, when $|Q_t| \in (r_1, r_2)$, we have by (μ4) that $|V_t| = 1$, hence $A^h_2(Q_t, V_t) = -\gamma$ and $A^r_2(Q_t, V_t, R_t) = -\frac{(Q_t,V_t)}{|Q_t|^2}R_t$. Moreover, in this domain, Q_t and V_t are deterministic.

The opposite is also true: if a probability satisfies all of the conditions stated here, it also satisfies (μ1) $\sim$ (μ5). Therefore, we can "divide" our limit process as follows.

Let

$$L_0 f(q,p) = \sum_{i=1}^d \frac{p^i}{\sqrt{1+|p|^2}}\frac{\partial}{\partial q_i}f(q,p) + \frac{1}{2}\sum_{i,j=1}^d(\sum_{k=1}^d \sigma_{ik}(q)\sigma_{jk}(q))\frac{\partial^2}{\partial p_i \partial p_j}f(q,p)$$
$$-\gamma\sum_{i=1}^d \frac{p^i}{\sqrt{1+|p|^2}}\frac{\partial}{\partial p_i}f(q,p),$$

and

$$L_u f(q,v,h,r) = \sum_{i=1}^d v_i \frac{\partial}{\partial q_i}f - \gamma\frac{\partial}{\partial h}f(q,v,h,r) - \sum_{i=1}^d \frac{(q,v)}{|q|^2}r_i\frac{\partial}{\partial r_i}f(q,v,h,r)$$
$$+\frac{1}{2}\left|{}^t\sigma(q)v\right|^2\nabla^2_h f(q,v,h,r) + \frac{1}{2}\sum_{i,j=1}^d\left[\left(\sigma(q) - \frac{{}^tqq\sigma(q)}{|q|^2}\right)^2\right]_{i,j}\frac{\partial^2}{\partial r_i \partial r_j}f(q,v,h,r)$$
$$+\sum_{i=1}^d\Big(\sum_{j,k=1}^d v_j\sigma_{jk}(q)\Big(\sigma_{ik}(q) - \frac{1}{|q|^2}q_i\sum_{l=1}^d q_l\sigma_{lk}(q)\Big)\Big)\frac{\partial^2}{\partial h \partial r_i}f(q,v,h,r).$$

Then our limit process can also be described by L_0 and L_u in the following way. Our limit process consists of two phases, a diffusion phase and a uniform motion phase. Precisely, it satisfies the following:

1. the particle keeps in the area $|Q_t| \geq r_1$;
2. when $|Q_t| > r_2$, $(Q_t, \frac{V_t}{\sqrt{1-|V_t|^2}})$ evolves according to the diffusion with generator L_0, and (H_t, R_t) are given by $H_t = \frac{1}{\sqrt{1-|V_t|^2}}$ and $R_t = \frac{1}{\sqrt{1-|V_t|^2}}\pi^{\perp}_{Q_t}V_t$;
3. the particle takes uniform motion in the area $|Q_t| \in (r_1, r_2)$ with $V_t = V_{t-} = \pm\frac{Q_t}{|Q_t|}$ and it reflects at $|Q_t| = r_1$ (hence (Q_t, V_t), the "visible" motion of the particle, is completely deterministic in this domain), and (Q_t, V_t, H_t, R_t) is a diffusion with generator L_u,
4. finally, its behavior at the boundary $|Q_t| = r_2$ of these two phases is determined as follows: when the particle arrives $|Q_t| = r_2$ from the diffusion phase, it simply enters the uniform motion phase by taking $V_t = -\frac{Q_t}{|Q_t|}$; when the particle arrives at $|Q_t| = r_2$ from the uniform motion phase, it either keeps in the uniform

motion phase by reflecting or re-enters the diffusion phase, depending on the value of H_t and R_t at that moment, according to ($\mu 5$).

Before going further, let us give some heuristical explanation. Our convergence in the area $|Q_t| > r_2$ is trivial: in this domain, we have $U(Q_t) = 0$, so the particle evolves according to the diffusion process without the term $-\lambda \nabla U(Q_t)dt$, so after taking $\lambda \to \infty$, we still have the same diffusion. This is our "diffusion phase". When $|Q_t| \in (r_1, r_2)$, the term $-\lambda \nabla U(Q_t)dt$ gives us a very strong "absorbing" force when $\lambda \to \infty$, which is parallel to Q_t, hence P_t becomes very large (and parallel to Q_t) in a very short time, therefore, heuristically V_t should be $\pm \frac{Q_t}{|Q_t|}$ in the area $|Q_t| \in (r_1, r_2)$. This is our second phase: the "uniform motion phase".

Since Q_t is continuous with respect to t, there is no problem with respect to the initial condition of the uniform motion phase when the particle enters it from the diffusion phase. The opposite is not so easy: when it re-reaches the boundary $|Q_t| = r_2$ from the uniform motion side, we have to determine whether it stays in the uniform motion side by taking $V_t = -\frac{Q_t}{|Q_t|}$ or re-enters the diffusion phase; and in the latter case, what is the new initial velocity V_t of the particle? In other words, what is the value of V_t (or P_t) at this moment? Notice that as just mentioned, when $\lambda \to \infty$, $|P_t|$ becomes ∞ in the domain $|Q_t| \in (r_1, r_2)$, so it is hopeless to track P_t (or V_t) directly.

We solve this problem with the help of H_t and R_t in the following way: Notice that for any $\lambda \geq 1$, whenever $U(Q_t^\lambda) = 0$, we always have $H_t^\lambda = \sqrt{1 + |P_t^\lambda|^2}$, so $|\pi_{Q_t^\lambda} P_t^\lambda|^2 = |H_t^\lambda|^2 - 1 - |R_t^\lambda|^2$, *i.e.*, $\pi_{Q_t^\lambda} P_t^\lambda$ is determined by $(Q_t^\lambda, H_t^\lambda, R_t^\lambda)$ up to ± 1. Especially, when the particle re-enters the diffusion domain $|Q_t^\lambda| > r_2$ from the uniform motion domain $|Q_t^\lambda| \in (r_1, r_2)$, we have that $\pi_{Q_t^\lambda} P_t^\lambda$ has the same direction as Q_t^λ, so P_t^λ and V_t^λ are uniquely determined by $(Q_t^\lambda, H_t^\lambda, R_t^\lambda)$. This fact keeps true when $\lambda \to \infty$. Moreover, as we show in Sections 3 and 4, H_t and R_t are continuous and trackable even after $\lambda \to \infty$. This enables us to determine V_t for $|Q_t| = r_2$ after taking limit $\lambda \to \infty$.

The rest of this paper is organized as follows: In Section 2, we prepare several basic results with respect to the proof of tightness. In Sections 3 $\sim$ 6, we prove that the distributions of $\{H_t^\lambda\}_{t\in[0,\infty)}$, $\{R_t^\lambda\}_{t\in[0,\infty)}$ and $\{V_t^\lambda\}_{t\in[0,\infty)}$ with $\lambda \geq 1$ are tight, by checking that the corresponding coefficients satisfy the conditions in the lemmas of Section 2. In Section 7, we give the proof of Theorem 1.1 (1), the uniqueness of the probability that satisfies ($\mu 1$) $\sim$ ($\mu 5$). Finally, we prove Theorem 1.1 (2) by showing that any cluster point of μ_λ, $\lambda \to \infty$, (the existence is ensured by Sections 3 $\sim$ 6), satisfies ($\mu 1$) $\sim$ ($\mu 5$). The proof of this fact is given in Section 8.

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2. **Basic lemmas for the proof of tightness**

In this section, we prepare several general results with respect to the tightness of measures of stochastic processes. These will be used in Sections 3 $\sim$ 6. All of the results of this section are already used in [2] and [3].

Let $\{\mathcal{F}_t\}_t$ denote the filtration generated by $\{B_t\}_t$. It is trivial that $(Q_t^\lambda, P_t^\lambda)$ is $\mathcal{F}_t$-measurable for any $t > 0$.

We first notice that although Theorem 1.1 (2) is a convergence with $t \in [0, \infty)$, it suffices to prove the corresponding weak convergence of the process with $t \in [0, T]$ for any $T > 0$. Choose any $T > 0$ and fix it from now on. It is trivial by definition that $|Q_t^\lambda| \leq |q_0| + T$ for any $t \in [0, T]$ and $\lambda \geq 1$.

Let us recall some basic facts about the Skorohod spaces $(D([0, T]; \mathbb{R}^d), d^0)$, and the tightness of the probability measures on it. (See Billingsley [1] for more details).

For any $T > 0$, let $D([0, T]; \mathbb{R}^d)$ be the Skorohod space:

$$
\begin{aligned}
D([0,T];\mathbb{R}^d) \;=\; \Big\{ w : [0,T] \to \mathbb{R}^d; \quad & w(t) = w(t+) := \lim_{s \downarrow t} w(s), t \in [0,T), \\
& \text{and } w(t-) := \lim_{s \uparrow t} w(s) \text{ exists}, t \in (0,T] \Big\},
\end{aligned}
$$

with the metric $d^0 = d_T^0$ given by

$$
d^0(w, \widetilde{w}) = \inf_{\lambda \in \Lambda} \Big\{ \|\lambda\|^0 \vee \|w - \widetilde{w} \circ \lambda\|_\infty \Big\}
$$

for any $w, \widetilde{w} \in D([0, T]; \mathbb{R}^d)$, where $\Lambda = \Big\{ \lambda : [0, T] \to [0, T];$ continuous, non-decreasing, $\lambda(0) = 0, \lambda(T) = T \Big\}$, $\|w\|_\infty = \sup_{0 \leq t \leq T} |w(t)|$, and $\|\lambda\|^0 = \sup_{0 \leq s < t \leq T} \left| \log \frac{\lambda(t) - \lambda(s)}{t - s} \right|$ for any $\lambda \in \Lambda$.

It is well-known that $(D([0, T]; \mathbb{R}^d), d^0)$ is a complete metric space. Also, $C([0, T]; \mathbb{R}^d) = \{w : [0, T] \to \mathbb{R}^d;$ continuous$\}$ is closed in $(D([0, T]; \mathbb{R}^d), d^0)$, and the Skorohod topology relativized to $C([0, T]; \mathbb{R}^d)$ coincides with the uniform topology there (See, *e.g.*, [1]).

Our base for the proof of tightness in $\wp(D([0, T]; \mathbb{R}^d))$ is the following. Here $\wp(D([0, T]; \mathbb{R}^d))$ means the space of all probabilities on $D([0, T]; \mathbb{R}^d)$.

Theorem 2.1 ([3])**.** *Let $(\Omega_n, \mathcal{F}_n, Q_n)$, $n \in \mathbb{N}$, be probability spaces, and let $X_n : \Omega_n \to D([0, T]; \mathbb{R}^d)$, $n \in \mathbb{N}$, be measurable. Let $\mu_{X_n} = Q_n \circ X_n^{-1}$. Suppose that there exist constants $\varepsilon, \beta, \gamma, C > 0$ such that*

1. $E^{P_n}\big[\|X_n(\cdot)\|_\infty^\varepsilon\big] \leq C$,
2. $E^{P_n}\Big[|X_n(r) - X_n(s)|^\beta |X_n(s) - X_n(t)|^\beta\Big] \leq C|t - r|^{1+\varepsilon}$ *for any* $0 \leq r \leq s \leq t \leq 1$,

3. $E^{P_n}\left[|X_n(s)-X_n(t)|^{\varepsilon}\right] \le C|t-s|^{\gamma}$ *for any* $0 \le s \le t \le 1$,

for any $n \in \mathbb{N}$. *Then* $\{\mu_{X_n}\}_{n=1}^{\infty}$ *is tight in* $\wp(D([0,T];\mathbb{R}^d))$.

The following is an easy consequence of Theorem 2.1.

Lemma 2.2 ([3]). *Let* X_t^λ *be any* d*-dimensional stochastic process given by*

$$dX_t^\lambda = \sigma^{X,\lambda}(t)dB_t + b^{X,\lambda}(t)dt.$$

If X_0^λ *and* $\sigma^{X,\lambda}(t)$ *are bounded for* $t \in [0,T]$ *and* $\lambda \ge 1$, *and*

$$\sup_{\lambda\ge 1}\sup_{t\in[0,T]} E\left[|b^{X,\lambda}(t)|^2\right] < \infty, \tag{2.1}$$

then we have that

1. $\sup_{\lambda\ge1} E\left[\sup_{t\in[0,T]}|X_t^\lambda|^2\right] < \infty$,
2. $\left\{\text{the distribution of } \{X_t^\lambda\}_{t\in[0,T]}; \lambda \ge 1\right\}$ *is tight in* $\wp(C([0,T],\mathbb{R}^d))$.

Notice that (2.1) of Lemma 2.2 is satisfied if $b^{X,\lambda}(t)$ is bounded.

Proof. Let $C := 3\sup_{\lambda\ge1}\left\{\|X_0^\lambda\|_\infty^2 \vee \|\sigma^{X,\lambda}\|_\infty^2 \vee \sup_{t\in[0,T]} E\left[|b^{X,\lambda}(t)|^2\right]\right\}$, which is finite by assumption. Then by Doob's inequality, we have that

$$\begin{aligned}
E\Big[\sup_{t\in[0,T]}|X_t^\lambda|^2\Big] &\le C + 3E\Big[\sup_{t\in[0,T]}\Big|\int_0^t \sigma^{X,\lambda}(s)dB_s\Big|^2\Big] + 3E\Big[\Big(\int_0^T |b^{X,\lambda}(s)|ds\Big)^2\Big]\\
&\le C + 12E\Big[\Big|\int_0^T \sigma^{X,\lambda}(s)dB_s\Big|^2\Big] + CT^2\\
&\le C + 4CT + CT^2 < \infty. \qquad (2.2)
\end{aligned}$$

So our first assertion holds.

Similarly, for any $0 \le t_1 < t_2 < t_3 \le T$, we have that

$$\begin{aligned}
&E[|X_{t_3}^\lambda - X_{t_2}^\lambda|^2\Big|\mathcal{F}_{t_2}]\\
&\quad = E\Big[\Big(\int_{t_2}^{t_3}\sigma^{X,\lambda}(s)dB_s + \int_{t_2}^{t_3} b^{X,\lambda}(s)ds\Big)^2\Big|\mathcal{F}_{t_2}\Big]\\
&\quad \le 2E\Big[\Big(\int_{t_2}^{t_3}\sigma^{X,\lambda}(s)dB_s\Big)^2\Big|\mathcal{F}_{t_2}\Big] + 2E\Big[\Big(\int_{t_2}^{t_3} b^{X,\lambda}(s)ds\Big)^2\Big|\mathcal{F}_{t_2}\Big]\\
&\quad \le 2C^2(t_3-t_2) + 2C^2(t_3-t_2)^2\\
&\quad \le 2C^2(1+T)(t_3-t_2),
\end{aligned}$$

hence

$$E[|X_{t_3}^\lambda - X_{t_2}^\lambda|] \le C\sqrt{2(T+1)}(t_3-t_2)^{1/2}, \tag{2.3}$$

and

$$E[|X_{t_3}^\lambda - X_{t_2}^\lambda|^2|X_{t_2}^\lambda - X_{t_1}^\lambda|^2]$$

$$\begin{aligned}
&\le 2C^2(1+T)(t_3-t_2)E[|X^\lambda_{t_2}-X^\lambda_{t_1}|^2]\\
&\le 2C^2(1+T)(t_3-t_2)\times 2C^2(1+T)(t_2-t_1)\\
&\le 4C^4(1+T)^2(t_3-t_1)^2. \qquad (2.4)
\end{aligned}$$

By Theorem 2.1 (with $\varepsilon = 1$, $\beta = 2$ and $\gamma = \frac{1}{2}$), (2.3), (2.2) and (2.4) imply our second assertion. □

We will also need the following to prove the tightness in $\wp(L^p([0,T],\mathbb{R}^d))$ for any $p>1$. This is an easy consequence of [2, Corollary 8].

Lemma 2.3 ([3]). *Let $b^\lambda : [0,T]\to\mathbb{R}^d$ ($\lambda\ge 1$) be a family of functions satisfying*

$$\sup_{\lambda\ge 1} E\Big[\Big(\lambda\int_0^T |b^\lambda(s)|ds\Big)^2\Big] < \infty.$$

Then we have that $\Big\{$the distribution of $\big\{\int_0^t b^\lambda(s)ds\big\}_{t\in[0,T]}; \lambda\ge 1\Big\}$ is tight in $\wp(L^p([0,T],$ $\mathbb{R}^d))$ for any $p>1$, with all of its cluster point(s) in $\wp(D([0,T],\mathbb{R}^d))$.

3. Tightness of H^λ_t

The tightness of $\Big\{$the distribution of $\big\{Q^\lambda_t\big\}_{t\in[0,T]}; \lambda\ge 1\Big\}$ in $\wp(C([0,T],\mathbb{R}^d))$ is trivial by Lemma 2.2, since $|\frac{dQ^\lambda_t}{dt}|\le 1$ for any $\lambda\ge 1$.

We prove the tightness of $\Big\{$the distribution of $\big\{H^\lambda_t\big\}_{t\in[0,T]}; \lambda\ge 1\Big\}$ in $\wp(C([0,T],\mathbb{R}^d))$ in this section. The tightnesses for R^λ_t and V^λ_t are given in Sections 4 ∼ 6.

Lemma 3.1. 1. *There exists a constant $C_1>0$ such that*

$$|A^h_1(Q^\lambda_t,V^\lambda_t)| + |A^h_2(Q^\lambda_t,V^\lambda_t)| \le C_1, \qquad \text{for any } t\in[0,T], \lambda\ge 1,$$

and we have that

$$dH^\lambda_t = A^h_1(Q^\lambda_t,V^\lambda_t)dB_t + A^h_2(Q^\lambda_t,V^\lambda_t)dt. \qquad (3.1)$$

2. *There exists a constant $C_2>0$ such that*

$$E\Big[\sup_{t\in[0,T]}|H^\lambda_t|\Big] < C_2 \qquad \text{for any } \lambda\ge 1.$$

3. $\Big\{$*the distribution of* $\big\{H^\lambda_t\big\}_{t\in[0,T]}; \lambda\ge 1\Big\}$ *is tight in* $\wp(C([0,T],\mathbb{R}^d))$.

Proof. The fact that $A^h_1(Q^\lambda_t,V^\lambda_t)$ and $A^h_2(Q^\lambda_t,V^\lambda_t)$ are bounded is trivial by definition, since σ is bounded and $|V^\lambda_t|\le 1$. Also, (3.1) is gotten by a simple calculation with the help of Ito's formula.

The assertions (2) and (3) are now trivial by Lemma 2.2. □

For any $\varepsilon \in (0, r_1)$, let $D_\varepsilon := \{x \in \mathbb{R}^d \big| |x| \geq r_1 - \varepsilon\}$ and $\tau_\varepsilon := \inf\{t \geq 0, Q(t) \in D_\varepsilon^C\} \wedge T$. Also, write $D := D_{\varepsilon_0}$ and

$$\tau := \tau_{\varepsilon_0} = \inf\{t \geq 0, |Q(t)| < r_1 - \varepsilon_0\} \wedge T.$$

As a corollary of Lemma 3.1, we have that $\lim_{\lambda\to\infty} \mu_\lambda(\tau_\varepsilon < T) = 0$ for any $\varepsilon > 0$.

Corollary 3.2. *For any $\varepsilon > 0$, we have that*

$$\lim_{\lambda\to\infty} P\Big(\inf_{t\in[0,T]} |Q_t^\lambda| \leq r_1 - \varepsilon\Big) = 0.$$

Proof. For any $\varepsilon > 0$, let $\delta = \inf_{|x|\leq r_1-\varepsilon} U(x)$. Then $\delta > 0$ by assumption. So by Lemma 3.1 (2), we have that

$$\begin{aligned} &P\Big(\inf_{t\in[0,T]} |Q_t^\lambda| \leq r_1 - \varepsilon\Big) \\ &\quad \leq P\Big(\sup_{t\in[0,T]} U(Q_t^\lambda) \geq \delta\Big) \leq P\Big(\sup_{t\in[0,T]} H_t^\lambda \geq \lambda\delta\Big) \\ &\quad \leq \lambda^{-1}\delta^{-1} E\Big[\sup_{t\in[0,T]} |H_t^\lambda|\Big] \leq \lambda^{-1} C_2, \end{aligned}$$

which converges to 0 as $\lambda \to \infty$. □

4. Tightness of R_t^λ

We prove the tightness for R_t^λ in this section. The main result of this section is the following two lemmas.

Lemma 4.1. *There exists a constant $C_3 > 0$ such that*

$$E\Big[\sup_{t\in[0,T\wedge\tau]} |R_t^\lambda|^2\Big] \leq C_3$$

for any $\lambda \geq 1$.

Lemma 4.2. 1. *We have that $A_1^r(Q_t^\lambda, V_t^\lambda)$ is bounded for $t \in [0,T]$ and $\lambda \geq 1$,*

$$\sup_{\lambda\geq 1} E\Big[\sup_{t\in[0,T\wedge\tau]} \Big|A_2^r(Q_t^\lambda, V_t^\lambda, R_t^\lambda)\Big|^2\Big] < \infty, \tag{4.1}$$

and the following holds:

$$dR_t^\lambda = A_1^r(Q_t^\lambda, V_t^\lambda)dB_t + A_2^r(Q_t^\lambda, V_t^\lambda, R_t^\lambda)dt. \tag{4.2}$$

2. $\Big\{$*the distribution of* $\{R_t^\lambda\}_{t\in[0,T\wedge\tau)}; \lambda \geq 1\Big\}$ *is tight in* $\wp(C([0,T], \mathbb{R}^d))$.

Before proving these two lemmas, let us first notice that for any $a, b \neq 0$, we have $|\pi_a^\perp b| = |b|\sqrt{1 - \frac{(a\cdot b)^2}{|a|^2|b|^2}}$, so by a simple calculation, we have that

$$\pi_b^\perp a = \frac{|\pi_a^\perp b|^2}{|b|^2} a - \frac{a \cdot b}{|b|^2} \pi_a^\perp b.$$

Therefore, for any $a, b \in \mathbb{R}^d$, we have that

$$|a||\pi_a^\perp b| = |b||\pi_b^\perp a|, \tag{4.3}$$

and

$$|b|^2 \pi_b^\perp a = |\pi_a^\perp b|^2 a - (a \cdot b)\pi_a^\perp b. \tag{4.4}$$

These facts will be used later.

Proof of Lemma 4.1. Let

$$f^\lambda(t) = |Q_t^\lambda|^2 P_t^\lambda - (Q_t^\lambda \cdot P_t^\lambda) Q_t^\lambda = |Q_t^\lambda|^2 R_t^\lambda.$$

Then by Ito's formula and a simple calculation, we have

$$df^\lambda(t) = \Big((Q_t^\lambda \cdot Q_t^\lambda) dP_t^\lambda - (Q_t^\lambda \cdot dP_t^\lambda) Q_t^\lambda\Big) - \frac{|P_t^\lambda|^2}{\sqrt{1+|P_t^\lambda|^2}} \pi_{P_t^\lambda}^\perp Q_t^\lambda dt, \tag{4.5}$$

hence

$$f^\lambda(t) = f^\lambda(0) + \int_0^t \Big((Q_s^\lambda \cdot Q_s^\lambda) dP_s^\lambda - (Q_s^\lambda \cdot dP_s^\lambda) Q_s^\lambda\Big) - \int_0^t \frac{|P_s^\lambda|^2}{\sqrt{1+|P_s^\lambda|^2}} \pi_{P_s^\lambda}^\perp Q_s^\lambda ds. \tag{4.6}$$

By (4.3), we have that

$$\begin{aligned} \frac{|P_s^\lambda|^2}{\sqrt{1+|P_s^\lambda|^2}} \Big|\pi_{P_s^\lambda}^\perp Q_s^\lambda\Big| &= \frac{|P_s^\lambda|}{\sqrt{1+|P_s^\lambda|^2}} |Q_s^\lambda| \Big|\pi_{Q_s^\lambda}^\perp P_s^\lambda\Big| \\ &= \frac{|P_s^\lambda|}{\sqrt{1+|P_s^\lambda|^2}} \cdot \frac{1}{|Q_s^\lambda|} |f^\lambda(s)| \le \frac{1}{|Q_s^\lambda|} |f^\lambda(s)|, \end{aligned}$$

this combined with (4.6) implies that

$$|f^\lambda(t)| \le |f^\lambda(0)| + \Big| \int_0^t \Big((Q_s^\lambda \cdot Q_s^\lambda) dP_s^\lambda - (Q_s^\lambda \cdot dP_s^\lambda) Q_s^\lambda\Big)\Big| + \int_0^t \frac{1}{|Q_s^\lambda|} |f^\lambda(s)| ds$$

for any $t \ge 0$. So for any $r \in [0, T]$ we have that

$$\begin{aligned} E\Big[\sup_{t\in[0,r]} |f^\lambda(t\wedge\tau)|^2\Big] &\le 3|f^\lambda(0)|^2 \\ &\quad + 3E\Big[\sup_{t\in[0,r]} \Big| \int_0^{t\wedge\tau} \Big((Q_s^\lambda, Q_s^\lambda) dP_s^\lambda - (Q_s^\lambda, dP_s^\lambda) Q_s^\lambda\Big)\Big|^2\Big] \\ &\quad + \frac{3T}{(r_1-\varepsilon_0)^2} \int_0^r E\Big[\sup_{u\in[0,s]} |f^\lambda(u\wedge\tau)|^2\Big] ds. \end{aligned} \tag{4.7}$$

Since $\nabla U(x)$ is parallel to x for any $x \in \mathbb{R}^d$, we have by (1.1) that

$$\begin{aligned} (Q_s^\lambda \cdot Q_s^\lambda) dP_s^\lambda - (Q_s^\lambda \cdot dP_s^\lambda) Q_s^\lambda &= |Q_s^\lambda|^2 \sigma(Q_s^\lambda) dB_s - ({}^t Q_s \sigma(Q_s^\lambda) dB_s) Q_s^\lambda \\ &\quad - \gamma\Big(|Q_s^\lambda|^2 \frac{P_s^\lambda}{\sqrt{1+|P_s^\lambda|^2}} - (Q_s^\lambda \cdot \frac{P_s^\lambda}{\sqrt{1+|P_s^\lambda|^2}}) Q_s^\lambda\Big) ds. \end{aligned} \tag{4.8}$$

Notice that

$$E\Big[\sup_{t\in[0,T]} \Big| \int_0^{t\wedge\tau} |Q_s^\lambda|^2 \sigma(Q_s^\lambda) dB_s\Big|^2\Big] \le 4E\Big[\Big| \int_0^{T\wedge\tau} |Q_s^\lambda|^2 \sigma(Q_s^\lambda) dB_s\Big|^2\Big]$$

$$\le 4T(|q_0|+T)^4\|\sigma\|_\infty^2. \tag{4.9}$$

Similarly,

$$E\Big[\sup_{t\in[0,T]}\Big|\int_0^{t\wedge\tau}({}^tQ_s^\lambda\sigma(Q_s^\lambda)dB_s)Q_s^\lambda\Big|^2\Big] \le 4T(|q_0|+T)^4\|\sigma\|_\infty^2. \tag{4.10}$$

Also, since

$$\begin{aligned}&\Big||Q_s^\lambda|^2\frac{P_s^\lambda}{\sqrt{1+|P_s^\lambda|^2}} - \left(Q_s^\lambda\cdot\frac{P_s^\lambda}{\sqrt{1+|P_s^\lambda|^2}}\right)Q_s^\lambda\Big|\\ &= |Q_s^\lambda|^2\Big|\pi^\perp_{Q_s^\lambda}\Big(\frac{P_s^\lambda}{\sqrt{1+|P_s^\lambda|^2}}\Big)\Big| \le |Q_s^\lambda|^2 \le (|q_0|+T)^2,\end{aligned}$$

we have that

$$E\Big[\sup_{t\in[0,T]}\Big|\int_0^{t\wedge\tau}\gamma\Big(|Q_s^\lambda|^2\frac{P_s^\lambda}{\sqrt{1+|P_s^\lambda|^2}}-\left(Q_s^\lambda\cdot\frac{P_s^\lambda}{\sqrt{1+|P_s^\lambda|^2}}\right)Q_s^\lambda\Big)ds\Big|^2\Big] \le (T\gamma(|q_0|+T)^2)^2. \tag{4.11}$$

Let $C_4 = 24T(|q_0|+T)^4\|\sigma\|_\infty^2 + 3(T\gamma(|q_0|+T)^2)^2$. Then (4.8), (4.9), (4.10) and (4.11) imply that

$$\begin{aligned}&E\Big[\sup_{t\in[0,T]}\Big|\int_0^{t\wedge\tau}\Big((Q_s^\lambda\cdot Q_s^\lambda)dP_s^\lambda - (Q_s^\lambda\cdot dP_s^\lambda)Q_s^\lambda\Big)\Big|^2\Big]\\ &\le 3E\Big[\sup_{t\in[0,T]}\Big|\int_0^{t\wedge\tau}|Q_s^\lambda|^2\sigma(Q_s^\lambda)dB_s\Big|^2\Big]\\ &\quad +3E\Big[\sup_{t\in[0,T]}\Big|\int_0^{t\wedge\tau}({}^tQ_s^\lambda\sigma(Q_s^\lambda)dB_s)Q_s^\lambda\Big|^2\Big]\\ &\quad +3E\Big[\sup_{t\in[0,T]}\Big|\int_0^{t\wedge\tau}\gamma\Big(|Q_s^\lambda|^2\frac{P_s^\lambda}{\sqrt{1+|P_s^\lambda|^2}} - (Q_s^\lambda\cdot\frac{P_s^\lambda}{\sqrt{1+|P_s^\lambda|^2}})Q_s^\lambda\Big)ds\Big|^2\Big]\\ &\le C_4. \end{aligned}\tag{4.12}$$

Let $g^\lambda(r) = E\Big[\sup_{t\in[0,r]}|f^\lambda(t\wedge\tau)|^2\Big]$. Then since $|f^\lambda(0)| = \Big||q_0|^2\pi^\perp_{q_0}p_0\Big| \le |q_0|^2|p_0|$, we get by (4.7) and (4.12) that

$$g^\lambda(r) \le 3|q_0|^4|p_0|^2 + 3C_4 + \frac{3T}{(r_1-\varepsilon_0)^2}\int_0^r g^\lambda(s)ds.$$

By Gronwall's Lemma, this implies that

$$g^\lambda(r) \le (3|q_0|^4|p_0|^2+3C_4)e^{\frac{3T}{(r_1-\varepsilon_0)^2}T}, \qquad \text{for all } r\in[0,T].$$

Since $|Q_t^\lambda| \ge r_1-\varepsilon_0$ for any $t\le\tau$ and $\lambda\ge 1$, we now get our assertion with $C_3 = \frac{1}{(r_1-\varepsilon_0)^2}(3|q_0|^4|p_0|^2+3C_4)e^{\frac{3T}{(r_1-\varepsilon_0)^2}T}$. □

Proof of Lemma 4.2. The fact that $A_1^r(Q_t^\lambda, V_t^\lambda)$ is bounded is easy since σ is bounded. For (4.1), notice that for any $t\le\tau$, we have that $|R_t^\lambda| = |\pi^\perp_{Q_t^\lambda}P_t^\lambda| \le |P_t^\lambda| \le$

$\sqrt{1+|P_t^\lambda|^2} = \frac{1}{\sqrt{1-|V_t^\lambda|^2}}$ and $|Q_t^\lambda| \geq r_1 - \varepsilon_0$, so $|A_2^r(Q_t^\lambda, V_t^\lambda, R_t^\lambda)| \leq \gamma + \frac{2}{r_1-\varepsilon_0}|R_t^\lambda|$. Therefore,

$$E\Big[\sup_{t\in[0,T\wedge\tau]} |A_2^r(Q_t^\lambda, V_t^\lambda, R_t^\lambda)|^2\Big] \leq 2\gamma^2 + \frac{8}{(r_1-\varepsilon_0)^2} E\Big[\sup_{t\in[0,T\wedge\tau]} |R_t^\lambda|^2\Big]$$
$$\leq 2\gamma^2 + \frac{8}{(r_1-\varepsilon_0)^2} C_3.$$

Finally, by (4.4), we have that

$$|P_t^\lambda|^2 \pi^\perp_{P_t^\lambda} Q_t^\lambda = \left|\pi^\perp_{Q_t^\lambda} P_t^\lambda\right|^2 Q_t^\lambda - (Q_t^\lambda \cdot P_t^\lambda)\pi^\perp_{Q_t^\lambda} P_t^\lambda.$$

This combined with Ito's formula implies (4.2), and completes the proof of our first assertion.

The second assertion is a direct consequence of the first one and Lemma 2.2. □

5. Tightness of $\pi_{Q_t^\lambda} V_t^\lambda$

We prove the tightness of V_t^λ by proving that its components that are parallel to Q_t^λ and perpendicular to Q_t^λ, respectively, are both tight. The tightness of the parallel part $\pi_{Q_t^\lambda} V_t^\lambda$ is proved in this section, and the tightness of $\pi^\perp_{Q_t^\lambda} V_t^\lambda$ will be proved in the next section.

Let us first prepare the following result with respect to the differential of V_t^λ.

Lemma 5.1. *We have that $A_1^v(Q_t^\lambda, V_t^\lambda)$ and $A_2^v(Q_t^\lambda, V_t^\lambda)$ are bounded for $t \in [0,T]$ and $\lambda \geq 1$, and the following holds:*

(5.1)
$$dV_t^\lambda = A_1^v(Q_t^\lambda, V_t^\lambda)dB_t + A_2^v(Q_t^\lambda, V_t^\lambda)dt - \lambda\sqrt{1-|V_t^\lambda|^2}\Big(\nabla U(Q_t^\lambda) - (V_t^\lambda \cdot \nabla U(Q_t^\lambda))V_t^\lambda\Big)dt.$$

Proof. The fact that $A_1^v(Q_t^\lambda, V_t^\lambda)$ and $A_2^v(Q_t^\lambda, V_t^\lambda)$ are bounded is trivial since σ is bounded and $|V_t^\lambda| \leq 1$. Also, (5.1) is a direct consequence of Ito's formula. □

Our main result of this section is the following.

Lemma 5.2. *We have that {the distribution of $\{(\pi_{Q_t^\lambda} V_t^\lambda)\}_{t\in[0,T\wedge\tau]}; \lambda \geq 1\}$ is tight as probabilities on L^p for any $p > 1$, with all of its cluster point(s) in $\wp(D([0,T];\mathbb{R}^d))$.*

Let

$$A^{\|,\lambda}_{jump}(t) = (1-|V_t^\lambda|^2)^{3/2}(1+|R_t^\lambda|^2)\nabla U(Q_t^\lambda).$$

Then Lemma 5.2 is a direct consequence of Lemmas 5.3 and 5.4 given below.

Lemma 5.3. 1. *There exist stochastic processes $A_1^{\|,\lambda}(t)$ and $A_2^{\|,\lambda}(t)$ such that they are bounded for $t \in [0, T\wedge\tau]$ and $\lambda \geq 1$, and*

$$d(\pi_{Q_t^\lambda} V_t^\lambda) = A_1^{\|,\lambda}(t)dB_t + A_2^{\|,\lambda}(t)dt - \lambda A^{\|,\lambda}_{jump}(t)dt. \quad (5.2)$$

2. *{the distribution of* $\{(\pi_{Q_t^\lambda} V_t^\lambda) + \lambda \int_0^t A^{\|,\lambda}_{jump}(s)ds\}_{t\in[0,T\wedge\tau]}; \lambda \geq 1\}$ *is tight as probabilities on* $D([0,T];\mathbb{R}^d)$.

Lemma 5.4. 1. $\sup_{\lambda\geq 1} E\Big[\Big(\lambda \int_0^{T\wedge\tau} |A^{\|,\lambda}_{jump}(s)|ds\Big)^2\Big] < \infty$.

2. *{the distribution of* $\{\lambda \int_0^t A^{\|,\lambda}_{jump}(s)ds\}_{t\in[0,T\wedge\tau]}; \lambda \geq 1\}$ *is tight as probabilities on* L^p *for any* $p > 1$, *with all of its cluster point(s) in* $\wp(D([0,T];\mathbb{R}^d))$.

We prove these two lemmas in the rest of this section.

Proof of Lemma 5.3. Since $\pi_{Q_t^\lambda} V_t^\lambda = \Big(V_t^\lambda \cdot Q_t^\lambda\Big)\frac{1}{|Q_t^\lambda|^2} Q_t^\lambda$, we get (5.2) by a direct calculation with the help of Lemma 5.1, with

$$\begin{aligned} A_1^{\|,\lambda}(t) &= \frac{1}{|Q_t^\lambda|^2} Q_t^\lambda ({}^t Q_t^\lambda) A_1^v(Q_t^\lambda, V_t^\lambda), \\ A_2^{\|,\lambda}(t) &= \frac{1}{|Q_t^\lambda|^2}\Big(Q_t^\lambda \cdot A_2^v(Q_t^\lambda, V_t^\lambda) + |V_t^\lambda|^2\Big)Q_t^\lambda - \frac{2(Q_t^\lambda\cdot V_t^\lambda)^2}{|Q_t^\lambda|^4} Q_t^\lambda + \frac{Q_t^\lambda \cdot V_t^\lambda}{|Q_t^\lambda|^2} V_t^\lambda. \end{aligned}$$

The fact that $A_1^{\|,\lambda}(t)$ and $A_2^{\|,\lambda}(t)$ are bounded for $t \in [0, T\wedge\tau]$ and $\lambda \geq 1$ is trivial.

The second assertion is an easy consequence of the first one and Lemma 2.2. □

Proof of Lemma 5.4. The second assertion is an easy consequence of the first assertion and Lemma 2.3. We prove the first assertion in the following.

Recall that $A = \{y \in \mathbb{R}^d \Big| \Big||y| - r_1\Big| \leq \varepsilon_0 \text{ or } |y| \geq r_2 - \varepsilon_0\}$. So by assumption, for any $x \in A$, we have that

$$\begin{aligned} |\nabla U(x)| &= |h'(|x|)| = h'(|x|)k(x) = k(x)h'(|x|)\frac{x}{|x|}\cdot\frac{x}{|x|} \\ &= \nabla U(x)\cdot k(x)\frac{x}{|x|}. \end{aligned}$$

Let g be a function in $C_b^1(\mathbb{R}^d)$ such that $g(x) = k(x)\frac{x}{|x|}$ for any $x \in A$. Then by the definition of $A^{\|,\lambda}_{jump}(t)$, we have that

$$|A^{\|,\lambda}_{jump}(t)| = \Big|\frac{1}{(1+|P_t^\lambda|^2)^{3/2}}(1 + |\pi^\perp_{Q_t^\lambda} P_t^\lambda|^2)\nabla U(Q_t^\lambda)\Big| = A^{\|,\lambda}_{jump}(t)\cdot g(Q_t^\lambda)$$

as long as $Q_t^\lambda \in A$.

By the formula of integration by parts and (5.2), we have that

$$\begin{aligned} &\lambda \int_0^t A^{\|,\lambda}_{jump}(s)\cdot g(Q_s^\lambda)ds \\ &= \int_0^t g(Q_s^\lambda)\cdot \frac{d}{ds}\Big(\int_0^s \lambda A^{\|,\lambda}_{jump}(r)dr\Big)ds \\ &= g(Q_t^\lambda)\cdot\Big(\int_0^t \lambda A^{\|,\lambda}_{jump}(r)dr\Big) - \int_0^t\Big(\int_0^s \lambda A^{\|,\lambda}_{jump}(r)dr\Big)\cdot\nabla g(Q_s^\lambda)V_s^\lambda ds \\ &= g(Q_t^\lambda)\cdot\Big[-\pi_{Q_t^\lambda}V_t^\lambda + \pi_{q_0}v_0 + \int_0^t A_1^{\|,\lambda}(s)dB_s + \int_0^t A_2^{\|,\lambda}(s)ds\Big] \end{aligned}$$

$$-\int_0^t \Big[-\pi_{Q_s^\lambda}V_s^\lambda+\pi_{q_0}v_0+\int_0^s A_1^{\|,\lambda}(r)dB_r+\int_0^s A_2^{\|,\lambda}(r)dr\Big]\cdot\nabla g(Q_s^\lambda)V_s^\lambda ds.$$

We have that $|V_t^\lambda|\le 1$ for any t and $\lambda\ge 1$. Let $C_5=(\|g\|_\infty+\|g'\|_\infty T)(2+T\sup_{\lambda\ge1}\|A_2^{\|,\lambda}\|_\infty)$ and $C_6=\|g\|_\infty+\|\nabla g\|_\infty T$. Then we get that

$$\begin{aligned}&\Big|\lambda\int_0^t A_{jump}^{\|,\lambda}(s)\cdot g(Q_s^\lambda)ds\Big|\\ &\le\|g\|_\infty\Big[2+T\|A_2^{\|,\lambda}\|_\infty+\Big|\int_0^t A_1^{\|,\lambda}(s)dB_s\Big|\Big]\\ &\quad+\|\nabla g\|_\infty T\Big[2+T\|A_2^{\|,\lambda}\|_\infty+\sup_{s\in[0,T]}\Big|\int_0^s A_1^{\|,\lambda}(r)dB_r\Big|\Big]\\ &\le C_5+C_6\sup_{s\in[0,T]}\Big|\int_0^s A_1^{\|,\lambda}(r)dB_r\Big|.\end{aligned}$$

Since

$$\begin{aligned}E\Big[\sup_{s\in[0,T]}\Big|\int_0^s A_1^{\|,\lambda}(r)dB_r\Big|^2\Big]&\le 4E\Big[\Big|\int_0^T A_1^{\|,\lambda}(r)dB_r\Big|^2\Big]\\ &\le 4T\|A_1^{\|,\lambda}\|_\infty^2,\end{aligned}$$

the calculation above implies that

$$E\Big[\Big|\lambda\int_0^t A_{jump}^{\|,\lambda}(s)\cdot g(Q_s^\lambda)ds\Big|^2\Big]\le 2C_5^2+2C_6^2\cdot 4T\|A_1^{\|,\lambda}\|_\infty^2.$$

We next estimate the difference between $\lambda\int_0^t|A_{jump}^{\|,\lambda}(s)|ds$ and $\lambda\int_0^t A_{jump}^{\|,\lambda}(s)\cdot g(Q_s^\lambda)ds$. Since $A^C\cap D=\{x\in\mathbb{R}^d\big|r_1+\varepsilon_0<|x|<r_2-\varepsilon_0\}$, we have by assumption that there exists a constant $\varepsilon_1>0$ such that $|\nabla U(x)|\le\varepsilon_1$ as long as $x\in A^C\cap D$. Since $\|g\|_\infty\le 1$ by assumption, and $|A_{jump}^{\|,\lambda}(s)|-A_{jump}^{\|,\lambda}(s)\cdot g(Q_s)=0$ as long as $Q_s\in A$, this implies that

$$\begin{aligned}\Big||A_{jump}^{\|,\lambda}(s)|-A_{jump}^{\|,\lambda}(s)\cdot g(Q_s^\lambda)\Big|&\le 2|A_{jump}^{\|,\lambda}(s)|1_{\{Q_s^\lambda\in A^C\}}\\ &\le\frac{2}{\sqrt{1+|P_s^\lambda|^2}}|\nabla U(Q_s^\lambda)|1_{\{Q_s^\lambda\in A^C\}}\\ &\le\frac{2\varepsilon_1}{\sqrt{1+|P_s^\lambda|^2}}1_{\{Q_s^\lambda\in A^C\}},\qquad s\le\tau.\end{aligned}$$

So

$$\begin{aligned}&E\Big[\Big|\lambda\int_0^{T\wedge\tau}|A_{jump}^{\|,\lambda}(s)|ds-\lambda\int_0^{T\wedge\tau}A_{jump}^{\|,\lambda}(s)\cdot g(Q_s^\lambda)ds\Big|^2\Big]\\ &\le\lambda^2E\Big[\Big(\int_0^T\frac{2\varepsilon_1}{\sqrt{1+|P_s^\lambda|^2}}1_{\{Q_s^\lambda\in A^C\cap D\}}ds\Big)^2\Big]\\ &\le 4T\varepsilon_1^2\int_0^T E\Big[\lambda^2\Big(\frac{1}{\sqrt{1+|P_s^\lambda|^2}}\Big)^2,Q_s^\lambda\in A^C\cap D\Big]ds.\end{aligned}$$

So our assertion is a consequence of Lemma 5.5 below. □

Lemma 5.5. *For any $\varepsilon > 0$, we have that*

$$\sup_{s\in[0,T],\lambda\geq 1} \lambda^2 E\Big[\frac{1}{1+|P_s^\lambda|^2}, |Q_s^\lambda| \in (r_1+\varepsilon, r_2-\varepsilon)\Big] < \infty.$$

Proof. Fix any $\varepsilon > 0$. By assumption, we have that there exists a constant $\varepsilon_2 > 0$ such that $U(x) \leq -\varepsilon_2$ whenenver $|Q_s^\lambda| \in (r_1+\varepsilon, r_2-\varepsilon)$. Therefore, if $|Q_s^\lambda| \in (r_1+\varepsilon, r_2-\varepsilon)$ and $H_s^\lambda > -\frac{1}{2}\varepsilon_2\lambda$, then $\sqrt{1+|P_s^\lambda|^2} = H_s^\lambda - \lambda U(Q_s^\lambda) \geq H_s^\lambda + \lambda\varepsilon_2 \geq \frac{1}{2}\varepsilon_2\lambda$. So

$$\lambda^2 E\Big[\frac{1}{1+|P_s^\lambda|^2}, \{Q_s^\lambda \in A^C \cap D_\varepsilon\} \cap \{H_s^\lambda > -\frac{1}{2}\varepsilon_2\lambda\}\Big] \leq \Big(\frac{\lambda}{\frac{1}{2}\varepsilon_2\lambda}\Big)^2 = \Big(\frac{2}{\varepsilon_2}\Big)^2.$$

Also, by Lemma 3.1 (1), we have that

$$H_s^\lambda \geq H_0 + \int_0^s A_1^h(Q_u^\lambda, V_u^\lambda)dB_u - C_1 s.$$

Therefore, if λ is big enough such that $-H_0 + C_1 s \leq -H_0 + C_1 T \leq \frac{1}{4}\varepsilon_2\lambda$, then

$$\begin{aligned}
&\lambda^2 E\Big[\frac{1}{1+|P_s^\lambda|^2}, \{|Q_s^\lambda| \in (r_1+\varepsilon, r_2-\varepsilon)\} \cap \{H_s^\lambda \leq -\frac{1}{2}\varepsilon_2\lambda\}\Big]\\
&\leq \lambda^2 P\Big(H_s^\lambda \leq -\frac{1}{2}\varepsilon_2\lambda\Big)\\
&\leq \lambda^2 P\Big(\int_0^s A_1^h(Q_u^\lambda, V_u^\lambda)dB_u \leq -\frac{1}{2}\varepsilon_2\lambda - H_0 + C_1 s\Big)\\
&\leq \lambda^2 P\Big(\int_0^s A_1^h(Q_u^\lambda, V_u^\lambda)dB_u \leq -\frac{1}{4}\varepsilon_2\lambda\Big)\\
&\leq \lambda^2 \Big(\frac{1}{4}\varepsilon_2\lambda\Big)^{-2} E\Big[\Big|\int_0^s A_1^h(Q_u^\lambda, V_u^\lambda)dB_u\Big|^2\Big]\\
&\leq \Big(\frac{\varepsilon_2}{4}\Big)^{-2} T C_1^2.
\end{aligned}$$

This completes the proof of our assertion. ☐

6. Tightness of $\pi^\perp_{Q_t^\lambda} V_t^\lambda$

We complete the proof of the tigthness of V_t^λ in this section. Our main result of this section is the following.

Lemma 6.1. *We have that $\Big\{$the distribution of $\Big\{V_{t\wedge\tau}^\lambda\Big\}_{t\in[0,T]}; \lambda \geq 1\Big\}$ is tight as probabilities on $L^p([0,T],\mathbb{R}^d)$ for any $p > 1$, with its cluster point(s) in $\wp(D([0,T];\mathbb{R}^d))$.*

By Lemma 5.2, it suffices to prove the following.

Lemma 6.2. *We have that $\{$the distribution of $\{(\pi^\perp_{Q_t^\lambda} V_t^\lambda)\}_{t\in[0,T\wedge\tau]}; \lambda \geq 1\}$ is tight as probabilities on $L^p([0,T],\mathbb{R}^d)$ for any $p > 1$, with all of its cluster point(s) in $\wp(D([0,T];\mathbb{R}^d))$.*

We prove Lemma 6.2 in the rest of this section. Let us first make some preparation.

Lemma 6.3.

$$\sup_{\lambda\geq 1} E\Big[\sup_{t\in[0,T]}\Big(\int_0^{t\wedge\tau}\frac{dP_s^\lambda}{1+|P_s^\lambda|^2}\Big)^2\Big]<\infty.$$

Proof. Let us first calculate $d\Big(\frac{P_t^\lambda}{1+|P_t^\lambda|^2}\Big)$. Let

$$\begin{aligned}
F_1^\lambda(t) &:= \frac{2}{(1+|P_t^\lambda|^2)^2}\Big(|P_t^\lambda|^2\sigma(Q_t^\lambda)-P_t^{\lambda t}P_t^\lambda\sigma(Q_t^\lambda)\Big),\\
F_2^\lambda(t) &:= \Big(-\frac{\sum_{i,j=1}^d\sigma_{ij}^2(Q_t^\lambda)}{(1+|P_t^\lambda|^2)^2}+\frac{4\,{}^tP_t^\lambda\sigma(Q_t^\lambda)^t\sigma(Q_t^\lambda)P_t^\lambda}{(1+|P_t^\lambda|^2)^3}\Big)P_t^\lambda-\frac{2\sigma(Q_t^\lambda)^t\sigma(Q_t^\lambda)P_t^\lambda}{(1+|P_t^\lambda|^2)^2}.
\end{aligned}$$

Then $F_1^\lambda(t)$ and $F_2^\lambda(t)$ are bounded for $t\in[0,T]$ and $\lambda\geq 1$, and by Ito's formula and a simple calculation, we have that

$$\begin{aligned}
d\Big(\frac{P_t^\lambda}{1+|P_t^\lambda|^2}\Big) &= -\frac{dP_t^\lambda}{1+|P_t^\lambda|^2}+\frac{2dP_t^\lambda}{(1+|P_t^\lambda|^2)^2}+F_1^\lambda(t)dB_t+F_2^\lambda(t)dt\\
&\quad -\frac{2}{(1+|P_t^\lambda|^2)^2}\lambda\Big(|P_t^\lambda|^2\nabla U(Q_t^\lambda)-(P_t^\lambda\cdot\nabla U(Q_t^\lambda))P_t^\lambda\Big)dt.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_0^{t\wedge\tau}\frac{dP_s^\lambda}{1+|P_s^\lambda|^2}\\
&=\frac{p_0}{1+|p_0|^2}-\frac{P_{t\wedge\tau}^\lambda}{1+|P_{t\wedge\tau}^\lambda|^2}+2\int_0^{t\wedge\tau}\frac{dP_s^\lambda}{(1+|P_s^\lambda|^2)^2}+\int_0^{t\wedge\tau}\Big(F_1^\lambda(s)dB_s+F_2^\lambda(s)ds\Big)\\
&\quad-\int_0^{t\wedge\tau}\frac{2}{(1+|P_s^\lambda|^2)^2}\lambda\Big(|P_s^\lambda|^2\nabla U(Q_s^\lambda)-(P_s^\lambda\cdot\nabla U(Q_s^\lambda))P_s^\lambda\Big)ds.
\end{aligned}$$

So in order to get our assertion, it suffices to prove the following three estimates.

$$\sup_{\lambda\geq 1} E\Big[\sup_{t\in[0,T]}\Big(\int_0^{t\wedge\tau}\frac{dP_s^\lambda}{(1+|P_s^\lambda|^2)^2}\Big)^2\Big]<\infty, \tag{6.1}$$

$$\sup_{\lambda\geq 1} E\Big[\sup_{t\in[0,T]}\Big(\int_0^{t\wedge\tau}(F_1^\lambda(s)dB_s+F_2^\lambda(s)ds)\Big)^2\Big]<\infty, \tag{6.2}$$

$$\begin{aligned}
&\sup_{\lambda\geq 1} E\Big[\sup_{t\in[0,T]}\Big(\int_0^{t\wedge\tau}\frac{\lambda}{(1+|P_s^\lambda|^2)^2}\cdot\\
&\qquad\cdot\Big(|P_s^\lambda|^2\nabla U(Q_s^\lambda)-(P_s^\lambda\cdot\nabla U(Q_s^\lambda))P_s^\lambda\Big)ds\Big)^2\Big]<\infty.
\end{aligned} \tag{6.3}$$

(6.2) is trivial by Lemma 2.2 (1) since $F_1^\lambda(t)$ and $F_2^\lambda(t)$ are bounded. We prove (6.1) and (6.3) in the following.

For (6.1), we have that

$$\int_0^{t\wedge\tau}\frac{dP_s^\lambda}{(1+|P_s^\lambda|^2)^2}=\int_0^{t\wedge\tau}\frac{\sigma(Q_s^\lambda)dB_s}{(1+|P_s^\lambda|^2)^2}-\gamma\int_0^{t\wedge\tau}\frac{V_s^\lambda ds}{(1+|P_s^\lambda|^2)^2}-\lambda\int_0^{t\wedge\tau}\frac{\nabla U(Q_s^\lambda)ds}{(1+|P_s^\lambda|^2)^2}.$$

The estimates for the first two terms on the right hand side above are trivial. Also, since $\left|\frac{\nabla U(Q_s^\lambda)}{(1+|P_s^\lambda|^2)^2}\right| \leq |A_{jump}^{\|,\lambda}(s)|$, we get

$$\sup_{\lambda\geq 1} E\Big[\sup_{t\in[0,T]}\Big(\lambda\int_0^{t\wedge\tau}\frac{\nabla U(Q_s^\lambda)ds}{(1+|P_s^\lambda|^2)^2}\Big)^2\Big] < \infty,$$

as an easy consequence of Lemma 5.4 (1).

Finally, for (6.3), we have by (4.3) that

$$\begin{aligned}
&\Big||P_s^\lambda|^2\nabla U(Q_s^\lambda) - (P_s^\lambda\cdot\nabla U(Q_s^\lambda))P_s^\lambda\Big| \\
&= \Big||P_s^\lambda|^2 Q_s^\lambda - (P_s^\lambda\cdot Q_s^\lambda)P_s^\lambda\Big|\cdot\frac{|\nabla U(Q_s^\lambda)|}{|Q_s^\lambda|} \\
&= |P_s^\lambda||\pi_{Q_s^\lambda}^\perp P_s^\lambda||\nabla U(Q_s^\lambda)| \\
&\leq |P_s^\lambda|(1+|\pi_{Q_s^\lambda}^\perp P_s^\lambda|^2)|\nabla U(Q_s^\lambda)| = |P_s^\lambda|(1+|P_s^\lambda|^2)^{3/2}|A_{jump}^{\|,\lambda}(s)|.
\end{aligned}$$

So (6.3) is also a direct consequence of Lemma 5.4 (1). This completes the proof of our assertion. □

The following is an easy corollary of Lemma 6.3.

Corollary 6.4.

$$\sup_{\lambda\geq 1} E\Big[\sup_{t\in[0,T]}\Big(\int_0^{t\wedge\tau}\frac{\lambda\nabla U(Q_s^\lambda)}{1+|P_s^\lambda|^2}ds\Big)^2\Big] < \infty.$$

We next use Corollary 6.4 to prove the following.

Lemma 6.5.

$$\sup_{\lambda\geq 1} E\Big[\Big(\int_0^{T\wedge\tau}\frac{\lambda|\nabla U(Q_s^\lambda)|}{1+|P_s^\lambda|^2}ds\Big)^2\Big] < \infty.$$

Proof. The idea is similar to that of the proof of Lemma 5.4.

Let g be the same one as in the proof of Lemma 5.4. Since

$$\begin{aligned}
&\int_0^t \frac{\lambda g(Q_s^\lambda)\cdot\nabla U(Q_s^\lambda)}{1+|P_s^\lambda|^2}ds \\
&= \int_0^t g(Q_s^\lambda)\cdot d\Big(\int_0^s\frac{\lambda\nabla U(Q_r^\lambda)}{1+|P_r^\lambda|^2}dr\Big)ds \\
&= g(Q_t^\lambda)\cdot\int_0^t\frac{\lambda\nabla U(Q_r^\lambda)}{1+|P_r^\lambda|^2}dr - \int_0^t\nabla g(Q_s^\lambda)V_s^\lambda\cdot\Big(\int_0^s\frac{\lambda\nabla U(Q_r^\lambda)}{1+|P_r^\lambda|^2}dr\Big)ds,
\end{aligned}$$

we have that

$$\begin{aligned}
&E\Big[\Big(\int_0^{T\wedge\tau}\frac{\lambda g(Q_s^\lambda)\cdot\nabla U(Q_s^\lambda)}{1+|P_s^\lambda|^2}ds\Big)^2\Big] \\
&\quad\leq\Big(2\|g\|_\infty^2 + 2T^2\|\nabla g\|_\infty^2\Big)E\Big[\sup_{t\in[0,T]}\Big(\int_0^{t\wedge\tau}\frac{\lambda\nabla U(Q_r^\lambda)}{1+|P_r^\lambda|^2}dr\Big)^2\Big],
\end{aligned}$$

which is bounded for $\lambda\geq 1$ by Corollary 6.4.

Also, as in the proof of Lemma 5.4, there exists a constant $\varepsilon_1 > 0$ such that

$$\Big||\nabla U(Q_s^\lambda)| - g(Q_s^\lambda)\cdot\nabla U(Q_s^\lambda)\Big| \le 2|\nabla U(Q_s^\lambda)|1_{\{Q_s^\lambda\in A^C\}} \le 2\varepsilon_1 1_{\{Q_s^\lambda\in A^C\}}, \quad s\le\tau,$$

so

$$\begin{aligned}
&E\Big[\Big(\int_0^{T\wedge\tau}\frac{\lambda|\nabla U(Q_s^\lambda)|}{1+|P_s^\lambda|^2}ds - \int_0^{T\wedge\tau}\frac{\lambda g(Q_s^\lambda)\cdot\nabla U(Q_s^\lambda)}{1+|P_s^\lambda|^2}ds\Big)^2\Big]\\
&\le \lambda^2 E\Big[\Big(\int_0^T\frac{2\varepsilon_1}{1+|P_s^\lambda|^2}1_{\{Q_s^\lambda\in A^C\cap D\}}ds\Big)^2\Big]\\
&\le 4\varepsilon_1^2 T\lambda^2 E\Big[\int_0^T\frac{1}{(1+|P_s^\lambda|^2)^2}1_{\{Q_s^\lambda\in A^C\cap D\}}ds\Big]\\
&= 4\varepsilon_1^2 T\lambda^2\int_0^T E\Big[\frac{1}{(1+|P_s^\lambda|^2)^2}, Q_s^\lambda\in A^C\cap D\Big]ds,
\end{aligned}$$

which is bounded by Lemma 5.5. □

Now we are ready to prove the tightness of $\pi^\perp_{Q_t^\lambda}V_t^\lambda$, the component of V_t^λ that is perpendicular to Q_t^λ. As in Section 5.2, we prove Lemma 6.2 by proving the following two lemmas.

Lemma 6.6. 1. *There exist stochastic processes* $A_1^{\perp,\lambda}(t)$ *and* $A_2^{\perp,\lambda}(t)$ *such that they are bounded for* $t\in[0,T\wedge\tau]$ *and* $\lambda\ge 1$*, and*

$$d(\pi^\perp_{Q_t^\lambda}V_t^\lambda) = A_1^{\perp,\lambda}(t)dB_t + A_1^{\perp,\lambda}(t)dt + \lambda(V_t^\lambda\cdot\nabla U(Q_t^\lambda))(1-|V_t^\lambda|^2)R_t^\lambda dt. \tag{6.4}$$

2. $\{$*the distribution of* $\{(\pi^\perp_{Q_t^\lambda}V_t^\lambda) - \lambda\int_0^t(V_s^\lambda\cdot\nabla U(Q_s^\lambda))(1-|V_s^\lambda|^2)R_s^\lambda ds\}_{t\in[0,T\wedge\tau]};\lambda\ge 1\}$ *is tight as probabilities on* $C([0,T];\mathbb{R}^d)$.

Lemma 6.7. 1. $\sup_{\lambda\ge1} E\Big[\lambda\int_0^{T\wedge\tau}|(V_s^\lambda\cdot\nabla U(Q_s^\lambda))(1-|V_s^\lambda|^2)R_s^\lambda|ds\Big]<\infty$.

2. $\{$*the distribution of* $\{\lambda\int_0^t(V_s^\lambda\cdot\nabla U(Q_s^\lambda))(1-|V_s^\lambda|^2)R_s^\lambda ds\}_{t\in[0,T\wedge\tau]};\lambda\ge1\}$ *is tight as probabilities on* L^p *for any* $p>1$*, with its cluster point(s) in* $\wp(D([0,T];\mathbb{R}^d))$.

Proof of Lemma 6.6. The first assertion is trivial by Lemmas 5.1 and 5.3, with

$$\begin{aligned}
A_1^{\perp,\lambda}(t) &= A_1^v(Q_t^\lambda,V_t^\lambda) - A_1^{\|,\lambda}(t),\\
A_2^{\perp,\lambda}(t) &= A_2^v(Q_t^\lambda,V_t^\lambda) - A_2^{\|,\lambda}(t).
\end{aligned}$$

The second assertion is trivial by the first assertion and Lemma 2.2. □

Proof of Lemma 6.7. We have that

$$\begin{aligned}
&E\Big[\lambda\int_0^{T\wedge\tau}|(V_s^\lambda,\nabla U(Q_s^\lambda))(1-|V_t^\lambda|^2)R_s^\lambda|ds\Big]\\
&\le E\Big[\Big(\lambda\int_0^{T\wedge\tau}\frac{|\nabla U(Q_s^\lambda)|}{1+|P_s^\lambda|^2}ds\Big)\Big(\sup_{s\in[0,T\wedge\tau]}|R_s^\lambda|\Big)\Big]\\
&\le E\Big[\Big(\lambda\int_0^{T\wedge\tau}\frac{|\nabla U(Q_s^\lambda)|}{1+|P_s^\lambda|^2}ds\Big)^2\Big]^{1/2}\times E\Big[\sup_{s\in[0,T\wedge\tau]}|R_s^\lambda|^2\Big]^{1/2}.
\end{aligned}$$

This combined with Lemma 6.5 and Lemma 4.1 gives us our first assertion. The second assertion is easy by the first assertion and Lemma 2.3. □

7. Proof of the uniqueness

In this section, we prove the uniqueness of the probability that satisfies $(\mu 1) \sim (\mu 5)$. The idea is as follows. We first prove that the particle only "passes through" and never "stays on" the boundary $|Q_t| = r_2$ of the two phases. This is a combination of [2, Corollary 5] and Lemma 7.1 in the following. Indeed, [2, Corollary 5] ensures that the particle never "stays on" the set $|Q_t| = r_2$ when it arrives from the diffusion phase, and Lemma 7.1 ensures that the same holds when it arrives from the uniform motion phase. Next, given any solution of the martingale problem, there exists a Brownian motion such that our solution can be represented as the distribution of the solution of the corresponding system of SDEs with jump (see Claim 1 in the proof of Theorem 1.1 (1)). As mentioned in Remark 1.2, we can convert the gotten SDE with respect to (Q_t, V_t, H_t, R_t) to SDEs with respect to (Q_t, P_t) and (H_t, R_t) in diffusion phase and uniform motion phase, respectively. Therefore, since the coefficients of the new SDEs are all Lipschitz continuous, we can prove the pathwise uniqueness of the solution (see Claim 2 in the proof of Theorem 1.1 (1)). Finally, in the same way as in Yamada-Watanabe [5], we prove that the pathwise uniqueness implies the uniqueness in the sense of the probability law, and this completes our proof of the uniqueness.

Notice that if a probability measure satisfies $(\mu 4)$, then we have that the particle is in uniform motion in $|Q_t| \in (r_1, r_2)$, *i.e.*, when considering the behavior of the particle in this phase, we can assume that V_t and Q_t are two non-random processes with both of them keeping in a common (or opposite) direction.

Lemma 7.1. *Let V_t and Q_t be two non-random processes with $V_t \parallel Q_t \parallel Q_0$ for any t, and that $|Q_t| \geq r_1$. Also, let $\nu \in \wp(C([0,\infty);\mathbb{R}) \times C([0,\infty);\mathbb{R}^d))$ be a solution of the martingale problem L_u with initial condition $R_0 \perp Q_0$. Then for any initial condition H_0, R_0 and any $t > 0$, we have that $\nu(H_t^2 = 1 + |R_t|^2) = 0$.*

Proof. Choose and fix any $H_0 \in \mathbb{R}$ and $R_0 \in \mathbb{R}^d$.

Since ν is a solution of the martingale problem L_u, and $\{(H_t, R_t)\}_t$ is continuous, we have by [4] that there exists a Brownian motion $\{B_t\}_t$ under ν such that (H_t, R_t) satisfies the following SDE.

$$\begin{aligned} dH_t &= {}^t V_t \sigma(Q_t) dB_t - \gamma dt \\ dR_t &= \left(\sigma(Q_t) - \tfrac{Q_t {}^t Q_t \sigma(Q_t)}{|Q_t|^2}\right) dB_t - \tfrac{Q_t \cdot V_t}{|Q_t|^2} R_t dt. \end{aligned}$$

By solving this SDE directly, we get that

$$H_t = H_0 + \int_0^t {}^t V_s \sigma(Q_s) dB_s - \gamma t,$$

$$R_t = \exp\Big(-\int_0^t \frac{Q_s\cdot V_s}{|Q_s|^2}ds\Big)R_0$$
$$+\int_0^t \Big(\sigma(Q_s)-\frac{Q_s{}^tQ_s\sigma(Q_s)}{|Q_s|^2}\Big)\exp\Big(-\int_s^t \frac{Q_u\cdot V_u}{|Q_u|^2}du\Big)dB_s.$$

Choose $e_1,\dots,e_{d-1}$ as an orthonormal basis of $Q_0^\perp$, the ortho-complement space of Q_0. Since Q_u, $u\in[0,t]$, keep in the same direction, it is trivial that R_t is orthogonal to Q_0 for any $t\geq 0$, so we can write $R_t = R_1^1e_1+\dots+R_t^{d-1}e_{d-1}$. Hence $|R_t|^2=|R_t^1|^2+\dots+|R_t^{d-1}|^2$. Choose an arbitrary $t>0$ and fix it from now on. Then the calculation above implies that ${}^t(H_t-H_0, R_t^1-\exp\Big(-\int_0^t\frac{Q_s\cdot V_s}{|Q_s|^2}ds\Big)R_0^1,\dots,R_t^{d-1}-\exp\Big(-\int_0^t\frac{Q_s\cdot V_s}{|Q_s|^2}ds\Big)R_0^{d-1})$ is a d-dimensional Gaussian random variable. Write it as $X\sim N(M,\Sigma^2)$. So in order to prove our lemma, it suffices to prove that Σ^2 is non-degenerate. Suppose not, then there exist $a,b_1,\dots,b_{d-1}\in\mathbb{R}$ such that $(a,b_1,\dots,b_{d-1})\neq\mathbf{0}\in\mathbb{R}^d$ and $(a,b_1,\dots,b_{d-1})\Sigma=\mathbf{0}$, hence $(a,b_1,\cdots,b_{d-1})X\sim N((a,{}^tb)M,0)$. Write $b=b_1e_1+\cdots b_{d-1}e_{d-1}$. Then we get that

$$E\Big[\Big(a\int_0^t {}^tV_s\sigma(Q_s)dB_s+{}^tb\int_0^t\Big(\sigma(Q_s)-\frac{Q_s{}^tQ_s\sigma(Q_s)}{|Q_s|^2}\Big)\exp\Big(-\int_s^t\frac{Q_u\cdot V_u}{|Q_u|^2}du\Big)dB_s\Big)^2\Big]=0.$$

The left hand side above is equal to

$$\int_0^t\Big|a{}^tV_s\sigma(Q_s)+{}^tb\Big(\sigma(Q_s)-\frac{Q_s{}^tQ_s\sigma(Q_s)}{|Q_s|^2}\Big)\exp\Big(-\int_s^t\frac{Q_u\cdot V_u}{|Q_u|^2}du\Big)\Big|^2ds$$
$$=\int_0^t\Big|\Big(a{}^tV_s+\exp\Big(-\int_s^t\frac{Q_u\cdot V_u}{|Q_u|^2}du\Big){}^tb\Big)\sigma(Q_s)\Big|^2ds.$$

Here in the last equality, we used the fact that $b\perp Q_s$. So

$$\big(a{}^tV_s+\exp\big(-\int_s^t\frac{Q_u\cdot V_u}{|Q_u|^2}du\big){}^tb\big)\sigma(Q_s)=0$$

for almost every $s\in[0,t]$. Since ${}^t\sigma\sigma$ is strictly positive-definite and b is perpendicular to V_s, we get that $a=0$ and $b=\mathbf{0}$, which contradicts the assumption that $(a,b_1,\dots,b_{d-1})\neq\mathbf{0}$. □

Proof of Theorem 1.1 (1). We complete the proof of Theorem 1.1 (1) in the rest of this section. First we have the following.

Claim 1. Let μ be a probability that satisfies $(\mu 1) \sim (\mu 5)$, and let $X = \{X_t\}_{t\geq 0} = \{(Q_t, V_t, H_t, R_t)\}_{t\geq 0}$ denote the canonical process. Then there exists a Brownian motion $\{B_t\}_{t\geq 0}$ such that X satisfies the following system of SDEs with jump.

$$
(7.1) \qquad \begin{cases} dQ_t = V_t dt \\ dV_t^c = 1_{\{|Q_t|>r_2\}}\Big(A_1^v(Q_t, V_t)dB_t + A_2^v(Q_t, V_t)dt\Big) \\ \Delta V_t = 1_{\{|Q_t|=r_1\}}\frac{2Q_t}{|Q_t|} + 1_{\{|Q_t|=r_2, Q_t\cdot V_{t-}<0\}}\Big(-\frac{Q_t}{|Q_t|} - V_{t-}\Big) \\ \qquad\qquad -1_{\{|Q_t|=r_2, Q_t\cdot V_{t-}>0, H_t<\sqrt{1+|R_t|^2}\}}\frac{2Q_t}{|Q_t|} \\ \qquad\qquad +1_{\{|Q_t|=r_2, Q_t\cdot V_{t-}>0, H_t>\sqrt{1+|R_t|^2}\}}\Big(\frac{\sqrt{H_t^2-1-|R_t|^2}Q_t/|Q_t|+R_t}{H_t} - \frac{Q_t}{|Q_t|}\Big) \\ dH_t = A_1^h(Q_t, V_t)dB_t + A_2^h(Q_t, V_t)dt \\ dR_t = A_1^r(Q_t, V_t)dB_t + A_2^r(Q_t, V_t, R_t)dt. \end{cases}
$$

Here V_t^c and ΔV_t stand for the continuous part and the jump part of V_t, respectively.

Proof of Claim 1. Let $\tau_0 = 0$ and for any $k \in \mathbb{N}$, let $\tau_k = \inf\{t > \tau_{k-1}; |Q_t| = r_1 \text{ or } r_2\}$. Then by [2, Corollary 5] and Lemma 7.1, we have that τ_k is strictly increasing with respect to k. Also, for any $k \in \mathbb{N}$, it is easy to be seen that the process $\{X_t; t \in [\tau_k, \tau_{k+2}]\}$ at least includes a piece of uniform motion, either from $|Q_t| = r_1$ to $|Q_t| = r_2$, or from $|Q_t| = r_2$ to $|Q_t| = r_1$, with the norm of velocity equal to 1, so $\tau_{k+2} - \tau_k \geq r_2 - r_1$. This is true for any $k \in \mathbb{N}$, so we get that $\lim_{k\to\infty} \tau_k = \infty$.

Let us prepare the notation $U_t^\eta = U_{t+\eta} - U_\eta$ for any $t > 0$, any stopping time η and any stochastic process U. Then we have that for any $k \geq 0$, $\{X_t^{\tau_k}; t \in [0, \tau_{k+1} - \tau_k]\}$ is a continuous solution of the martingale problem L. Therefore, by Revuz-Yor [4], there exists a Brownian motion $\{B_t^{(k)}\}_t$ such that

$$
dX_t^{\tau_k} = K_1(Q_t, V_t)dB_t^{(k)} + K_2(Q_t, V_t, R_t)dt, \qquad t \in (0, \tau_{k+1} - \tau_k).
$$

By enlarging the probability space if necessary, we may assume that $\{B_t^{(k)}\}_t, k \geq 0$, are independent. For any $t \geq 0$, define B_t by $B_t = B_{t-\tau_k}^{(k)} + B_{\tau_k-\tau_{k-1}}^{(k-1)} + \cdots + B_{\tau_1}^{(0)}$ if $t \in [\tau_k, \tau_{k+1})$. Then $\{B_t\}_{t\geq 0}$ is a Brownian motion, and it is trivial that (Q_t, V_t, H_t, R_t) satisfies all of the equations in (7.1) except the one with respect to ΔV_t. The fact that ΔV_t satisfies the third equation in (7.1) is a simple consequence of $(\mu 4)$ and $(\mu 5)$. $\square$

Claim 2. Pathwise uniqueness of the solution of (7.1) holds.

Proof of Claim 2. Let $\{Y_t = (Q_t^Y, V_t^Y, H_t^Y, R_t^Y)\}$ and $\{Z_t = (Q_t^Z, V_t^Z, H_t^Z, R_t^Z)\}$ be two strong solutions of (7.1). We prove that $P(Y_t = Z_t, t \geq 0) = 1$.

Let $\tau_0 = 0$ and for any $k \in \mathbb{N}$, let $\tau_k = \inf\{t > \tau_{k-1}; |Q_t^Y| = r_2 \text{ or } |Q_t^Z| = r_2\}$. (Notice that the definition of τ_k is different from before.) By [2, Corollary 5] and Lemma 7.1, we have that τ_k is strictly increasing with respect to t. Also, we use a_k as a flag to clarify in which phase the particle evolves right after τ_k, precisely, we define

$$a_k = \begin{cases} 1; & \text{if } Q_{\tau_k} \cdot V_{\tau_k} > 0, \\ 0; & \text{if } Q_{\tau_k} \cdot V_{\tau_k} < 0. \end{cases}$$

So $a_k = 1$ if the particle is in the diffusion phase right after τ_k, and $a_k = 0$ if the particle is in the uniform motion phase right after τ_k. By (μ5), a_k is given by

$$a_k = \begin{cases} 1; & \text{if } Q_{\tau_k} \cdot V_{\tau_k -} > 0 \text{ and } H_t > \sqrt{1+|R_t|^2}, \\ 0; & \text{if (1) } Q_{\tau_k} \cdot V_{\tau_k -} < 0, \\ & \text{or (2) } Q_{\tau_k} \cdot V_{\tau_k -} > 0 \text{ and } H_t < \sqrt{1+|R_t|^2}. \end{cases}$$

In order to prove Claim 2, it suffices to prove that $P(Y_t = Z_t, t < \tau_k)$ for any $k \in \mathbb{N}$. We prove this by induction.

First, for $k = 1$, we have that until τ_1, both Y and Z keep in the diffusion phase, hence both $|V_t^Y| < 1$ and $|V_t^Z| < 1$ almost surely by definition. So as claimed in Remark 1.2, if we define $P_t^Y = \frac{V_t^Y}{\sqrt{1-|V_t^Y|^2}}$ and $P_t^Z = \frac{V_t^Z}{\sqrt{1-|V_t^Z|^2}}$, then both $\{(Q_t^Y, P_t^Y); t < \tau_1\}$ and $\{(Q_t^Z, P_t^Z); t < \tau_1\}$ satisfy the following system of SDEs

$$\begin{cases} dQ_t = \frac{P_t}{\sqrt{1+|P_t|^2}} dt \\ dP_t = \sigma(Q_t) dB_t - \gamma \frac{P_t}{\sqrt{1+|P_t|^2}} dt, \end{cases} \tag{7.2}$$

and (H_t^Y, R_t^Y) and (H_t^Z, R_t^Z) are given by $H_t^U = \sqrt{1+|P_t^U|^2}$ and $R_t^U = \pi_{Q_t^U}^{\perp} P_t^U$, $U \in \{Y, Z\}$. Since all of the coefficients of (7.2) are Lipschitz continuous, the pathwise uniqueness of the solution of (7.2) holds. Therefore,

$$P(Y_t = Z_t, t < \tau_1) = 1.$$

Next, for any $k \in \mathbb{N}$, if $P(Y_t = Z_t, t < \tau_k) = 1$, we prove in the following that $P(Y_t = Z_t, t < \tau_{k+1}) = 1$. Indeed, since for $U \in \{Y, Z\}$, we have that (Q_t^U, H_t^U, R_t^U) is continuous in t and ΔV_t^U is determined by Q_t^U, H_t^U, R_t^U and V_{t-}^U, our assumption $P(Y_t = Z_t, t < \tau_k) = 1$ implies that

$$P(Y_{\tau_k} = Z_{\tau_k}) = 1.$$

In particular, since by the definition of τ_k, at least one of $Q_{\tau_k}^Y$ and $Q_{\tau_k}^Z$ has norm r_2, we get that $P(|Q_{\tau_k}^Y| = |Q_{\tau_k}^Z| = r_2) = 1$. Depending on whether $a_k = 1$ or $a_k = 0$. We now have that the particle stays in either the diffusion phase or the uniform motion phase, *i.e.*, $|Q_t^Y|, |Q_t^Z| \in (r_2, \infty), t \in (\tau_k, \tau_{k+1})$ or $|Q_t^Y|, |Q_t^Z| \in [r_1, r_2), t \in (\tau_k, \tau_{k+1})$, respectively. In particular, notice that in the latter case, we have that $\tau_{k+1} = \tau_k + 2(r_2 - r_1)$.

As in the proof of Claim 1, we use the notation $U_t^\eta = U_{t+\eta} - U_\eta$ for any $t > 0$, any stopping time η and any stochastic process U. Then by Remark 1.2, we have that either of the following two cases holds, depending on $a_k = 1$ or $a_k = 0$: (1) let

$P_t^Y := \frac{V_t^Y}{\sqrt{1-|V_t^Y|^2}}$ and $P_t^Z := \frac{V_t^Z}{\sqrt{1-|V_t^Z|^2}}$, then both $\{(Q_t^{Y,\tau_k}, P_t^{Y,\tau_k}); t < \tau_{k+1}-\tau_k\}$ and $\{(Q_t^{Z,\tau_k}, P_t^{Z,\tau_k}); t < \tau_{k+1}-\tau_k\}$ satisfy the SDE (7.2) and (H_t^Y, R_t^Y) and (H_t^Z, R_t^Z) are given by $H_t^U = \sqrt{1+|P_t^U|^2}$ and $R_t^U = \pi^{\perp}_{Q_t^U} P_t^U$, $U \in \{Y, Z\}$, $t \in (\tau_k, \tau_{k+1})$; (2) both (Q_t^Y, V_t^Y) and (Q_t^Z, V_t^Z), $t \in (\tau_k, \tau_{k+1})$ are given by

$$dQ_t = V_t dt, \quad V_t = -1_{\{t\in(\tau_k,\tau_k+r_2-r_1)\}}\frac{Q_t}{|Q_t|} + 1_{\{t\in[\tau_k+r_2-r_1,\tau_{k+1})\}}\frac{Q_t}{|Q_t|},$$

and both $(H_t^{Y,\tau_k}, R_t^{Y,\tau_k})$ and $(H_t^{Z,\tau_k}, R_t^{Z,\tau_k})$ satisfy the SDE

$$\left\{\begin{array}{l} dH_t = {}^tV_t\sigma(Q_t)dB_t - \gamma dt, \\ dR_t = \Big(\sigma(Q_t) - \frac{1}{|Q_t|^2}Q_t\{{}^tQ_t\}\sigma(Q_t)\Big)dB_t - \frac{Q_t\cdot V_t}{|Q_t|^2}R_t dt. \end{array}\right. \tag{7.3}$$

Since all of the coefficients in (7.2) and (7.3) are Lipschitz continuous, the pathwise uniqueness of the solution of (7.2) and (7.3) holds. Therefore,

$$P(Y_t = Z_t, t < \tau_{k+1}) = 1.$$

This completes the proof of Claim 2 by induction. □

Now, we are ready to complete the proof of Theorem 1.1 (1). By Claims 1 and 2, it suffices to prove that the pathwise uniqueness of the solution of (7.1) implies that the uniqueness in the sense of the probability law holds for the solution of the same equation. We prove this in the following. The idea is similar to that of [5]. Let (Y_t, B_t) and (Y_t', B_t') be two weak solutions of (7.1), and let $P(dw_1dw_2)$ and $P'(dw_1dw_2)$ be the probability laws of them on $(\widetilde{W}\times W, \mathcal{B}(\widetilde{W}\times W))$. Here $\widetilde{W}$ is as before, and $W = C([0,\infty);\mathbb{R}^d)$. Let $P_{w_2}(dw_1)$ be the regular conditional distribution of $P(dw_1dw_2)$ given w_2, and define $P'_{w_2}(dw_1)$ in the same way. Finally, define a probability measure $Q(dw_1dw_2dw_3)$ on $(\widetilde{W}\times\widetilde{W}\times W, \mathcal{B}(\widetilde{W}\times\widetilde{W}\times W))$, by

$$Q(dw_1dw_2dw_3) = P_{w_3}(dw_1)P'_{w_3}(dw_2)R(dw_3),$$

where R is the probability law of Brownian motion $\{B_t\}$ on $(W, \mathcal{B}(W))$.

Define $\mathcal{B}_t(W) = \sigma\{w(s), s\le t\}$, and define $\mathcal{B}_t(\widetilde{W})$, $\mathcal{B}_t(\widetilde{W}\times W)$ and $\mathcal{B}_t(\widetilde{W}\times\widetilde{W}\times W)$ in the same way. As in [5, Lemma 1], for any $B\in\mathcal{B}_t(\widetilde{W})$, we have that $P_w(B)$ and $P'_w(B)$ are $\mathcal{B}_t(W)$-measurable. So for any $t > s > 0$, any $\mathcal{B}_s(\widetilde{W})$-measurable functions F_1, F_2 and $\mathcal{B}_s(W)$-measurable function F_3, we have that $\int_{\widetilde{W}} F_1(w_1)P_w(dw_1)$ and $\int_{\widetilde{W}} F_2(w_2)P'_w(dw_2)$ are $\mathcal{B}_s(W)$-measurable, hence

$$\begin{aligned}&\int_{\widetilde{W}\times\widetilde{W}\times W}[w_3^i(t)-w_3^i(s)]F_1(w_1)F_2(w_2)F_3(w_3)Q(dw_1dw_2dw_3)\\ &= \int_W [w_3^i(t)-w_3^i(s)]\Big(\int_{\widetilde{W}}F_1(w_1)P_w(dw_1)\Big)\Big(\int_{\widetilde{W}}F_2(w_2)P'_w(dw_2)\Big)F_3(w)R(dw)\\ &= 0.\end{aligned}$$

Similarly,

$$\int_{\widetilde{W}\times\widetilde{W}\times W}\Big\{[w_3^i(t)-w_3^i(s)][w_3^j(t)-w_3^j(s)]-\delta_{ij}(t-s)\Big\}$$
$$F_1(w_1)F_2(w_2)F_3(w_3)Q(dw_1dw_2dw_3)=0.$$

Therefore, $\{w_3(t)\}_t$ is a $\{\mathcal{B}_t(\widetilde{W}\times\widetilde{W}\times W)\}$-BM under Q.

Since (Y_t, B_t) and (w_1, w_3) are the equivalent processes and so are (Y'_t, B'_t) and (w_2, w_3), we have two solutions (w_1, w_3) and (w_2, w_3) on the same filtered space $(\widetilde{W}\times\widetilde{W}\times W, \mathcal{B}(\widetilde{W}\times\widetilde{W}\times W), Q; \mathcal{B}_t(\widetilde{W}\times\widetilde{W}\times W))$. Since the initial conditions are the same, we get from the pathwise uniqueness of the solution that $w_1(t) = w_2(t)$, Q-almost surely. This implies that $P(dw_1dw_2) = P'(dw_1dw_2)$, and completes our proof of the uniqueness. □

8. **The convergence**

In this section, we prove that any cluster point of μ_λ as $\lambda\to\infty$ satisfies (μ1) $\sim$ (μ5). This combined with Theorem 1.1 (1) completes the proof of Theorem 1.1 (2). First, it suffices to prove the assertion with $t\in[0,T]$ for any $T>0$. Also, since $\lim_{\lambda\to\infty}\mu_\lambda(\tau<T)=0$ by Corollary 3.2, it suffices to consider the processes with $t\wedge\tau$ instead of t.

Let μ_∞ be any such cluster point, *i.e.*, there exists a sequence $\lambda_n\to\infty$ $(n\to\infty)$ such that $\mu_{\lambda_n}\to\mu_\infty$ weakly as $n\to\infty$.

The fact that μ_∞ satisfies (μ1) is trivial, and the fact that μ_∞ satisfies (μ2) is a direct consequence of Corollary 3.2.

Also, if we can prove that it satisfies (μ4), then $dV_t=0$ in the domain $|Q_t|\in(r_1,r_2)$; while when $|Q_t|>r_2$, we have that $\nabla U(Q_t)=0$, so by a simple calculation, we get that in the domain $|Q_t|>r_2$, $dV_t^\lambda = A_1^v(Q_t^\lambda, V_t^\lambda)dB_t + A_2^v(Q_t^\lambda, V_t^\lambda)dt$. Combining this with Lemmas 3.1 and 4.2, we get that μ_∞ satisfies (μ3).

Therefore, in order to get our assertion, it suffices to prove that μ_∞ satisfies (μ4) and (μ5).

We first prepare the following.

Lemma 8.1. *$\int_0^{T\wedge\tau} f(Q_t)g(V_t)dt$ is continuous in $(Q_\cdot, V_\cdot)\in D([0,T])\times L^p([0,T])$ for any $p\geq 1$ and any $f, g\in C_b^1(\mathbb{R}^d;\mathbb{R})$.*

Proof. For any (Q^1, V^1) and (Q^2, V^2), we have that

$$\Big|\int_0^T f(Q_t^1)g(V_t^1)dt - \int_0^T f(Q_t^2)g(V_t^2)dt\Big|$$
$$\leq \Big|\int_0^T f(Q_t^1)(g(V_t^1)-g(V_t^2))dt\Big| + \Big|\int_0^T \Big(f(Q_t^1)-f(Q_t^2)g(V_t^2)\Big)dt\Big|$$

$$\leq \|f\|_\infty \|g'\|_\infty \Big| \int_0^T \Big| V_t^1 - V_t^2 \Big| dt + \|g\|_\infty \|f'\|_\infty \int_0^T |Q_t^1 - Q_t^2| dt.$$

So if $\{Q_t^1 - Q_t^2; t \in [0,T]\} \to 0$ in $D([0,T])$, hence in $L^p([0,T])$, and if $\{V_t^1 - V_t^2; t \in [0,T]\} \to 0$ in $L^1([0,T])$, then $\int_0^{T\wedge\tau} f(Q_t^1) g(V_t^1) dt - \int_0^{T\wedge\tau} f(Q_t^2) g(V_t^2) dt \to 0$. □

Notice that since $V_t = \sqrt{1-|V_t|^2} R_t + \pi_{Q_t} V_t$, and R_t is almost surely finite, we have that $\{V_t \neq \pm \frac{Q_t}{|Q_t|}\} \subset \{|V_t| < 1\}$. Therefore, the fact that μ_∞ satisfies (μ4) is a consequence of the following two lemmas.

Lemma 8.2. *For any $g \in C_0^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ with supp$(g) \in \Big(B(0,r_2) \setminus \overline{B(0,r_1)}\Big) \times \mathbb{R}^d$, we have that $g(Q_t, V_t)$ is continuous under μ_∞.*

Proof. Choose any such g. Then there exist constants $\varepsilon_1, \varepsilon_2 > 0$ such that $g(Q_t, V_t) \neq 0 \Rightarrow |Q_t| \in (r_1 + \varepsilon_1, r_2 - \varepsilon_1)$, hence $U(Q_t) < -\varepsilon_2$.

By Lemma 5.1 and Ito's formula, we have that

$$\begin{aligned} & dg(Q_t^\lambda, V_t^\lambda) \\ & = g_1(Q_t^\lambda, V_t^\lambda) \cdot V_t^\lambda dt + g_2(Q_t^\lambda, V_t^\lambda) \cdot \Big(A_1^v(Q_t^\lambda, V_t^\lambda) dB_t + A_2^v(Q_t^\lambda, V_t^\lambda) dt \Big) \\ & \quad - g_2(Q_t^\lambda, V_t^\lambda) \cdot \lambda \Big((1 - |V_t^\lambda|^2)^{1/2} \nabla U(Q_t^\lambda) - (V_t^\lambda \cdot \nabla U(Q_t^\lambda))(1 - |V_t^\lambda|^2) P_t \Big) dt \\ & \quad + \frac{1}{2} g_{22}(Q_t^\lambda, V_t^\lambda) A_1^v(Q_t^\lambda, V_t^\lambda)^2 dt. \end{aligned}$$

By Lemma 5.5, we have that

$$\sup_{\lambda \geq 1} \sup_{s \in [0,T]} \lambda^2 E^{\mu_\lambda} \Big[(1 - |V_t^\lambda|^2) 1_{\{|Q_t^\lambda| \in (r_1+\varepsilon_1, r_2-\varepsilon_1)\}} \Big] < \infty. \tag{8.1}$$

The other coefficients are all bounded. Therefore, we get by Lemma 2.2 that $\{$the distribution of $\{g(Q_t^\lambda, V_t^\lambda), t \in [0,T]\}; \lambda \geq 1\}$ is tight in $D([0,T])$. Since $g(Q_t^\lambda, V_t^\lambda)$ is continuous for any $\lambda \geq 1$, and $C([0,T])$ is closed in $D([0,T])$, we get our assertion. □

Lemma 8.3. *We have μ_∞-almost surely that $|V_t| = 1$ whenever $|Q_t| < r_2$.*

Proof. We first prove the assertion for $|Q_t| \in (r_1, r_2)$, *i.e.*, we prove that

$$\mu_\infty(\Big\{ |Q_t| \in (r_1, r_2) \text{ and } |V_t| < 1 \text{ for some } t \in [0,T] \Big\}) = 0.$$

It suffices to prove that

$$\mu_\infty(\Big\{ |Q_t| \in (r_1 + \varepsilon, r_2 - \varepsilon) \text{ and } |V_t| < 1 - \delta \text{ for some } t \in [0,T] \Big\}) = 0$$

for any $\varepsilon, \delta > 0$. Since by Lemma 8.2, we have μ_∞-almost surely that V_t is continuous with respect to t when $|Q_t| \in (r_1, r_2)$, it suffices, in turn, to prove that

$$\mu_\infty(\Big\{ \int_0^T 1_{\{|Q_t| \in (r_1+\varepsilon, r_2-\varepsilon)\}} 1_{\{|V_t| < 1-\delta\}} dt > 0 \Big\}) = 0. \tag{8.2}$$

Fix any $\varepsilon, \delta > 0$. Choose $g_1, g_2 \in C_0^1(\mathbb{R}^d)$ such that $1_{\{|q| \in (r_1+\varepsilon, r_2-\varepsilon)\}} \leq g_1(q) \leq 1_{\{|q| \in (r_1+\frac{\varepsilon}{2}, r_2-\frac{\varepsilon}{2})\}}$ and $1_{\{|v|<1-\delta\}} \leq g_2(v) \leq 1_{\{|v|<1-\frac{\delta}{2}\}}$. Then in order to prove (8.2), it suffices to prove that

$$\mu_\infty\left(\left\{\int_0^{T\wedge\tau} g_1(Q_t)g_2(V_t)dt > 0\right\}\right) = 0.$$

By Lemma 8.1, we have that $\{\int_0^T g_1(Q_t)g_2(V_t)dt > 0\}$ is an open set. So

$$\begin{aligned}
&\mu_\infty(\Big\{\int_0^{T\wedge\tau} g_1(Q_t)g_2(V_t)dt > 0\Big\}) \\
&\leq \lim_{n\to\infty} \mu_{\lambda_n}(\Big\{\int_0^{T\wedge\tau} g_1(Q_t)g_2(V_t)dt > 0\Big\}) \\
&\leq \lim_{n\to\infty} \mu_{\lambda_n}(\Big\{\int_0^{T\wedge\tau} 1_{\{|Q_t| \in (r_1+\frac{\varepsilon}{2}, r_2-\frac{\varepsilon}{2})\}} 1_{\{|V_t|<1-\frac{\delta}{2}\}} dt > 0\Big\}) \\
&\leq \lim_{n\to\infty} \mu_{\lambda_n}(\Big\{|Q_t| \in (r_1+\frac{\varepsilon}{2}, r_2-\frac{\varepsilon}{2}) \text{ and } |V_t| < 1-\frac{\delta}{2} \text{ for some } t \in [0, T\wedge\tau]\Big\}).
\end{aligned}$$

On the other hand, by assumption, there exists a constant $\eta > 0$ such that $|q| \in (r_1+\frac{\varepsilon}{2}, r_2-\frac{\varepsilon}{2}) \Rightarrow U(q) < -\eta$. Also, for any $\lambda \geq 1$, we have that

$$|V_t^\lambda| < 1-\frac{\delta}{2} \Leftrightarrow |P_t^\lambda| < \frac{1-\frac{\delta}{2}}{\sqrt{1-(1-\frac{\delta}{2})^2}} \Rightarrow |P_t^\lambda| < \sqrt{\frac{1}{\delta}}.$$

So

$$\begin{aligned}
&|Q_t^\lambda| \in (r_1+\frac{\varepsilon}{2}, r_2-\frac{\varepsilon}{2}), |V_t^\lambda| < 1-\delta \\
&\quad \Rightarrow H_t^\lambda = \sqrt{1+|P_t^\lambda|^2} + \lambda U(Q_t^\lambda) < \sqrt{1+\frac{1}{\delta}} - \lambda\eta.
\end{aligned}$$

Therefore, for any $\lambda \geq 1$ large enough such that $\sqrt{1+\frac{1}{\delta}} - \lambda\eta < -\frac{\eta}{2}\lambda$, we have by the definition of μ_λ and Lemma 3.1 (2) that

$$\begin{aligned}
&\mu_\lambda(\Big\{|Q_t| \in (r_1+\frac{\varepsilon}{2}, r_2-\frac{\varepsilon}{2}) \text{ and } |V_t| < 1-\frac{\delta}{2} \text{ for some } t \in [0, T\wedge\tau]\Big\}) \\
&= P(\Big\{|Q_t^\lambda| \in (r_1+\frac{\varepsilon}{2}, r_2-\frac{\varepsilon}{2}) \text{ and } |V_t^\lambda| < 1-\frac{\delta}{2} \text{ for some } t \in [0, T\wedge\tau]\Big\}) \\
&\leq P(\Big\{H_t^\lambda < \sqrt{1+\frac{1}{\delta}} - \lambda\eta \text{ for some } t \in [0, T\wedge\tau]\Big\}) \\
&\leq P(\Big\{\sup_{t\in[0,T]} |H_t^\lambda| > \frac{\eta}{2}\lambda\Big\}) \\
&\leq \lambda^{-1}\frac{2}{\eta} E[\sup_{t\in[0,T]} |H_t^\lambda|] \\
&\leq \lambda^{-1}\frac{2}{\eta} C_2 \to 0, \qquad \lambda \to \infty.
\end{aligned}$$

This completes the proof of our assertion for $|Q_t| \in (r_1, r_2)$.

Since $|Q_t| \geq r_1$ μ_∞-almost surely by (μ2), the only thing left to be proven is that under μ_∞, $|V_t| = 1$ when $|Q_t| = r_1$. It suffices to prove that there exists an $\varepsilon > 0$ small enough such that $\int_0^{T\wedge\tau}(1-|V_t|^2)1_{\{|Q_t|\in(r_1-\varepsilon,r_1+\varepsilon)\}}dt = 0$, μ_∞-almost surely.

By assumption, there exists a function $g \in C_b^1(\mathbb{R}^d)$ and constants $\delta, \varepsilon > 0$ such that $\delta 1_{\{|q|\in(r_1-\varepsilon,r_1+\varepsilon)\}} \leq g(q) \leq |\nabla U(q)|$ for any $q \in \mathbb{R}^d$. By Lemma 8.1, we have that $\{\int_0^{T\wedge\tau}(1-|V_t|^2)g(Q_t)dt > a\}$ is an open set in $D([0,T]) \times L^p([0,T])$ for any $a > 0$. So for any $a > 0$, we have that

$$
\begin{aligned}
&\mu_\infty\Big(\int_0^{T\wedge\tau}(1-|V_t|^2)1_{\{|Q_t|\in(r_1-\varepsilon,r_1+\varepsilon)\}}dt > a\Big)\\
&\quad\leq \mu_\infty\Big(\int_0^{T\wedge\tau}(1-|V_t|^2)g(Q_t)dt > \delta a\Big)\\
&\quad\leq \lim_{n\to\infty}\mu_{\lambda_n}\Big(\int_0^{T\wedge\tau}(1-|V_t|^2)g(Q_t)dt > \delta a\Big)\\
&\quad\leq (\delta a)^{-1}\lim_{n\to\infty}\lambda_n^{-1}E^{\mu_{\lambda_n}}\Big[\lambda_n\int_0^{T\wedge\tau}(1-|V_t|^2)|\nabla U(Q_t)|dt\Big].
\end{aligned}
$$

The expectation on the right hand side above is bounded for $n \in \mathbb{N}$ by Lemma 6.5, so

$$
\mu_\infty\Big(\int_0^{T\wedge\tau}(1-|V_t|^2)1_{\{|Q_t|\in(r_1-\varepsilon,r_1+\varepsilon)\}}dt > a\Big) = 0, \qquad a > 0.
$$

Therefore,

$$
\mu_\infty\Big(\int_0^{T\wedge\tau}(1-|V_t|^2)1_{\{|Q_t|\in(r_1-\varepsilon,r_1+\varepsilon)\}}dt = 0\Big) = 1.
$$

□

Finally, we check that μ_∞ satisfies (μ5). This is a consequence of the following three Lemmas.

Lemma 8.4. *There exists a constant $\varepsilon > 0$ such that*

$$
\int_0^{T\wedge\tau} 1_{\{H_t<\sqrt{1+|R_t|^2}\}}1_{\{|Q_t|\in(r_2-\varepsilon,r_2+\varepsilon)\}}(1-|V_t|^2)dt = 0, \qquad \mu_\infty - \textit{almost surely.}
$$

Proof. Let $\varepsilon > 0$ be a constant such that $|\nabla U(x)| > 0$ as long as $|x| \in (r_2 - 2\varepsilon, r_2)$. It suffices to prove that

$$
\mu_\infty\Big(\int_0^{T\wedge\tau} 1_{\{H_t<\sqrt{1+|R_t|^2}-2\varepsilon_1\}}1_{\{|Q_t|\in(r_2-\varepsilon,r_2+\varepsilon)\}}(1-|V_t|^2)dt > \varepsilon_3\Big) = 0
$$

for any $\varepsilon_1, \varepsilon_3 > 0$.

Choose $f_1 \in C_b^\infty(\mathbb{R})$ and $f_2 \in C_0^\infty(\mathbb{R}^d)$ such that

$$
1_{\{x>2\varepsilon_1\}} \leq f_1(x) \leq 1_{\{x>\varepsilon_1\}}, \qquad 1_{\{|x|\in(r_2-\varepsilon,r_2+\varepsilon)\}} \leq f_2(x) \leq 1_{\{|x|\in(r_2-2\varepsilon,r_2+2\varepsilon)\}}.
$$

Since $\int_0^{T\wedge\tau} f_1(\sqrt{1+|R_t|^2}-H_t)f_2(Q_t)(1-|V_t|^2)dt$ is continuous with respect to (Q,V,H,R), we have that

$$\begin{aligned}&\mu_\infty\Big(\int_0^{T\wedge\tau} 1_{\{H_t<\sqrt{1+|R_t|^2}-2\varepsilon_1\}}1_{\{|Q_t|\in(r_2-\varepsilon,r_2+\varepsilon)\}}(1-|V_t|^2)dt>\varepsilon_3\Big)\\ &\quad\le \mu_\infty\Big(\int_0^{T\wedge\tau} f_1(\sqrt{1+|R_t|^2}-H_t)f_2(Q_t)(1-|V_t|^2)dt>\varepsilon_3\Big)\\ &\quad\le \lim_{n\to\infty}\mu_{\lambda_n}\Big(\int_0^{T\wedge\tau} f_1(\sqrt{1+|R_t|^2}-H_t)f_2(Q_t)(1-|V_t|^2)dt>\varepsilon_3\Big).\end{aligned}$$

Therefore, in order to get our assertion, it suffices to prove that

$$\lim_{\lambda\to\infty}P\Big(\int_0^{T\wedge\tau} f_1(\sqrt{1+|R_t^\lambda|^2}-H_t^\lambda)f_2(Q_t^\lambda)(1-|V_t^\lambda|^2)dt>\varepsilon_3\Big)=0. \tag{8.3}$$

We prove this in the following.

Notice that for any $\lambda\ge 1$, we have that $H_t^\lambda=\sqrt{1+|P_t^\lambda|^2}+\lambda U(Q_t^\lambda)\ge\sqrt{1+|R_t^\lambda|^2}+\lambda U(Q_t^\lambda)$, therefore, if $H_t^\lambda<\sqrt{1+|R_t^\lambda|^2}-\varepsilon_1$, then $U(Q_t^\lambda)<0$, which implies that $Q_t^\lambda\in(r_1,r_2)$, so if $|Q_t^\lambda|\in(r_2-2\varepsilon,r_2+2\varepsilon)$ in addition, we get that $|Q_t^\lambda|\in(r_2-2\varepsilon,r_2)$. So for any $\delta\in(0,2\varepsilon)$ and $\lambda\ge1$, we have that

$$\begin{aligned}&P\Big(\int_0^{T\wedge\tau} f_1(\sqrt{1+|R_t^\lambda|^2}-H_t^\lambda)f_2(Q_t^\lambda)(1-|V_t^\lambda|^2)dt>\varepsilon_3\Big)\\ &\quad\le P\Big(\int_0^{T\wedge\tau} 1_{\{H_t^\lambda<\sqrt{1+|R_t^\lambda|^2}-\varepsilon_1\}}1_{\{|Q_t^\lambda|\in(r_2-2\varepsilon,r_2-\delta]\}}(1-|V_t^\lambda|^2)dt>\varepsilon_3/2\Big)\\ &\qquad+P\Big(\int_0^{T\wedge\tau} 1_{\{H_t^\lambda<\sqrt{1+|R_t^\lambda|^2}-\varepsilon_1\}}1_{\{|Q_t^\lambda|\in(r_2-\delta,r_2)\}}(1-|V_t^\lambda|^2)dt>\varepsilon_3/2\Big).\end{aligned}$$

Let us first deal with the second term on the right hand side above. Since $|Q_t^\lambda|\in(r_2-\delta,r_2)$, we have that $H_t^\lambda+\lambda|U(Q_t^\lambda)|=H_t^\lambda-\lambda U(Q_t^\lambda)=\sqrt{1+|P_t^\lambda|^2}=(1-|V_t^\lambda|^2)^{-1/2}$, so

$$\lambda|\nabla U(Q_t^\lambda)|(1-|V_t^\lambda|^2)^{1/2}=\frac{|\nabla U(Q_t^\lambda)|}{|U(Q_t^\lambda)|}\cdot\frac{\lambda|U(Q_t^\lambda)|}{H_t+\lambda|U(Q_t^\lambda)|}.$$

Let $a(\delta):=\inf_{|x|\in(r_2-\delta,r_2)}\frac{|\nabla U(x)|}{|U(x)|}$. Then by assumption, we have that $a(\delta)\to\infty$ as $\delta\to0$. Moreover, since $\lambda|U(Q_t^\lambda)|=\sqrt{1+|P_t^\lambda|^2}-H_t^\lambda\ge\sqrt{1+|R_t^\lambda|^2}-H_t^\lambda$, we have that

$$\begin{aligned}\frac{\lambda|U(Q_t^\lambda)|}{H_t^\lambda+\lambda|U(Q_t^\lambda)|}&=1-\frac{H_t^\lambda}{H_t^\lambda+\lambda|U(Q_t^\lambda)|}\\ &\ge 1-\frac{H_t^\lambda}{H_t^\lambda+\sqrt{1+|R_t^\lambda|^2}-H_t^\lambda}=\frac{\sqrt{1+|R_t^\lambda|^2}-H_t^\lambda}{\sqrt{1+|R_t^\lambda|^2}},\end{aligned}$$

so

$$\frac{\lambda|U(Q_t^\lambda)|}{H_t^\lambda+\lambda|U(Q_t^\lambda)|}\ge\frac{\varepsilon_1}{\sqrt{1+|R_t^\lambda|^2}},\qquad \text{if } H_t^\lambda<\sqrt{1+|R_t^\lambda|^2}-\varepsilon_1.$$

Combining the above, we get that

$$\begin{aligned}&\lambda|\nabla U(Q^\lambda_t)|(1-|V^\lambda_t|^2)^{3/2}\\&\geq \lambda|\nabla U(Q^\lambda_t)|(1-|V^\lambda_t|^2)^{1/2}1_{\{H^\lambda_t<\sqrt{1+|R^\lambda_t|^2}-\varepsilon_1\}}1_{\{|Q^\lambda_t|\in(r_2-\delta,r_2)\}}(1-|V^\lambda_t|^2)\\&\geq a(\delta)\frac{\varepsilon_1}{\sqrt{1+|R^\lambda_t|^2}}1_{\{H^\lambda_t<\sqrt{1+|R^\lambda_t|^2}-\varepsilon_1\}}1_{\{|Q^\lambda_t|\in(r_2-\delta,r_2)\}}(1-|V^\lambda_t|^2).\end{aligned}$$

Therefore,

$$\begin{aligned}&P\Big(\int_0^{T\wedge\tau}1_{\{H^\lambda_t<\sqrt{1+|R^\lambda_t|^2}-\varepsilon_1\}}1_{\{|Q^\lambda_t|\in(r_2-\delta,r_2)\}}(1-|V^\lambda_t|^2)dt>\varepsilon_3/2\Big)\\&\leq P\Big(\int_0^{T\wedge\tau}\frac{1}{a(\delta)\varepsilon_1}\lambda|\nabla U(Q^\lambda_t)|(1-|V^\lambda_t|^2)^{3/2}\sqrt{1+|R^\lambda_t|^2}dt>\varepsilon_3/2\Big)\\&\leq a(\delta)^{-1}\frac{2}{\varepsilon_1\varepsilon_3}E\Big[\int_0^{T\wedge\tau}\lambda|\nabla U(Q^\lambda_t)|(1-|V^\lambda_t|^2)^{3/2}\sqrt{1+|R^\lambda_t|^2}dt\Big].\end{aligned}$$

The expectation on the right hand side above is bounded for $\lambda\geq 1$ by Lemma 5.4. Therefore, we get that

$$\lim_{\delta\to 0}\sup_{\lambda\geq 1}P\Big(\int_0^{T\wedge\tau}1_{\{H^\lambda_t<\sqrt{1+|R^\lambda_t|^2}-\varepsilon_1\}}1_{\{|Q^\lambda_t|\in(r_2-\delta,r_2)\}}(1-|V^\lambda_t|^2)dt>\varepsilon_3/2\Big)=0.$$

Therefore, in order to prove (8.3), it suffices to prove that

$$\lim_{\lambda\to\infty}\mu_\lambda\Big(\int_0^{T\wedge\tau}1_{\{H_t<\sqrt{1+|R_t|^2}-\varepsilon_1\}}1_{\{|Q_t|\in(r_2-2\varepsilon,r_2-\delta]\}}(1-|V_t|^2)dt>\varepsilon_3/2\Big)=0$$

for any $\delta>0$. We prove it in the following. By Lemma 5.5, we have that $C_\delta:=\sup_{\lambda\geq 1}\sup_{s\in[0,T]}\lambda^2E\Big[(1-|V^\lambda_t|^2)1_{\{|Q^\lambda_t|\in(r_2-2\varepsilon,r_2-\delta]\}}\Big]<\infty$. Therefore,

$$\begin{aligned}&P\Big(\int_0^{T\wedge\tau}1_{\{H^\lambda_t<\sqrt{1+|R^\lambda_t|^2}-\varepsilon_1\}}1_{\{|Q^\lambda_t|\in(r_2-2\varepsilon,r_2-\delta]\}}(1-|V^\lambda_t|^2)dt>\varepsilon_3/2\Big)\\&\leq(\varepsilon_3/2)^{-1}E\Big[\int_0^{T\wedge\tau}1_{\{|Q^\lambda_t|\in(r_2-2\varepsilon,r_2-\delta]\}}(1-|V^\lambda_t|^2)dt\Big]\\&\leq(\varepsilon_3/2)^{-1}TC_\delta\lambda^{-2},\end{aligned}$$

which converges to 0 as $\lambda\to\infty$ for any $\delta>0$. □

Lemma 8.5. *We have μ_∞-almost surely that if $|Q_t|=r_2$, $Q_t\cdot V_{t-}>0$ and $H_t>\sqrt{1+|R_t|^2}$, then $V_t=\frac{\sqrt{H_t^2-1-|R_t|^2}Q_t/|Q_t|+R_t}{H_t}$.*

Proof. It suffices to prove the assertion with the condition $H_t>\sqrt{1+|R_t|^2}$ substituted by $H_t>\sqrt{1+|R_t|^2+\varepsilon^2}$ for any $\varepsilon>0$. We prove the latter in the following.

Choose $\varepsilon_1>0$ such that $U(x)\leq 0$ whenever $|x|\geq r_2-\varepsilon_1(>r_1)$. Let B be the set of ω's that satisfy the following: there exist $t_1,t_2,t_3\in[0,T\wedge\tau]$ such that $t_1<t_2<t_3$, $H_s>\sqrt{1+|R_s|^2+\varepsilon^2}$ and $|Q_s|>r_2-\varepsilon_1$ for any $s\in[t_1,t_3]$, $|Q_{t_2}|^2-|Q_{t_1}|^2>\int_{t_1}^{t_2}\frac{2\varepsilon r_1}{\sqrt{1+|R_u|^2+\varepsilon^2}}du$ and $|Q_{t_3}|^2-|Q_{t_2}|^2<\int_{t_2}^{t_3}\frac{2\varepsilon r_1}{\sqrt{1+|R_u|^2+\varepsilon^2}}du$. Then B is an open set.

We prove in the following that

$$\mu_\lambda(B) = 0, \qquad \text{for any } \lambda \geq 1. \tag{8.4}$$

Indeed, for any $\lambda \geq 1$, if $|Q^\lambda_{t_2}|^2 - |Q^\lambda_{t_1}|^2 > \int_{t_1}^{t_2} \frac{2\varepsilon r_1}{\sqrt{1+|R^\lambda_u|^2+\varepsilon^2}} du$ and $|Q^\lambda_{t_3}|^2 - |Q^\lambda_{t_2}|^2 < \int_{t_2}^{t_3} \frac{2\varepsilon r_1}{\sqrt{1+|R^\lambda_u|^2+\varepsilon^2}} du$, then there exists a $t \in [t_1, t_3]$ such that

$$\frac{d}{dt}\Big(|Q^\lambda_t|^2 - \int_0^t \frac{2\varepsilon r_1}{\sqrt{1+|R^\lambda_u|^2+\varepsilon^2}} du\Big) = 0. \tag{8.5}$$

In particular, $Q^\lambda_t \cdot V^\lambda_t > 0$. On the other hand, if $H^\lambda_s > \sqrt{1+|R^\lambda_s|^2+\varepsilon^2}$ and $|Q_s| > r_2 - \varepsilon_1$ for any $s \in [t_1, t_3]$, then

$$\begin{aligned} |Q^\lambda_s \cdot V^\lambda_s|^2 &= |\pi_{Q^\lambda_s} V^\lambda_s|^2 |Q^\lambda_s|^2 = \Big(1 - \frac{1+|R^\lambda_s|^2}{1+|P^\lambda_s|^2}\Big)|Q^\lambda_s|^2 \\ &= \Big(1 - \frac{1+|R^\lambda_s|^2}{(H^\lambda_s + \lambda|U(Q^\lambda_s)|)^2}\Big)|Q^\lambda_s|^2 \\ &\geq \Big(\frac{\varepsilon}{\sqrt{1+|R^\lambda_s|^2+\varepsilon^2}} \cdot r_1\Big)^2, \qquad \text{for any } s \in [t_1, t_3]. \end{aligned}$$

Since $V^\lambda_\cdot$ and $Q^\lambda_\cdot$ are continuous, and $(Q^\lambda_t, V^\lambda_t) > 0$ with some $t \in [t_1, t_3]$, this implies

$$Q^\lambda_s \cdot V^\lambda_s > \frac{\varepsilon}{\sqrt{1+|R_t|^2+\varepsilon^2}} \cdot r_1, \qquad \text{for any } s \in [t_1, t_3].$$

Therefore,

$$\frac{d}{ds}\Big(|Q^\lambda_s|^2 - \int_0^s \frac{2\varepsilon r_1}{\sqrt{1+|R^\lambda_u|^2+\varepsilon^2}} du\Big) = 2(Q^\lambda_s, V^\lambda_s) - \frac{2\varepsilon r_1}{\sqrt{1+|R^\lambda_s|^2+\varepsilon^2}} > 0, \quad s \in [t_1, t_3].$$

This contradicts (8.5). Therefore, $\mu_\lambda(B) = 0$.

Since B is open, (8.4) implies that

$$\mu_\infty(B) = 0. \tag{8.6}$$

Now, under μ_∞, if $|Q_t| = r_2$, $Q_t \cdot V_{t-} > 0$ and $H_t > \sqrt{1+|R_t|^2+\varepsilon^2}$, then we have that there exists a $\delta > 0$ small enough such that $H_s > \sqrt{1+|R_s|^2+\varepsilon^2}$ and $|Q_s| > r_2 - \varepsilon_1$ for any $s \in [t-\delta, t+\delta]$, and $V_s = \frac{Q_t}{|Q_t|}$ for any $s \in [t-\delta, t)$. Without loss of generality, we assume that $\delta < 2(r_2 - \varepsilon r_1)$. Then for any $s \in [t-\delta, t]$, we have that $Q_s = Q_t - (t-s)\frac{Q_t}{|Q_t|} = (1 - \frac{t-s}{r_2})Q_t$, hence

$$\begin{aligned} |Q_t|^2 - |Q_s|^2 &= \Big(1 - \Big(1 - \frac{t-s}{r_2}\Big)^2\Big)|Q_t|^2 \\ &= 2(t-s)r_2 - (t-s)^2 = (2r_2 - (t-s))(t-s) \\ &> 2\varepsilon r_1 (t-s) \\ &\geq \int_s^t \frac{2\varepsilon r_1}{\sqrt{1+|R_u|^2+\varepsilon^2}} du. \end{aligned}$$

This combined with (8.6) implies that under our condition, for any $t_3 \in (t, t+\delta)$, we have $|Q_{t_3}|^2 - |Q_t|^2 \geq \int_t^{t_3} \frac{2\varepsilon r_1}{\sqrt{1+|R_u|^2+\varepsilon^2}} du$, in particular, $|Q_{t_3}| > r_2$, therefore, $V_{t_3} = \frac{\sqrt{H_{t_3}^2 - 1 - |R_{t_3}|^2} Q_{t_3}/|Q_{t_3}| + R_{t_3}}{H_{t_3}}$. Taking $t_3 \to t+0$ in this equation, since $H_\cdot$ and $R_\cdot$ are continuous and $V_\cdot$ is right-continuous, we get our assertion. □

Lemma 8.6. *We have μ_∞-almost surely that if $|Q_t| = r_2$ and $Q_t \cdot V_{t-} < 0$, then $V_t = -\frac{Q_t}{|Q_t|}$.*

Proof. The proof is similar to that of Lemma 8.5. For any $\varepsilon > 0$, let B_ε be the set of ω's that satisfy the following: there exist $t_1, t_2, t_3 \in [0, T \wedge \tau]$ such that $t_1 < t_2 < t_3$, $H_s > \sqrt{1+|R_s|^2+\varepsilon^2}$ for any $s \in [t_1, t_3]$, $|Q_{t_2}|^2 - |Q_{t_1}|^2 < -\int_{t_1}^{t_2} \frac{2\varepsilon(r_1-\varepsilon_0)}{\sqrt{1+|R_u|^2+\varepsilon^2}} du$ and $|Q_{t_3}|^2 - |Q_{t_2}|^2 > -\int_{t_2}^{t_3} \frac{2\varepsilon(r_1-\varepsilon_0)}{\sqrt{1+|R_u|^2+\varepsilon^2}} du$. Then B_ε is an open set, and by exactly the same method as in the proof of Lemma 8.5, we get that $\mu_\infty(B_\varepsilon) = 0$.

Choose any $\varepsilon > 0$ and suppose that $|Q_t| = r_2$ and $Q_t \cdot V_{t-} < -\varepsilon$. Then since Q is continuous, we have that there exists a $\delta > 0$ such that $|Q_s| \in (r_2, 2r_2)$ and $Q_s \cdot V_s < -\varepsilon$ for any $s \in [t-\delta, t)$. Similar as in the proof of Lemma 8.5, since $|Q_s| > r_2$ implies $H_s = \sqrt{1+|R_s|^2+\frac{|\pi_{Q_s} V_s|^2}{1-|V_s|^2}}$, this implies that $H_s > \sqrt{1+|R_s|^2+(\frac{\varepsilon}{2r_2})^2}$ for any $s \in [t-\delta, t)$. Since H and R are continuous, we get that there exists a $\delta' > 0$ such that $H_s > \sqrt{1+|R_s|^2+(\frac{\varepsilon}{4r_2})^2}$ for any $s \in [t-\delta', t+\delta']$. On the other hand, for any $s \in [t-\delta', t)$, we have that

$$|Q_t|^2 - |Q_s|^2 = \int_s^t 2Q_u \cdot V_u du < -\int_s^t 2\varepsilon du < -\int_s^t \frac{2\frac{\varepsilon}{4r_2}(r_1-\varepsilon_0)}{\sqrt{1+|R_u|^2+(\frac{\varepsilon}{4r_2})^2}} du.$$

Combining this with the fact that $\mu_\infty(B_{\frac{\varepsilon}{4r_2}}) = 0$, we get μ_∞-almost surely that for any $t_3 \in (t, t+\delta']$, we have $|Q_{t_3}|^2 - |Q_t|^2 < -\int_t^{t_3} \frac{2\frac{\varepsilon}{4r_2}(r_1-\varepsilon_0)}{\sqrt{1+|R_u|^2+(\frac{\varepsilon}{4r_2})^2}} du < 0$, in particular, $|Q_{t_3}| < r_2$. By (μ4), this implies that $V_{t_3} = -\frac{Q_{t_3}}{|Q_{t_3}|}$. Since Q is continuous and V is right-continuous, by taking $t_3 \to t+0$, we get that $V_t = -\frac{Q_t}{|Q_t|}$. □

We have now completed the proof of the fact that for any converging subsequence $\{\mu_{\lambda_n}; n \in \mathbb{N}\}$ of $\{\mu_\lambda; \lambda \geq 1\}$ with $\lim_{n\to\infty} \lambda_n = \infty$, its limit satisfies conditions (μ1) $\sim$ (μ5). By uniqueness, this completes the proof of Theorem 1.1 (2).

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